Geometry

CK-12 Foundation

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Chapter 1

Basics of Geometry

1.1 Points, Lines, and Planes

Learning Objectives

- Understand the undefined terms *point*, *line*, and *plane*.
- Understand defined terms, including *space*, *segment*, and *ray*.
- Identify and apply basic postulates of points, lines, and planes.
- Draw and label terms in a diagram.

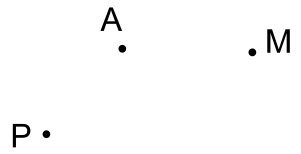
Introduction

Welcome to the exciting world of geometry! Ahead of you lie many exciting discoveries that will help you learn more about the world. Geometry is used in many areas—from art to science. For example, geometry plays a key role in construction, fashion design, architecture, and computer graphics. This course focuses on the main ideas of geometry that are the foundation of applications of geometry used everywhere. In this chapter, you'll study the basic elements of geometry. Later you will prove things about geometric shapes using the vocabulary and ideas in this chapter—so make sure that you completely understand each of the concepts presented here before moving on.

Undefined Terms

The three basic building blocks of geometry are **points**, **lines**, and **planes**. These are undefined terms. While we cannot define these terms precisely, we can get an idea of what they are by looking at examples and models.

A **point** is a location that has no size. To imagine a point, look at the period at the end of this sentence. Now imagine that period getting smaller and smaller until it disappears. A point describes a location, such as the location of the period, but a point has no size. We use dots (like periods) to represent points, but since the dots themselves occupy space, these dots are not points—we only use dots as representations. Points are labeled with a capital letter, as shown below.

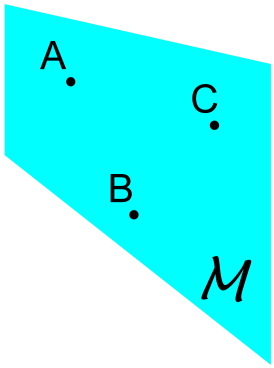


A **line** is an infinite series of points in a row. A line does not occupy space, so to imagine a line you can imagine the thinnest string you can think of, and shrink it until it occupies no space at all. A line has direction and location, but still does not take up space. Lines are sometimes referred to by one italicized letter, but they can also be identified by two points that are on the line. Lines are called one-dimensional, since they have direction in one dimension.

The last undefined term is **plane**. You can think of a plane as a huge sheet of paper—so big that it goes on forever! Imagine the paper as thin as possible, and extend it up, down, left, and right. Planes can be named by letter, or by three points that lie in the plane. You already know one plane from your algebra class—the $xy-$coordinate plane. Planes are called two-dimensional, since any point on a plane can be described by two numbers, called coordinates, as you learned in algebra.

Notation Notes: As new terms are introduced, notation notes will help you learn how to write and say them.

1. Points are named using a single capital letter. The first image shows points A, M, and P.
2. In the image of a line, the same line has several names. It can be called "line g," $\overleftrightarrow{PQ}$, or $\overleftrightarrow{QP}$. The order of the letters does not matter when naming a line, so the same line can have many names. When using two points to name a line, you must use the line symbol $\leftrightarrow$ above the letters.
3. Planes are named using a script (cursive) letter or by naming three points contained in the plane. The illustrated plane can be called plane M or "the plane defined by points A, B, and C."



Example 1

Which term best describes how San Diego, California, would be represented on a globe?

A. point

B. line

C. plane

A city is usually labeled with a dot, or point, on a globe. Though the city of San Diego occupies space, it is reduced when placed on the globe. Its label is merely to show a location with reference to the other cities, states, and countries on a globe. So, the correct answer is A.

Example 2

Which geometric object best models the surface of a movie screen?

A. point

B. line

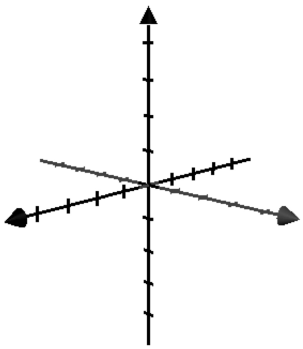
C. plane

The surface of a movie screen extends in two dimensions: up and down and left to right. This description most closely resembles a plane. So, the correct answer is C. Note that a plane is a model of the movie screen, but the screen is not actually a plane. In geometry, planes extend infinitely, but the movie screen does not.

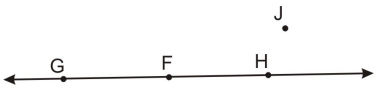
Figure 1.1: Airscreen auf dem James Dean Festival in Marion, USA

Defined Terms

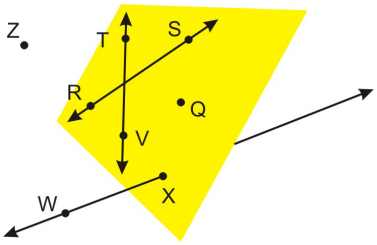
Now we can use **point**, **line**, and **plane** to define new terms. One word that has already been used is **space.** Space is the set of all points expanding in three dimensions. Think back to the plane. It extended along two different lines: up and down, and side to side. If we add a third direction, we have something that looks like three-dimensional space. In algebra, the $x - y$ plane is adapted to model space by adding a third axis coming out of the page. The image below shows three perpendicular axes.



Points are said to be **collinear** if they lie along the same line. The picture below shows points F, G, and H are collinear. Point J is **non-collinear** with the other three since it does not lie in the same line.



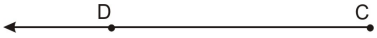
Similarly, points and lines can be **coplanar** if they lie within the same plane. The diagram below shows two lines ($\overleftrightarrow{RS}$ and $\overleftrightarrow{TV}$) and one point (Q) that are coplanar. It also shows line $\overleftrightarrow{WX}$ and point Z that are **non-coplanar** with $\overleftrightarrow{RS}$ and Q.



A **segment** designates a portion of a line that has two endpoints. Segments are named by their endpoints.

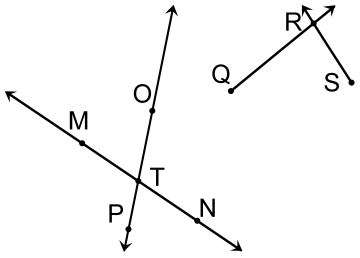
Notation Notes: Just like lines, segments are written with two capital letters. For segments we use a bar on top with no arrows. Segments can also be named in any order, so the segment above could be named $\overline{EF}$ or $\overline{FE}$.

A **ray** is a portion of a line that has only one endpoint and extends infinitely in the other direction. Rays are named by their endpoints and another point on the line. The endpoint always comes first in the name of a ray.



Like segments, rays are named with two capital letters, and the symbol on top has one arrow. The ray is always named with the endpoint first, so we would write $\overrightarrow{CD}$ for the figure above.

An **intersection** is the point or set of points where lines, planes, segments, or rays cross each other. Intersections are very important since you can study the different regions they create.



In the image above, R is the point of intersection of $\overrightarrow{QR}$ and $\overrightarrow{SR}$. T is the intersection of $\overleftrightarrow{MN}$ and $\overleftrightarrow{PO}$.

Example 3

Which geometric object best models a straight road connecting two cities?

A. ray

B. line

C. segment

D. plane

Since the straight road connects two distinct points (cities), and we are interested in the section between those two endpoints, the best term is *segment*. A segment has two endpoints. So, the correct answer is C.

Example 4

Which term best describes the relationship among the strings on a tennis racket?

Figure 1.2: Photograph of a tennis racket and two balls

A. collinear

B. coplanar

C. non-collinear

D. non-coplanar

The strings of a tennis racket are like intersecting segments. They also are all located on the plane made by the head of the racket. So, the best answer is B. Note that the strings are not really the same as segments and they are not exactly coplanar, but we can still use the geometric model of a plane for the head of a tennis racket, even if the model is not perfect.

Basic Postulates

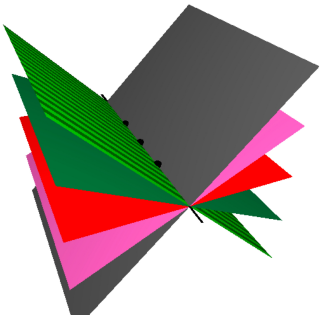
Now that we have some basic vocabulary, we can talk about the rules of geometry. Logical systems like geometry start with basic rules, and we call these basic rules **postulates.** We assume that a postulate is true and by definition a postulate is a statement that cannot be proven.

A **theorem** is a statement that can be proven true using postulates, definitions, logic, and other theorems we've already proven. Theorems are the "results" that are true given postulates and definitions. This section introduces a few basic postulates that you must understand as you move on to learn other theorems.

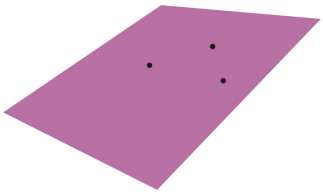
The first of five postulates you will study in this lesson states that there is exactly one line through any two points. You could test this postulate easily with a ruler, a piece of paper, and a pencil. Use your pencil to draw two points anywhere on the piece of paper. Use your ruler to connect these two points. You'll find that there is only one possible straight line that goes through them.

Line Postulate: There is exactly one line through any two points.

Similarly, there is exactly one plane that contains any three non-collinear points. To illustrate this, ask three friends to hold up the tips of their pencils, and try and lay a piece of paper on top of them. If your friends line up their pencils (making the points collinear), there are an infinite number of possible planes. If one hand moves out of line, however, there is only one plane that will contain all three points. The following image shows five planes passing through three collinear points.

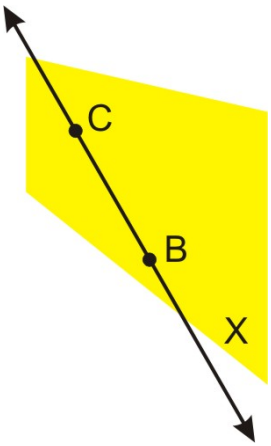


Plane Postulate: There is exactly one plane that contains any three non-collinear points.



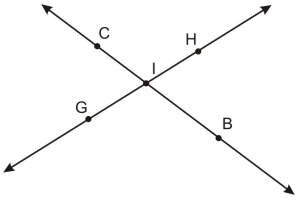
If two coplanar points form a line, that line is also within the same plane.

Postulate: A line connecting points in a plane also lies within the plane.

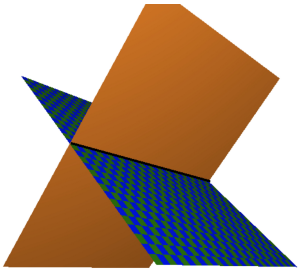


Sometimes lines intersect and sometimes they do not. When two lines do intersect, the intersection will be a single point. This postulate will be especially important when looking at angles and relationships between lines. As an extension of this, the final postulate for this lesson states that when two planes intersect they meet in a single line. The following diagrams show these relationships.

Postulate: The intersection of any two distinct lines will be a single point.



Postulate: The intersection of two planes is a line.



Example 5

How many non-collinear points are required to identify a plane?

A. 1

B. 2

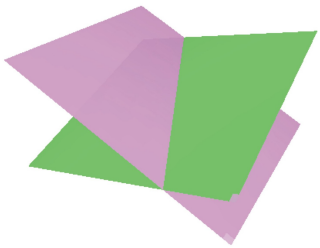
C. 3

D. 4

The second postulate listed in this lesson states that you can identify a plane with three non-collinear points. It is important to label them as non-collinear points since there are infinitely many planes that contain collinear points. The answer is C.

Example 6

What geometric figure represents the intersection of the two planes below?



A. point

B. line

C. ray

D. plane

The fifth postulate presented in this lesson says that the intersection of two planes is a line. This makes sense from the diagram as well. It is a series of points that extends infinitely in both directions, so it is definitely a line. The answer is B.

Drawing and Labeling

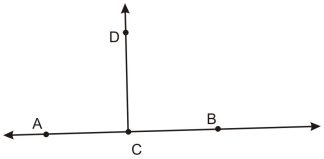
It is important as you continue your study of geometry to practice drawing geometric shapes. When you make geometric drawings, you need to be sure to follow the conventions of geometry so other people can "read" your drawing. For example, if you draw a line, be sure to include arrows at both ends. With only one arrow, it will appear as a ray, and without any arrows, people will assume that it is a line segment. Make sure you label your points, lines, and planes clearly, and refer to them by name when writing explanations. You will have many opportunities to hone your drawing skills throughout this geometry course.

Example 7

Draw and label the intersection of line $\overleftrightarrow{AB}$ *and ray* $\overrightarrow{CD}$ *at point* C.

To begin making this drawing, make a line with two points on it. Label the points A and B.

Next, add the ray. The ray will have an endpoint C and another point D. The description says that the ray and line will intersect at C, so point C should be on $\overleftrightarrow{AB}$. It is not important from this description in what direction $\overrightarrow{CD}$ points.



The diagram above satisfies the conditions in the problem.

Lesson Summary

In this lesson, we explored points, lines, and planes. Specifically, we have learned:

- The significance of the undefined terms point, line, and plane.
- The significance of defined terms including space, segment, and ray.
- How to identify and apply basic postulates of points, lines, and planes.
- How to draw and label the terms you have studied in a diagram.

These skills are the building blocks of geometry. It is important to have these concepts solidified in your mind as you explore other topics of geometry and mathematics.

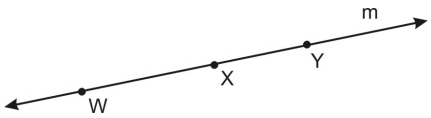
Points to Consider

You can think of postulates as the basic rules of geometry. Other activities also have basic rules. For example, in the game of soccer one of the basic rules is that players are not allowed to use their hands to move the ball. How do the rules shape the way that the game is played? As you become more familiar with the geometric postulates, think about how the basic "rules of the game" in geometry determine what you can and cannot do.

Now that you know some of the basics, we are going to look at how measurement is used in geometry.

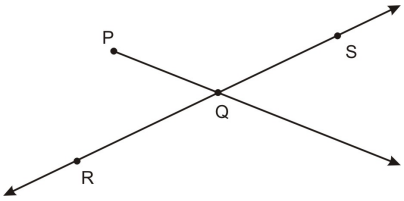
Review Questions

1. Draw an image showing all of the following:
   (a) $\overline{AB}$
   (b) $\overleftrightarrow{CD}$ intersecting $\overline{AB}$
   (c) Plane P containing $\overline{AB}$ but not $\overleftrightarrow{CD}$
2. Name this line in three ways.



3. What is the best possible geometric model for a soccer field? (See figure 1.3.) Explain your answer.
4. What type of geometric object is the intersection of a line and a plane? Draw your answer.
5. What type of geometric object is made by the intersection of three planes? Draw your answer.
6. What type of geometric object is made by the intersection of a sphere (a ball) and a plane? Draw your answer.
7. Use geometric notation to explain this picture in as much detail as possible.

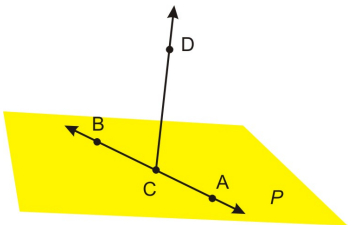
Figure 1.3



8. True or false: Any two distinct points are collinear. Justify your answer.
9. True or false: Any three distinct points **determine** a plane (or in other words, there is exactly one plane passing through any three points). Justify your answer.
10. One of the statements in 8 or 9 is false. Rewrite the false statement to make it true.

Review Answers

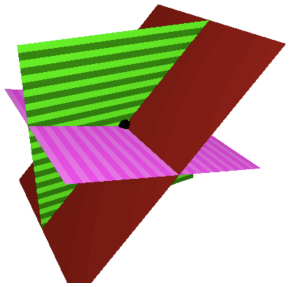
1. Answers will vary, one possible example:



2. $\overleftrightarrow{WX}$,$\overleftrightarrow{YW}$,m (and other answers are possible).
3. A soccer field is like a plane since it is a flat two-dimensional surface.
4. A line and a plane intersect at a point. See the diagram for answer 1 for an illustration.

If $\overrightarrow{CD}$ were extended to be a line, then the intersection of $\overrightarrow{CD}$ and plane P would be point C.

5. Three planes intersect at one point.



6. A circle.

7. $\overrightarrow{PQ}$ intersects $\overleftrightarrow{RS}$ at point Q.
8. True: The Line Postulate implies that you can always draw a line between any two points, so they must be collinear.
9. False. Three collinear points could be at the intersection of an infinite number of planes. See the images of intersecting planes for an illustration of this.
10. For 9 to be true, it should read: "Any three non-collinear points determine a plane."

1.2 Segments and Distance

Learning Objectives

- Measure distances using different tools.
- Understand and apply the ruler postulate to measurement.
- Understand and apply the segment addition postulate to measurement.
- Use endpoints to identify distances on a coordinate grid.

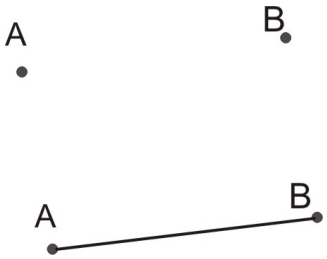
Introduction

You have been using measurement for most of your life to understand quantities like weight, time, distance, area, and volume. Any time you have cooked a meal, bought something,

or played a sport, measurement has played an important role. This lesson explores the postulates about measurement in geometry.

Measuring Distances

There are many different ways to identify measurements. This lesson will present some that may be familiar, and probably a few that are new to you. Before we begin to examine distances, however, it is important to identify the meaning of **distance** in the context of geometry. The distance between two points is defined by the length of the line segment that connects them.

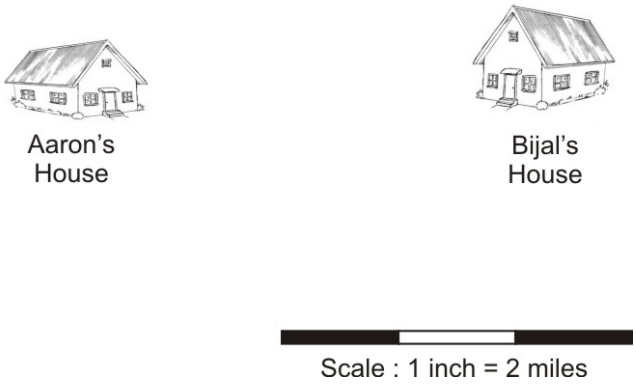


The most common way to measure distance is with a ruler. Also, distance can be estimated using scale on a map. Practice this skill in the example below.

Notation Notes: When we name a segment we use the endpoints and and overbar with no arrows. For example, "Segment AB" is written $\overline{AB}$. The length of a segment is named by giving the endpoints without using an overline. For example, the length of $\overline{AB}$ is written AB. In some books you may also see $m\overline{AB}$, which means the same as AB, that is, it is the length of the segment with endpoints A and B.

Example 1

Use the scale to estimate the distance between Aaron's house and Bijal's house. Assume that the first third of the scale in black represents one inch.



You need to find the distance between the two houses in the map. The scale shows a sample

distance. Use the scale to estimate the distance. You will find that approximately three segments the length of the scale fit between the two points. Be careful—three is not the answer to this problem! As the scale shows one unit equal to two miles, you must multiply three units by two miles.

$$3 \text{ units} \times \frac{2 \text{ miles}}{1 \text{ unit}} = 6 \text{ miles}$$

The distance between the houses is about six miles.

You can also use estimation to identify measurements in other geometric figures. Remember to include words like *approximately, about,* or *estimation* whenever you are finding an estimated answer.

Ruler Postulate

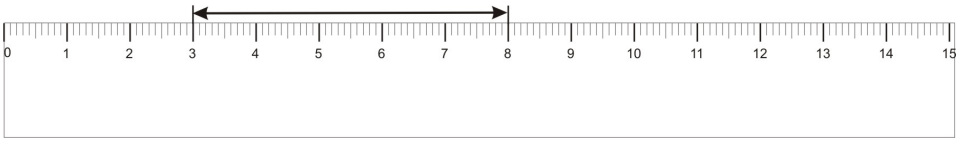
You have probably been using rulers to measure distances for a long time and you know that a ruler is a tool with measurement markings.

Ruler Postulate: If you use a ruler to find the distance between two points, the distance will be the absolute value of the difference between the numbers shown on the ruler.

The ruler postulate implies that you do not need to start measuring at the zero mark, as long as you use subtraction to find the distance. *Note, we say "absolute value" here since distances in geometry must always be positive, and subtraction can yield a negative result.*

Example 2

What distance is marked on the ruler in the diagram below? Assume that the scale is marked in centimeters.



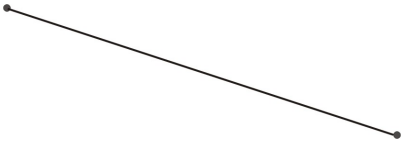
The way to use the ruler is to find the absolute value of difference between the numbers shown. The line segment spans from 3 cm to 8 cm.

$$|3 - 8| = |-5| = 5$$

The absolute value of the difference between the two numbers shown on the ruler is 5 cm. So, the line segment is 5 cm long.

Example 3

Use a ruler to find the length of the line segment below.

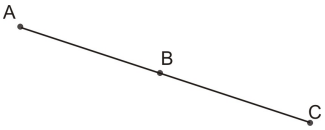


Line up the endpoints with numbers on your ruler and find the absolute value of the difference between those numbers. If you measure correctly, you will find that this segment measures 2.5 inches or 6.35 centimeters.

Segment Addition Postulate

Segment Addition Postulate: The measure of any line segment can be found by adding the measures of the smaller segments that comprise it.

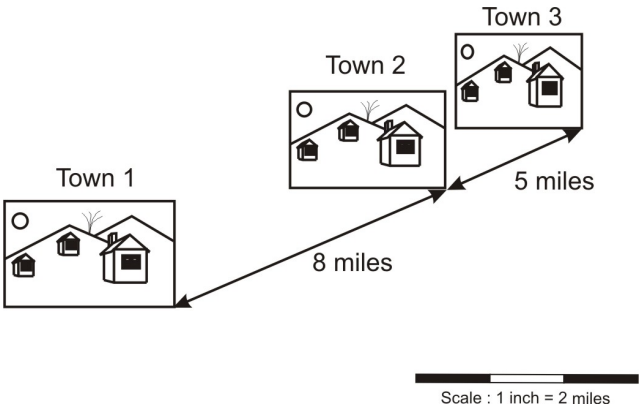
That may seem like a lot of confusing words, but the logic is quite simple. In the diagram below, if you add the lengths of $\overline{AB}$ and $\overline{BC}$, you will have found the length of $\overline{AC}$. In symbols, $AB + BC = AC$.



Use the segment addition postulate to put distances together.

Example 4

The map below shows the distances between three collinear towns. Assume that the first third of the scale in black represents one inch.



What is the distance between town 1 *and town* 3?

You can see that the distance between town 1 and town 2 is eight miles. You can also see that the distance between town 2 and town 3 is five miles. Using the segment addition postulate, you can add these values together to find the total distance between town 1 and town 3.

$$8 + 5 = 13$$

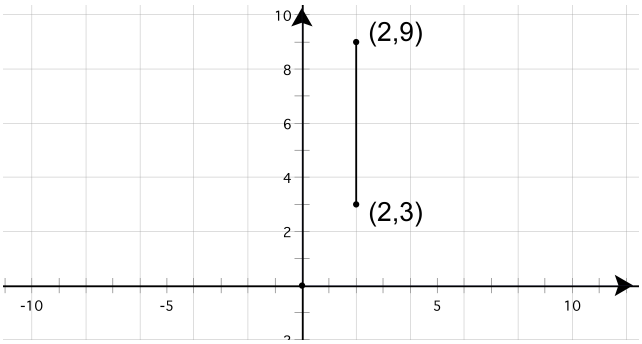
The total distance between town 1 and town 3 is 13 miles .

Distances on a Grid

In algebra you most likely worked with graphing lines in the $x-y$ coordinate grid. Sometimes you can find the distance between points on a coordinate grid using the values of the coordinates. If the two points line up horizontally, look at the change of value in the $x-$coordinates. If the two points line up vertically, look at the change of value in the $y-$coordinates. The change in value will show the distance between the points. Remember to use absolute value, just like you did with the ruler. Later you will learn how to calculate distance between points that do not line up horizontally or vertically.

Example 5

What is the distance between the two points shown below?



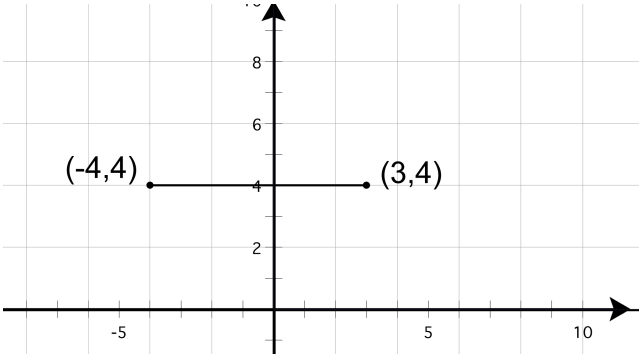
The two points shown on the grid are at $(2, 9)$ and $(2, 3)$. As these points line up vertically, look at the difference in the $y-$values.

$$|9 - 3| = |6| = 6$$

So, the distance between the two points is 6 units.

Example 6

What is the distance between the two points shown below?



The two points shown on the grid are at $(-4, 4)$ and $(3, 4)$. These points line up horizontally, so look at the difference in the $x-$values. Remember to take the absolute value of the difference between the values to find the distance.

$$|(-4) - 3| = |-7| = 7$$

The distance between the two points is 7 units.

Lesson Summary

In this lesson, we explored segments and distances. Specifically, we have learned:

- How to measure distances using many different tools.
- To understand and apply the Ruler Postulate to measurement.
- To understand and apply the Segment Addition Postulate to measurement.
- How to use endpoints to identify distances on a coordinate grid.

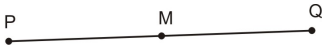
These skills are useful whenever performing measurements or calculations in diagrams. Make sure that you fully understand all concepts presented here before continuing in your study.

Review Questions

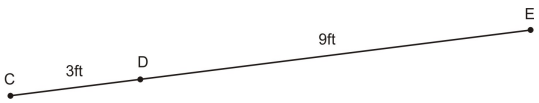
1. Use a ruler to measure the length of $\overline{AB}$ below.

Figure 1.4

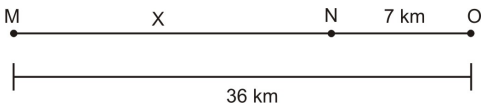
2. According to the ruler in the following image, how long is the cockroach?
3. The ruler postulate says that we could have measured the cockroach in 2 without using the 0 cm marker as the starting point. If the same cockroach as the one in 2 had its head at 6.5 cm, where would its tail be on the ruler?
4. Suppose M is exactly in the middle of $\overline{PQ}$ and $PM = 8$ cm. What is PQ?



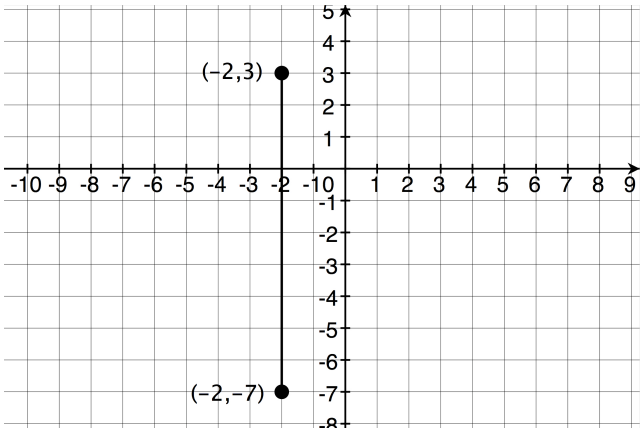
5. What is CE in the diagram below?



6. Find x in the diagram below:



7. What is the length of the segment connecting $(-2, 3)$ and $(-2, -7)$ in the coordinate plane? Justify your answer.



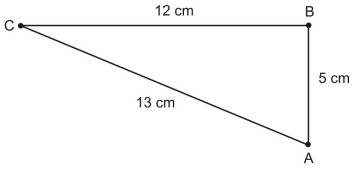
8. True or false: If $AB = 5$ cm and $BC = 12$ cm, then $AC = 17$ cm.
9. True or false: $|a - b| = |b - a|$.
10. One of the statements in 8 or 9 is false. Show why it is false, and then change the statement to make it true.

Review Answers

1. Answers will vary depending on scaling when printed and the units you use.
2. 4.5 cm(yuck!).
3. The tail would be at either 11 cm or 2 cm, depending on which way the cockroach was facing.
4. $PQ = 2$ (PM) $= 16$ cm.
5. $CE = 3$ ft $+ 9$ ft $= 12$ ft.
6. $x = 36$ km $- 7$ km $= 29$ km.
7. Since the points are at the same $x-$coordinate, we find the absolute value of the difference of the $y-$coordinates.

$$|-7-3| = |-10| = 10$$

8. False.
9. True. $a - b = -(b - a)$, but the absolute value sign makes them both positive.
10. Number 8 is false. See the diagram below for a counterexample. To make 8 true, we need to add something like: "If points A, B, and C are collinear, and B is between A and C, then if $AB = 5$ cm and $BC = 12$ cm, then $AC = 17$ cm."



1.3 Rays and Angles

Learning Objectives

- Understand and identify rays.
- Understand and classify angles.
- Understand and apply the protractor postulate.
- Understand and apply the angle addition postulate.

Introduction

Now that you know about line segments and how to measure them, you can apply what you have learned to other geometric figures. This lesson deals with rays and angles, and you can apply much of what you have already learned. We will try to help you see the connections between the topics you study in this book instead of dealing with them in isolation. This will give you a more well-rounded understanding of geometry and make you a better problem solver.

Rays

A ray is a part of a line with exactly one endpoint that extends infinitely in one direction. Rays are named by their endpoint and a point on the ray.

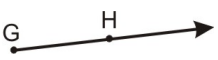
The ray above is called $\overrightarrow{AB}$. The first letter in the ray's name is always the endpoint of the ray, it doesn't matter which direction the ray points.

Rays can represent a number of different objects in the real world. For example, the beam of light extending from a flashlight that continues forever in one direction is a ray. The flashlight would be the endpoint of the ray, and the light continues as far as you can imagine so it is the infinitely long part of the ray. Are there other real-life objects that can be represented as rays?

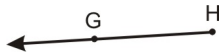
Example 1

Which of the figures below shows $\overrightarrow{GH}$?

A.

B. 

C.

D. 

Remember that a ray has one endpoint and extends infinitely in one direction. Choice A is a line segment since it has two endpoints. Choice B has one endpoint and extends infinitely in one direction, so it is a ray. Choice C has no endpoints and extends infinitely in two directions — it is a line. Choice D also shows a ray with endpoint H. Since we need to identify $\overrightarrow{GH}$ with endpoint G, we know that choice B is correct.

Example 2

Use this space to draw $\overrightarrow{RT}$.

Remember that you are not expected to be an artist. In geometry, you simply need to draw figures that accurately represent the terms in question. This problem asks you to draw a ray. Begin with a line segment. Use your ruler to draw a straight line segment of any length.

Now draw an endpoint on one end and an arrow on the other.

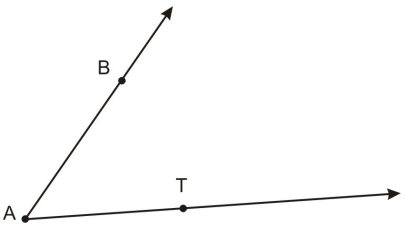
Finally, label the endpoint R and another point on the ray T.

The diagram above shows $\overrightarrow{RT}$.

Angles

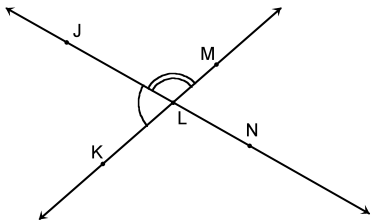
An **angle** is formed when two rays share a common endpoint. That common endpoint is called the **vertex** and the two rays are called the **sides** of the angle. In the diagram below,$\overrightarrow{AB}$ and$\overrightarrow{AT}$ form an angle, $\angle BAT$, or $\angle A$ for short. The symbol $\angle$ is used for naming angles.

The same basic definition for angle also holds when lines, segments, or rays intersect.



Notation Notes:

1. Angles can be named by a number, a single letter at the vertex, or by the three points that form the angle. When an angle is named with three letters, the middle letter will always be the vertex of the angle. In the diagram above, the angle can be written $\angle BAT$, or $\angle TAB$, or $\angle A$. You can use one letter to name this angle since point A is the vertex and there is only one angle at point A.
2. If two or more angles share the same vertex, you MUST use three letters to name the angle. For example, in the image below it is unclear which angle is referred to by $\angle L$. To talk about the angle with one arc, you would write $\angle KLJ$. For the angle with two arcs, you'd write $\angle JLM$.

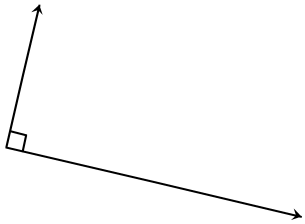


We use a ruler to measure segments by their *length.* But how do we measure an angle? The length of the sides does not change how wide an angle is "open." Instead of using length, the size of an angle is measured by the amount of *rotation* from one side to another. By definition, a full turn is defined as 360 degrees. Use the symbol ° for degrees. You may have heard "360" used as slang for a "full circle" turn, and this expression comes from the fact that a full rotation is 360°.

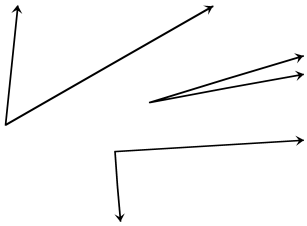
The angle that is made by rotating through one-fourth of a full turn is very special. It measures $\frac{1}{4} \times 360^\circ = 90^\circ$ and we call this a **right angle.** Right angles are easy to identify, as they look like the corners of most buildings, or a corner of a piece of paper.

A **right angle** measures exactly 90°.

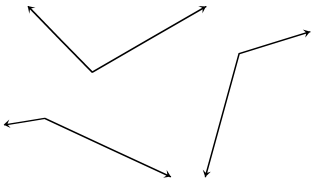
Right angles are usually marked with a small square. When two lines, two segments, or two rays intersect at a right angle, we say that they are **perpendicular**. The symbol $\perp$ is used for two perpendicular lines.



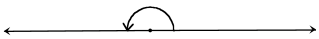
An **acute angle** measures between 0° and 90°.



An **obtuse angle** measures between 90° and 180°.



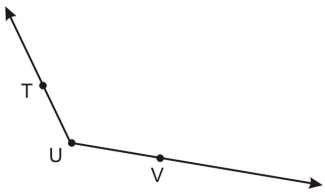
A **straight angle** measures exactly 180°. These are easy to spot since they look like straight lines.



You can use this information to classify any angle you see.

Example 3

What is the name and classification of the angle below?

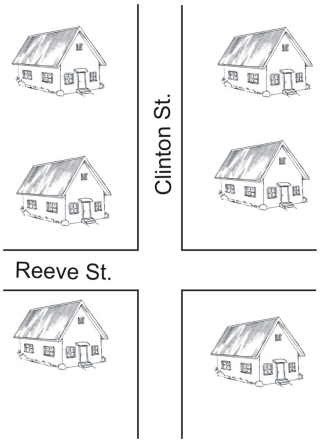


Begin by naming this angle. It has three points labeled and the vertex is U. So, the angle

will be named $\angle TUV$ or just $\angle U$. For the classification, compare the angle to a right angle. $\angle TUV$ opens wider than a right angle, and less than a straight angle. So, it is **obtuse**.

Example 4

What term best describes the angle formed by Clinton and Reeve streets on the map below?



The intersecting streets form a **right angle**. It is a square corner, so it measures 90°.

Protractor Postulate

In the last lesson, you studied the ruler postulate. In this lesson, we'll explore the **Protractor Postulate.** As you can guess, it is similar to the ruler postulate, but applied to angles instead of line segments. A protractor is a half-circle measuring device with angle measures marked for each degree. You measure angles with a protractor by lining up the vertex of the angle on the center of the protractor and then using the protractor postulate (see below). Be careful though, most protractors have two sets of measurements—one opening clockwise and one opening counterclockwise. Make sure you use the same scale when reading the measures of the angle.

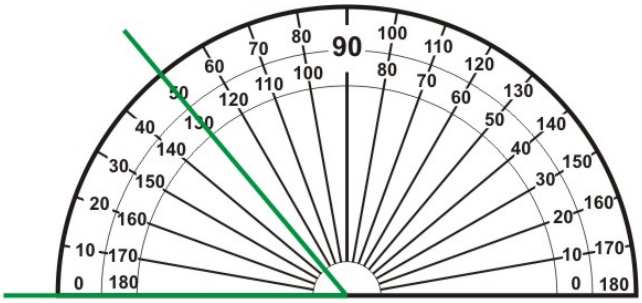
Protractor Postulate: For every angle there is a number between 0 and 180 that is the measure of the angle in degrees. You can use a protractor to measure an angle by aligning the center of the protractor on the vertex of the angle. The angle's measure is then the absolute value of the difference of the numbers shown on the protractor where the sides of the angle intersect the protractor.

It is probably easier to understand this postulate by looking at an example. The basic idea is that you do not need to start measuring an angle at the zero mark, as long as you find the absolute value of the difference of the two measurements. Of course, starting with one side at zero is usually easier. Examples 5 and 6 show how to use a protractor to measure angles.

Notation Note: When we talk about the measure of an angle, we use the symbols $m\angle$. So for example, if we used a protractor to measure $\angle TUV$ in example 3 and we found that it measured 120°, we could write $m\angle TUV = 120°$.

Example 5

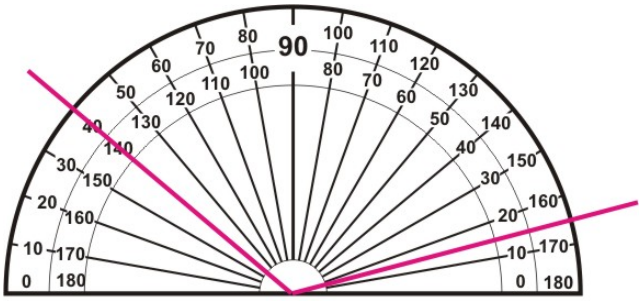
What is the measure of the angle shown below?



This angle is lined up with a protractor at 0°, so you can simply read the final number on the protractor itself. Remember you can check that you are using the correct scale by making sure your answer fits your angle. If the angle is acute, the measure of the angle should be less than 90°. If it is obtuse, the measure will be greater than 90°. In this case, the angle is acute, so its measure is 50°.

Example 6

What is the measure of the angle shown below?



This angle is not lined up with the zero mark on the protractor, so you will have to use subtraction to find its measure.

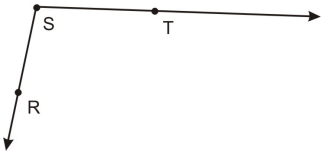
Using the inner scale, we get $|140 - 15| = |125| = 125°$.

Using the outer scale, $|40 - 165| = |-125| = 125°$.

Notice that it does not matter which scale you use. The measure of the angle is 125°.

Example 7

Use a protractor to measure $\angle RST$ below.



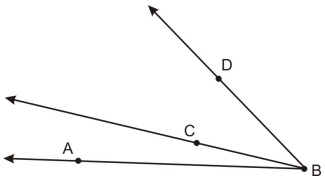
You can either line it up with zero, or line it up with another number and find the absolute value of the differences of the angle measures at the endpoints. Either way, the result is 100°. The angle measures 100°.

Multimedia Link The following applet gives you practice measuring angles with a protractor Measuring Angles Applet.

Angle Addition Postulate

You have already encountered the ruler postulate and the protractor postulate. There is also a postulate about angles that is similar to the Segment Addition Postulate.

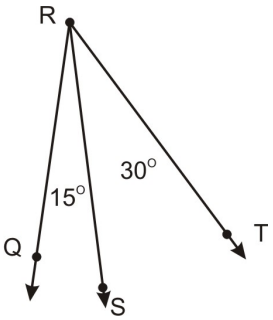
Angle Addition Postulate: The measure of any angle can be found by adding the measures of the smaller angles that comprise it. In the diagram below, if you add $m\angle ABC$ and $m\angle CBD$, you will have found $m\angle ABD$.



Use this postulate just as you did the segment addition postulate to identify the way different angles combine.

Example 8

What is $m\angle QRT$ in the diagram below?



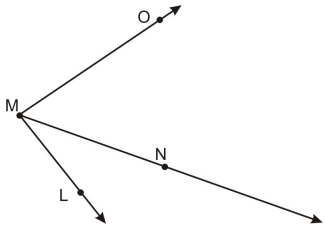
You can see that $m\angle QRS$ is 15°. You can also see that $m\angle SRT$ is 30°. Using the angle addition postulate, you can add these values together to find the total $m\angle QRT$.

$$15 + 30 = 45$$

So, $m\angle QRT$ is 45°.

Example 9

What is $m\angle LMN$ in the diagram below given $m\angle LMO = 85°$ and $m\angle NMO = 53°$?



To find $m\angle LMN$, you must subtract $m\angle NMO$ from $m\angle LMO$.

$$85 - 53 = 32$$

So, $m\angle LMN = 32°$.

Lesson Summary

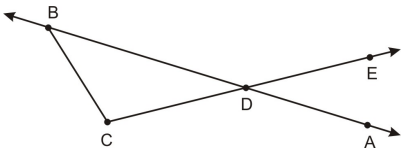
In this lesson, we explored rays and angles. Specifically, we have learned:

- To understand and identify rays.
- To understand and classify angles.
- To understand and apply the Protractor Postulate.
- To understand and apply the Angle Addition Postulate.

These skills are useful whenever studying rays and angles. Make sure that you fully understand all concepts presented here before continuing in your study.

Review Questions

Use this diagram for questions 1-4.



1. Give two possible names for the ray in the diagram.
2. Give four possible names for the line in the diagram.
3. Name an acute angle in the diagram.
4. Name an obtuse angle in the diagram.
5. Name a straight angle in the diagram.
6. Which angle can be named using only one letter?
7. Explain why it is okay to name some angles with only one angle, but with other angles this is not okay.
8. Use a protractor to find $m\angle PQR$ below:

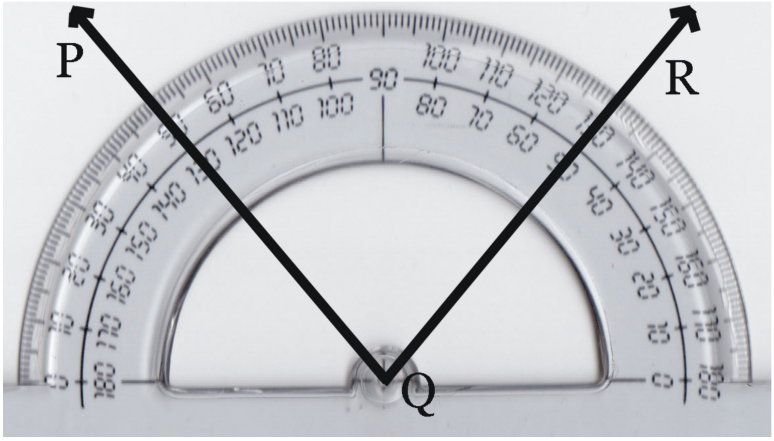
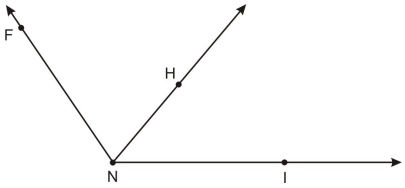


Figure 1.5

9. Given $m\angle FNI = 125°$ and $m\angle HNI = 50°$, find $m\angle FNH$.

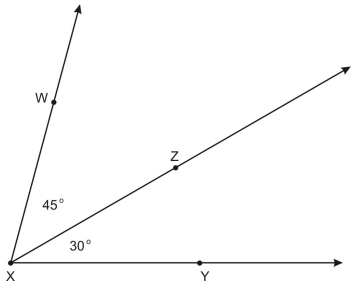


10. True or false: Adding two acute angles will result in an obtuse angle. If false, provide a counterexample.

Review Answers

1. CD or CE

2. BD, DB, AB, or BA are four possible answers. There are more (how many?)
3. BDC
4. BDE or BCD or CDA
5. BDA
6. Angle C
7. If there is more than one angle at a given vertex, then you must use three letters to name the angle. If there is only one angle at a vertex (as in angle C above) then it is permissible to name the angle with one letter.
8. $|(50 - 130)| = |(-80)| = 80$.
9. $m\angle FNH = |125 - 50| = |75| = 75°$.
10. False. For a counterexample, suppose two acute angles measure 30° and 45°, then the sum of those angles is 75°, but 75° is still acute. See the diagram for a counterexample:



1.4 Segments and Angles

Learning Objectives

- Understand and identify congruent line segments.
- Identify the midpoint of line segments.
- Identify the bisector of a line segment.
- Understand and identify congruent angles.
- Understand and apply the Angle Bisector Postulate.

Introduction

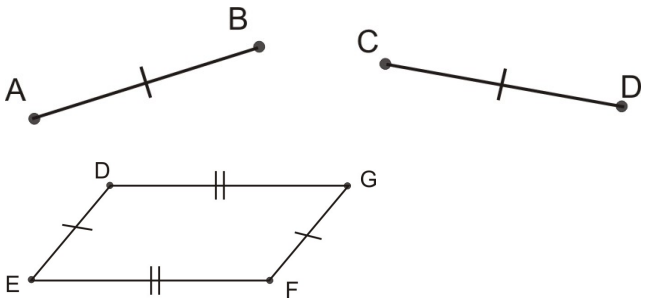
Now that you have a better understanding of segments, angles, rays, and other basic geometric shapes, we can study the ways in which they can be divided. Any time you come across a segment or an angle, there are different ways to separate it into parts.

Congruent Line Segments

One of the most important words in geometry is **congruent.** This term refers to geometric objects that have exactly the same size and shape. Two segments are congruent if they have the same length.

Notation Notes:

1. When two things are congruent we use the symbol $\cong$. For example if $\overline{AB}$ is congruent to $\overline{CD}$, then we would write $\overline{AB} \cong \overline{CD}$.
2. When we draw congruent segments, we use tic marks to show that two segments are congruent.
3. If there are multiple pairs of congruent segments (which are not congruent to each other) in the same picture, use two tic marks for the second set of congruent segments, three for the third set, and so on. See the two following illustrations.



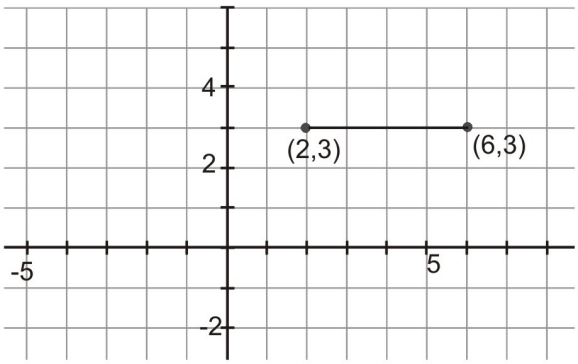
Recall that the length of segment $\overline{AB}$ can be written in two ways: $m\overline{AB}$ or simply AB. This might be a little confusing at first, but it will make sense as you use this notation more and more. Let's say we used a ruler and measured $\overline{AB}$ and we saw that it had a length of 5 cm. Then we could write $m\overline{AB} = 5$ cm, or $AB = 5$ cm.

If we know that $\overline{AB} \cong \overline{CD}$, then we can write $m\overline{AB} = m\overline{CD}$ or simply $AB = CD$.

You can prove two segments are congruent in a number of ways. You can measure them to find their lengths using any units of measurement—the units do not matter as long as you use the same units for both measurements. Or, if the segments are drawn in the $x - y$ plane, you can also find their lengths on the coordinate grid. Later in the course you will learn other ways to prove two segments are congruent.

Example 1

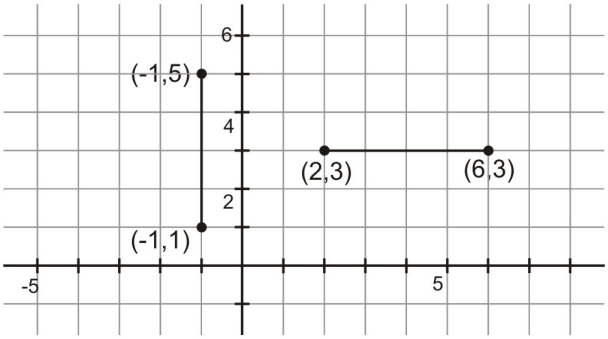
Henrietta drew a line segment on a coordinate grid as shown below.



She wants to draw another segment congruent to the first that begins at $(-1, 1)$ *and travels straight up (that is, in the* $+y$ *direction). What will be the coordinates of its second endpoint?*

You will have to solve this problem in stages. The first step is to identify the length of the segment drawn onto the grid. It begins at $(2, 3)$ and ends at $(6, 3)$. So, its length is 4 units.

The next step is to draw the second segment. Use a pencil to create the segment according to the specifications in the problem. You know that the segment needs to be congruent to the first, so it will be 4 units long. The problem also states that it travels straight up from the point $(-1, 1)$. Draw in the point at $(-1, 1)$ and make a line segment 4 units long that travels straight up.

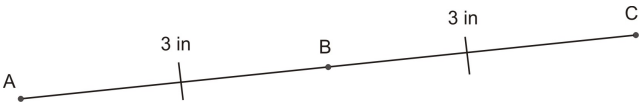


Now that you have drawn in the new segment, use the grid to identify the new endpoint. It has an $x-$coordinate of -1 and a $y-$coordinate of 5. So, its coordinates are $(-1, 5)$.

Segment Midpoints

Now that you understand congruent segments, there are a number of new terms and types of figures you can explore. A **segment midpoint** is a point on a line segment that divides the segment into two congruent segments. So, each segment between the midpoint and an endpoint will have the same length. In the diagram below, point B is the midpoint of

segment $\overline{AC}$ since $\overline{AB}$ is congruent to $\overline{BC}$.

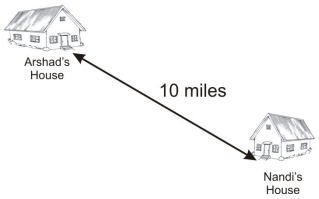


There is even a special postulate dedicated to midpoints.

Segment Midpoint Postulate: Any line segment will have exactly one midpoint—no more, and no less.

Example 2

Nandi and Arshad measure and find that their houses are 10 miles *apart. If they agree to meet at the midpoint between their two houses, how far will each of them travel?*



The easiest way to find the distance to the midpoint of the imagined segment connecting their houses is to divide the length by 2.

$$10 \div 2 = 5$$

So, each person will travel five miles to meet at the midpoint between Nandi's and Arshad's houses.

Segment Bisectors

Now that you know how to find midpoints of line segments, you can explore **segment bisectors.** A bisector is a line, segment, or ray that passes through a midpoint of another segment. You probably know that the prefix "bi" means two (think about the two wheels of a **bi**cycle). So, a bisector cuts a line segment into two congruent parts.

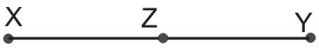
Example 3

Use a ruler to draw a bisector of the segment below.

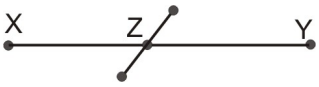
The first step in identifying a bisector is finding the midpoint. Measure the line segment to find that it is 4 cm long. To find the midpoint, divide this distance by 2.

$$4 \div 2 = 2$$

So, the midpoint will be 2 cm from either endpoint on the segment. Measure 2 cm from an endpoint and draw the midpoint.



To complete the problem, draw a line segment that passes through the midpoint. It doesn't matter what angle this segment travels on. As long as it passes through the midpoint, it is a bisector.

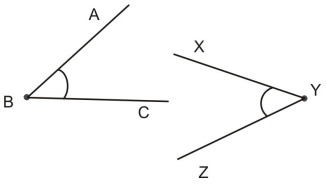


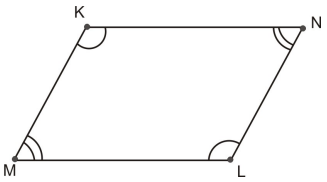
Congruent Angles

You already know that congruent line segments have exactly the same length. You can also apply the concept of congruence to other geometric figures. When angles are congruent, they have exactly the same measure. They may point in different directions, have different side lengths, have different names or other attributes, but their measures will be equal.

Notation Notes:

1. When writing that two angles are congruent, we use the congruent symbol: $\angle ABC \cong \angle ZYX$. Alternatively, the symbol $m\angle ABC$ refers to the measure of $\angle ABC$, so we could write $m\angle ABC = m\angle ZYX$ and that has the same meaning as $\angle ABC \cong \angle ZYX$. You may notice then, that *numbers* (such as measurements) are equal while *objects* (such as angles and segments) are congruent.
2. When drawing congruent angles, you use an arc in the middle of the angle to show that two angles are congruent. If two different pairs of angles are congruent, use one set of arcs for one pair, then two for the next pair and so on.

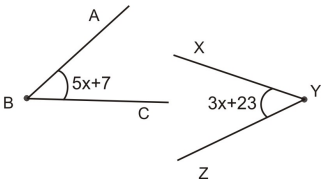




Use algebra to find a way to solve the problem below using this information.

Example 4

The two angles shown below are congruent.



What is the measure of each angle?

This problem combines issues of both algebra and geometry, so make sure you set up the problem correctly. It is given that the two angles are congruent, so they must have the same measurements. Therefore, you can set up an equation in which the expressions representing the angle measures are equal to each other.

$$5x + 7 = 3x + 23$$

Now that you have an equation with one variable, you can solve for the value of x.

$$\begin{aligned} 5x + 7 &= 3x + 23 \\ 5x - 3x &= 23 - 7 \\ 2x &= 16 \\ x &= 8 \end{aligned}$$

So, the value of x is 8. You are not done, however. Use this value of x to find the measure of one of the angles in the problem.

$$\begin{aligned} m\angle ABC &= 5x + 7 \\ &= 5(8) + 7 \\ &= 40 + 7 \\ &= 47 \end{aligned}$$

Finally, we know $m\angle ABC = m\angle XYZ$, so both of the angles measure 47°.

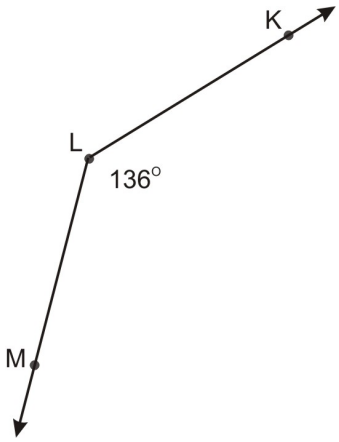
Angle Bisectors

If a segment bisector divides a segment into two congruent parts, you can probably guess what an **angle bisector** is. An angle bisector divides an angle into two congruent angles, each having a measure exactly half of the original angle.

Angle Bisector Postulate: Every angle has exactly one bisector.

Example 5

The angle below measures 136°.



If a bisector is drawn in this angle, what will be the measure of the new angles formed?

This is similar to the problem about the midpoint between the two houses. To find the measurements of the smaller angles once a bisector is drawn, divide the original angle measure by 2:

$$136 \div 2 = 68$$

So, each of the newly formed angles would measure 68° when the 136° angle is bisected.

Lesson Summary

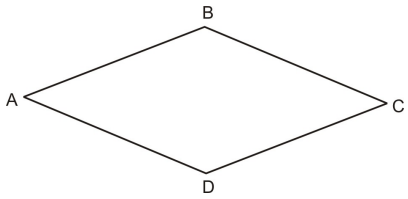
In this lesson, we explored segments and angles. Specifically, we have learned:

- How to understand and identify congruent line segments.
- How to identify the midpoint of line segments.
- How to identify the bisector of a line segment.
- How to understand and identify congruent angles.
- How to understand and apply the Angle Bisector Postulate.

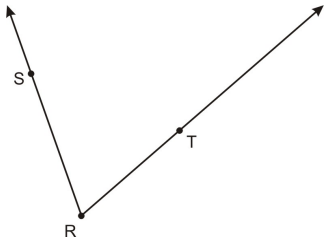
These skills are useful whenever performing measurements or calculations in diagrams. Make sure that you fully understand all concepts presented here before continuing in your study.

Review Questions

1. Copy the figure below and label it with the following information:
   (a) $\angle A \cong \angle C$
   (b) $\angle B \cong \angle D$
   (c) $\overline{AB} \cong \overline{AD}$

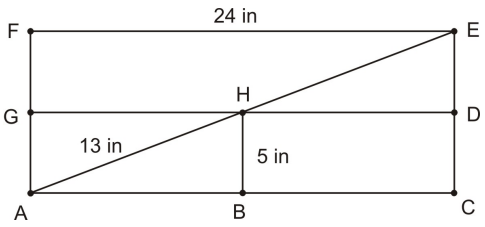


2. Sketch and label an angle bisector $\overrightarrow{RU}$ of $\angle SRT$ below.



3. If we know that $m\angle SRT = 64°$, what is $m\angle SRU$?

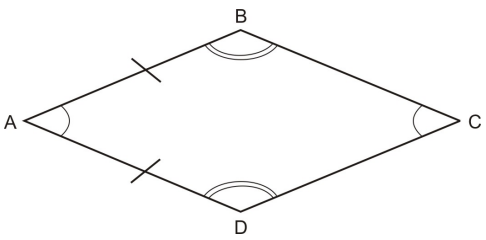
Use the following diagram of rectangle $ACEF$ for questions 4-10. (For these problems you can assume that opposite sides of a rectangle are congruent—later you will prove this is true.)

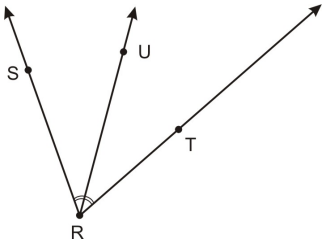


Given that H is the midpoint of $\overline{AE}$ and $\overline{DG}$, find the following lengths:

4. $GH =$
5. $AB =$
6. $AC =$
7. $HE =$
8. $AE =$
9. $CE =$
10. $GF =$
11. How many copies of $\triangle ABH$ can fit inside rectangle $ACEF$?

Review Answers

1. 

2. 

3. $32°$
4. $GH = 12$ in
5. $AB = 12$ in
6. $AC = 24$ in
7. $HE = 12$ in
8. $AE = 26$ in
9. $CE = 10$ in
10. $GF = 5$ in
11. 8

1.5 Angle Pairs

Learning Objectives

- Understand and identify complementary angles.
- Understand and identify supplementary angles.
- Understand and utilize the Linear Pair Postulate.
- Understand and identify vertical angles.

Introduction

In this lesson you will learn about special angle pairs and prove the vertical angles theorem, one of the most useful theorems in geometry.

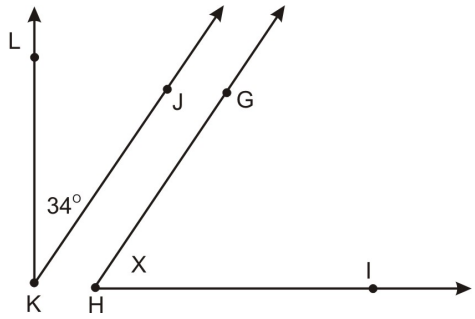
Complementary Angles

A pair of angles are **Complementary angles** if the sum of their measures is 90°.

Complementary angles do not have to be congruent to each other. Rather, their only defining quality is that the sum of their measures is equal to the measure of a right angle: 90°. If the outer rays of two adjacent angles form a right angle, then the angles are complementary.

Example 1

The two angles below are complementary. $m\angle GHI = x$. What is the value of x?



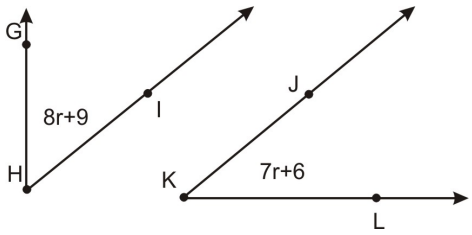
Since you know that the two angles must sum to 90°, you can create an equation. Then solve for the variable. In this case, the variable is x.

$$34 + x = 90$$
$$34 + x - 4 = 90 - 34$$
$$x = 56$$

Thus, the value of x is 56°.

Example 2

The two angles below are complementary. What is the measure of each angle?



This problem is a bit more complicated than the first example. However, the concepts are the same. If you add the two angles together, the sum will be 90°. So, you can set up an algebraic equation with the values presented.

$$(7r + 6) + (8r + 9) = 90$$

The best way to solve this problem is to solve the equation above for r. Then, you must substitute the value for r back into the original expressions to find the value of each angle.

$$\begin{aligned}(7r + 6) + (8r + 9) &= 90\\ 15r + 15 &= 90\\ 15r + 15 - 15 &= 90 - 15\\ 15r &= 75\\ \frac{15r}{15} &= \frac{75}{15}\\ r &= 5\end{aligned}$$

The value of r is 5. Now substitute this value back into the expressions to find the measures of the two angles in the diagram.

$$\begin{aligned}7r + 6\\ 7(5) + 6\\ 35 + 6\\ 41\end{aligned}$$

$$\begin{aligned}8r + 9\\ 8(5) + 9\\ 40 + 9\\ 49\end{aligned}$$

$m\angle JKL = 41°$ and $m\angle GHI = 49°$. You can check to make sure these numbers are accurate by verifying if they are complementary.

$$41 + 49 = 90$$

Since these two angle measures sum to 90°, they are complementary.

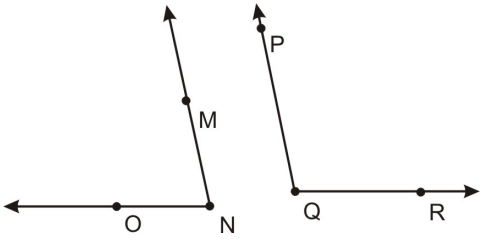
Supplementary Angles

Two angles are **supplementary** if their measures sum to 180°.

Just like complementary angles, supplementary angles need not be congruent, or even touching. Their defining quality is that when their measures are added together, the sum is 180°. You can use this information just as you did with complementary angles to solve different types of problems.

Example 3

The two angles below are supplementary. If $m\angle MNO = 78°$, what is $m\angle PQR$?



This process is very straightforward. Since you know that the two angles must sum to 180°, you can create an equation. Use a variable for the unknown angle measure and then solve for the variable. In this case, let's substitute y for $m\angle PQR$.

$$\begin{aligned} 78 + y &= 180 \\ 78 + y - 78 &= 180 - 78 \\ y &= 102 \end{aligned}$$

So, the measure of $y = 102$ and thus $m\angle PQR = 102°$.

Example 4

What is the measure of two congruent, supplementary angles?

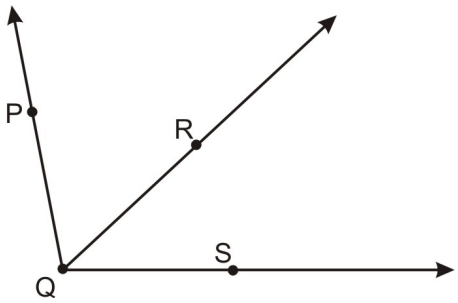
There is no diagram to help you visualize this scenario, so you'll have to imagine the angles (or even better, draw it yourself by translating the words into a picture!). Two supplementary angles must sum to 180°. Congruent angles must have the same measure. So, you need to find two congruent angles that are supplementary. You can divide 180° by two to find the value of each angle.

$$180 \div 2 = 90$$

Each congruent, supplementary angle will measure 90°. In other words, they will be right angles.

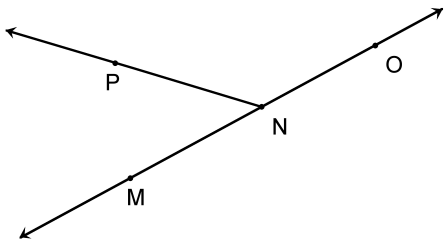
Linear Pairs

Before we talk about a special pair of angles called **linear pairs**, we need to define **adjacent angles.** Two angles are adjacent if they share the same vertex and one side, but they do not overlap. In the diagram below, $\angle PQR$ and $\angle RQS$ are adjacent.



However, $\angle PQR$ and $\angle PQS$ are not adjacent since they overlap (i.e. they share common points in the interior of the angle).

Now we are ready to talk about linear pairs. A **linear pair** is two angles that are adjacent and whose non-common sides form a straight line. In the diagram below, $\angle MNP$ and $\angle PNO$ are a linear pair. Note that $\overleftrightarrow{MO}$ is a line.

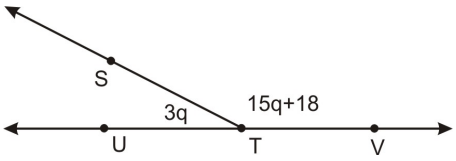


Linear pairs are so important in geometry that they have their own postulate.

Linear Pair Postulate: If two angles are a linear pair, then they are supplementary.

Example 5

The two angles below form a linear pair. What is the value of each angle?



If you add the two angles, the sum will be 180°. So, you can set up an algebraic equation with the values presented.

$$(3q) + (15q + 18) = 180$$

The best way to solve this problem is to solve the equation above for q. Then, you must plug the value for q back into the original expressions to find the value of each angle.

$$\begin{aligned}(3q) + (15q + 18) &= 180\\ 18q + 18 &= 180\\ 18q &= 180 - 18\\ 18q &= 162\\ \frac{18q}{18} &= \frac{162}{18}\\ q &= 9\end{aligned}$$

The value of q is 9. Now substitute this value back into the expressions to determine the measures of the two angles in the diagram.

$3q$	$15q + 18$
$3(9)$	$15(9) + 18$
27	$135 + 18$
	153

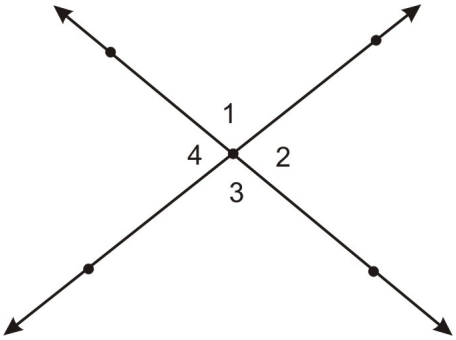
The two angles in the diagram measure 27° and 153°. You can check to make sure these numbers are accurate by verifying if they are supplementary.

$$27 + 153 = 180$$

Vertical Angles

Now that you understand supplementary and complementary angles, you can examine more complicated situations. Special angle relationships are formed when two lines intersect, and you can use your knowledge of linear pairs of angles to explore each angle further.

Vertical angles are defined as two non-adjacent angles formed by intersecting lines. In the diagram below, $\angle 1$ and $\angle 3$ are vertical angles. Also, $\angle 4$ and $\angle 2$ are vertical angles.



Suppose that you know $m\angle 1 = 100°$. You can use that information to find the measurement of all the other angles. For example, $\angle 1$ and$\angle 2$ must be supplementary since they are a linear pair. So, to find $m\angle 2$, subtract 100° from 180°.

$$\begin{aligned} m\angle 1 + m\angle 2 &= 180 \\ 100 + m\angle 2 &= 180 \\ m\angle 2 &= 180 - 100 \\ m\angle 2 &= 80 \end{aligned}$$

So $\angle 2$ measures 80°. Knowing that angles 2 and 3 are also supplementary means that $m\angle 3 = 100°$, since the sum of 100° and 80° is 180°. If angle 3 measures 100°, then the measure of angle 4 must be 80°, since 3 and 4 are also supplementary. Notice that angles 1 and 3 are congruent (100°) and 2 and 4 are congruent (80°).

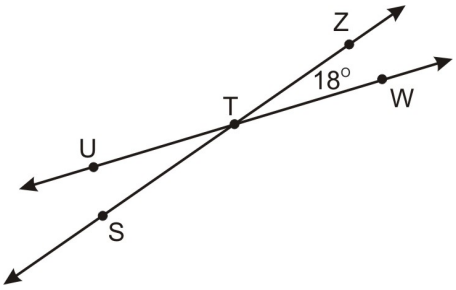
The **Vertical Angles Theorem** states that if two angles are vertical angles then they are congruent.

We can prove the vertical angles theorem using a process just like the one we used above. There was nothing special about the given measure of $\angle 1$. Here is proof that vertical angles will always be congruent: Since $\angle 1$ and $\angle 2$ form a linear pair, we know that they are supplementary: $m\angle 1 + m\angle 2 = 180°$. For the same reason, $\angle 2$ and$\angle 3$ are supplementary: $m\angle 2 + m\angle 3 = 180°$. Using a substitution, we can write $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$. Finally, subtracting $m\angle 2$ on both sides yields $m\angle 1 = m\angle 3$. Or, by the definition of congruent angles, $\angle 1 \cong \angle 3$.

Use your knowledge of vertical angles to solve the following problem.

Example 6

What is $m\angle STU$ in the diagram below?



Using your knowledge of intersecting lines, you can identify that $\angle STU$ is vertical to the angle marked 18°. Since vertical angles are congruent, they will have the same measure. So, $m\angle STU$ is also equal to 18°.

Lesson Summary

In this lesson, we explored angle pairs. Specifically, we have learned:

- How to understand and identify complementary angles.
- How to understand and identify supplementary angles.
- How to understand and utilize the Linear Pair Postulate.
- How to understand and identify vertical angles.

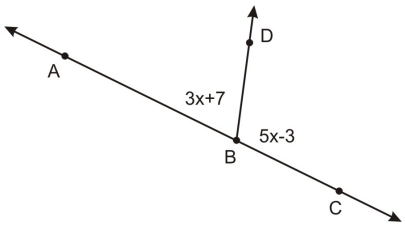
The relationships between different angles are used in almost every type of geometric application. Make sure that these concepts are retained as you progress in your studies.

Review Questions

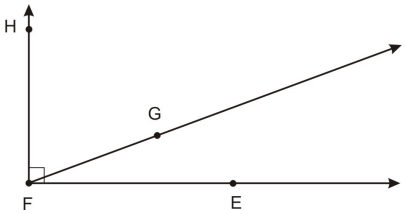
1. Find the measure of the angle complementary to $\angle A$ if $m\angle A =$
   (a) 45°
   (b) 82°
   (c) 19°
   (d) $z°$
2. Find the measure of the angle supplementary to $\angle B$ if
   (a) 45°

(b) 118°
(c) 32°
(d) $x°$

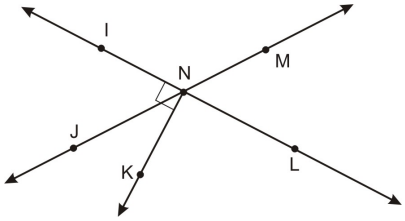
3. Find $m\angle ABD$ and $m\angle DBC$.



4. Given $m\angle EFG = 20°$, Find $m\angle HFG$.



Use the diagram below for exercises 5 and 6. Note that $\overline{NK} \perp \overleftrightarrow{IL}$.



5. Identify each of the following (there may be more than one correct answer for some of these questions).
   (a) Name one pair of vertical angles.
   (b) Name one linear pair of angles.
   (c) Name two complementary angles.
   (d) Nam two supplementary angles.
6. Given that $m\angle IJN = 63°$, find
   (a) $m\angle JNK$.
   (b) $m\angle KNL$.
   (c) $m\angle MNL$.
   (d) $m\angle MNI$.

Review Answers

1. (a) 45°
   (b) 8°

(c) 81°
(d) $(90 - z)^\circ$

2. (a) 135°
(b) 62°
(c) 148°
(d) $(180 - x)^\circ$
3. $m\angle ABD = 73^\circ$, $m\angle DBC = 107^\circ$
4. $m\angle HFG = 70^\circ$
5. (a) $\angle JNI$ and $\angle MNL$ (or $\angle INM$ and $\angle JNL$ also works);
(b) $\angle INM$ and $\angle MNL$ (or $\angle INK$ and $\angle KNL$ also works);
(c) $\angle INK$ and $\angle JNK$;
(d) same as (b) $\angle INM$ and $\angle MNL$ (or $\angle INK$ and $\angle KNL$ also works).
6. (a) 27°
(b) 90°
(c) 63°
(d) 117°

1.6 Classifying Triangles

Learning Objectives

- Define triangles.
- Classify triangles as acute, right, obtuse, or equiangular.
- Classify triangles as scalene, isosceles, or equilateral.

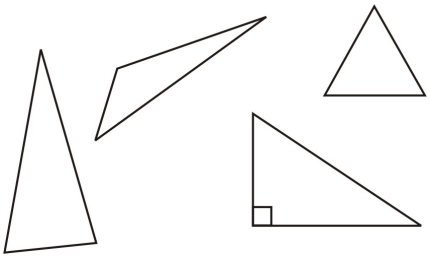
Introduction

By this point, you should be able to readily identify many different types of geometric objects. You have learned about lines, segments, rays, planes, as well as basic relationships between many of these figures. Everything you have learned up to this point is necessary to explore the classifications and properties of different types of shapes. The next two sections focus on two-dimensional shapes—shapes that lie in one plane. As you learn about polygons, use what you know about measurement and angle relationships in these sections.

Defining Triangles

The first shape to examine is the **triangle.** Though you have probably heard of triangles before, it is helpful to review the formal definition. A triangle is any closed figure made by three line segments intersecting at their endpoints. Every triangle has three **vertices**

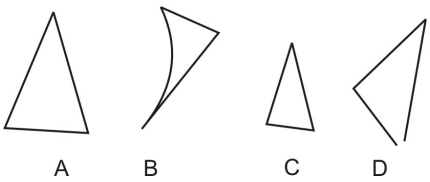
(points at which the segments meet), three **sides** (the segments themselves), and three **interior angles** (formed at each vertex). All of the following shapes are triangles.



You may have learned in the past that the sum of the interior angles in a triangle is always 180°. Later we will prove this property, but for now you can use this fact to find missing angles. Other important properties of triangles will be explored in later chapters.

Example 1

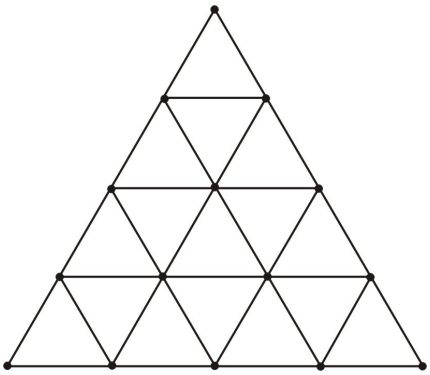
Which of the figures below are not triangles?



To solve this problem, you must carefully analyze the four shapes in the answer choices. Remember that a triangle has three sides, three vertices, and three interior angles. Choice A fits this description, so it is a triangle. Choice B has one curved side, so its sides are not exclusively line segments. Choice C is also a triangle. Choice D, however, is not a closed shape. Therefore, it is not a triangle. Choices B and D are not triangles.

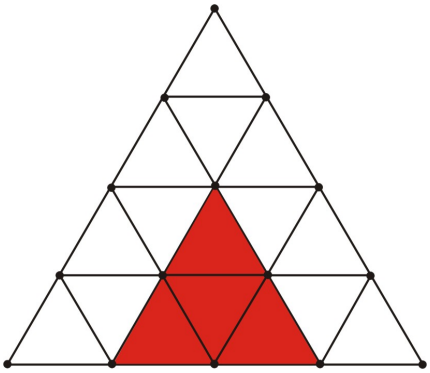
Example 2

How many triangles are in the diagram below?



To solve this problem, you must carefully count the triangles of different size. Begin with the smallest triangles. There are 16 small triangles.

Now count the triangles that are formed by four of the smaller triangles, like the one below.



There are a total of seven triangles of this size, if you remember to count the inverted one in the center of the diagram.

Next, count the triangles that are formed by nine of the smaller triangles. There are three of these triangles. And finally, there is one triangle formed by 16 smaller triangles.

Now, add these numbers together.

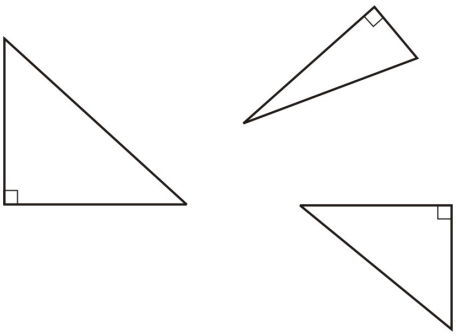
$$16 + 7 + 3 + 1 = 27$$

So, there are a total of 27 triangles in the figure shown.

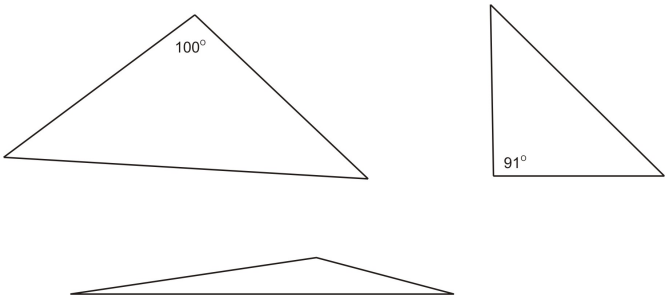
Classifications by Angles

Earlier in this chapter, you learned how to classify angles as acute, obtuse, or right. Now that you know how to identify triangles, we can separate them into classifications as well. One way to classify a triangle is by the measure of its angles. In any triangle, two of the angles will always be acute. This is necessary to keep the total sum of the interior angles at 180°. The third angle, however, can be acute, obtuse, or right.

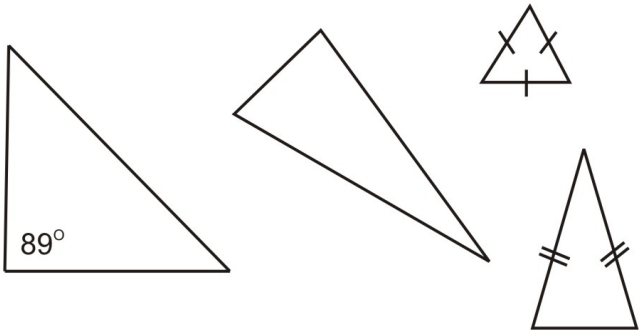
This is how triangles are classified. If a triangle has one right angle, it is called a **right triangle**.



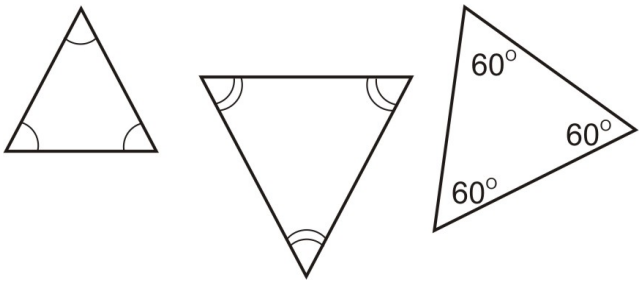
If a triangle has one obtuse angle, it is called an **obtuse triangle**.



If all of the angles are acute, it is called an **acute triangle**.

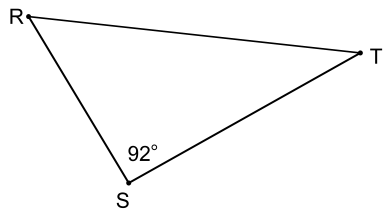


The last type of triangle classifications by angles occurs when all angles are congruent. This triangle is called an **equiangular triangle**.



Example 3

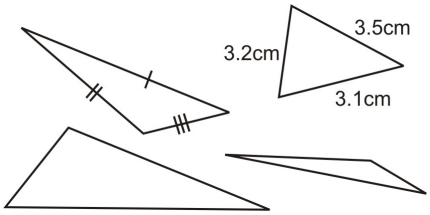
Which term best describes $\triangle RST$ below?



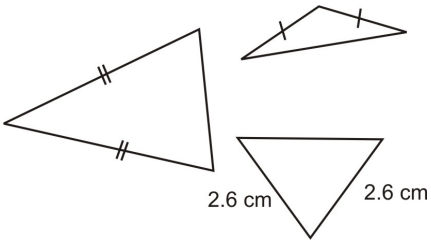
The triangle in the diagram has two acute angles. But, $m\angle RST = 92°$ so$\angle RST$ is an obtuse angle. If the angle measure were not given, you could check this using the corner of a piece of notebook paper or by measuring the angle with a protractor. An obtuse angle will be greater than 90° (the square corner of a paper) and less than 180° (a straight line). Since one angle in the triangle above is obtuse, it is an obtuse triangle.

Classifying by Side Lengths

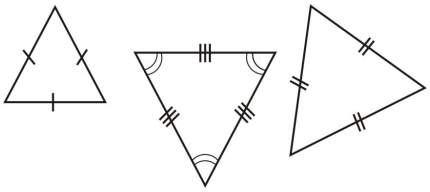
There are more types of triangle classes that are not based on angle measure. Instead, these classifications have to do with the sides of the triangle and their relationships to each other. When a triangle has all sides of different length, it is called a **scalene triangle.**



When at least two sides of a triangle are congruent, the triangle is said to be an **isosceles triangle**.

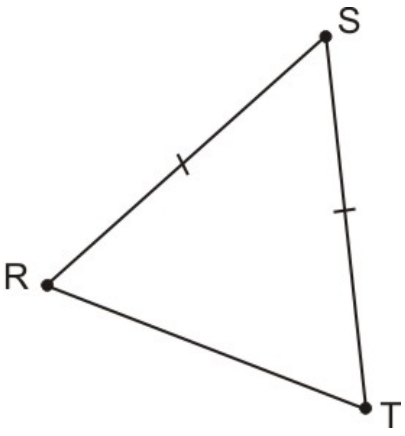


Finally, when a triangle has sides that are all congruent, it is called an **equilateral triangle**. Note that by the definitions, an equilateral triangle is also an isosceles triangle.



Example 4

Which term best describes the triangle below?



A. scalene

B. isosceles

C. equilateral

To classify the triangle by side lengths, you have to examine the relationships between the sides. Two of the sides in this triangle are congruent, so it is an isosceles triangle. The correct answer is B.

Lesson Summary

In this lesson, we explored triangles and their classifications. Specifically, we have learned:

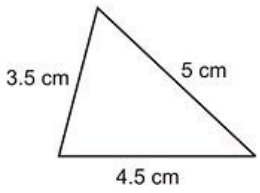
- How to define triangles.
- How to classify triangles as acute, right, obtuse, or equiangular.
- How to classify triangles as scalene, isosceles, or equilateral.

These terms or concepts are important in many different types of geometric practice. It is important to have these concepts solidified in your mind as you explore other topics of geometry and mathematics.

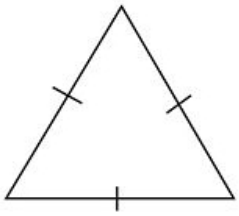
Review Questions

Exercises 1-5: Classify each triangle by its sides and by its angles. If you do not have enough information to make a classification, write "not enough information."

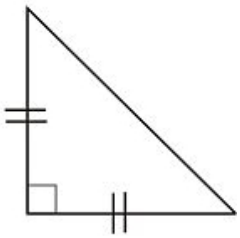
1. A)



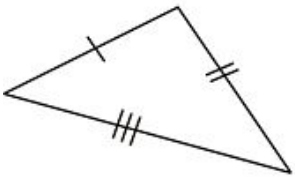
2. B)



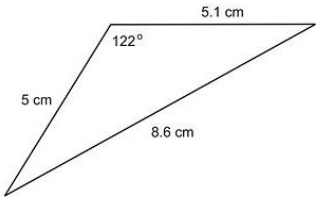
3. C)



4. D)



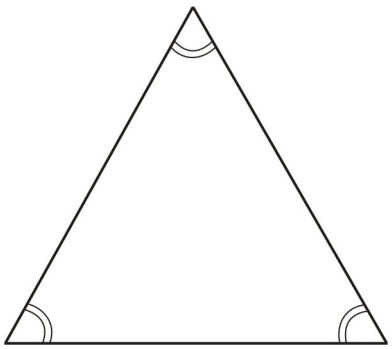
5. E)



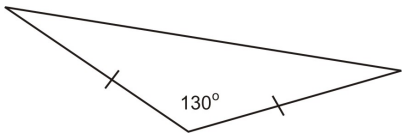
6. Sketch an equiangular triangle. What must be true about the sides?
7. Sketch an obtuse isosceles triangle.
8. True or false: A right triangle can be scalene.
9. True or false: An obtuse triangle can have more than one obtuse angle.
10. One of the answers in 8 or 9 is false. Sketch an illustration to show why it is false, and change the false statement to make it true.

Review Answers

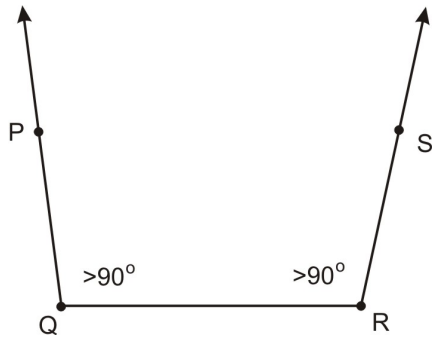
1. A is an acute scalene triangle.
2. B is an equilateral triangle.
3. C is a right isosceles triangle.
4. D is a scalene triangle. Since we don't know anything about the angles, we cannot assume it is a right triangle, even though one of the angles looks like it may be 90°.
5. E is an obtuse scalene triangle.
6. If a triangle is equiangular then it is also equilateral, so the sides are all congruent.



7. Sketch below:



8. True.
9. False.
10. 9 is false since the three sides would not make a triangle. To make the statement true, it should say: "An obtuse triangle has exactly one obtuse angle."



1.7 Classifying Polygons

Learning Objectives

- Define polygons.
- Understand the difference between convex and concave polygons.
- Classify polygons by number of sides.
- Use the distance formula to find side lengths on a coordinate grid.

Introduction

As you progress in your studies of geometry, you can examine different types of shapes. In the last lesson, you studied the triangle, and different ways to classify triangles. This lesson presents other shapes, called **polygons**. There are many different ways to classify and analyze these shapes. Practice these classification procedures frequently and they will get easier and easier.

Defining Polygons

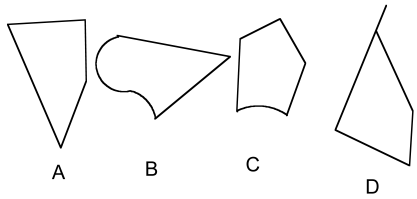
Now that you know what a triangle is, you can learn about other types of shapes. Triangles belong to a larger group of shapes called **polygons.** A polygon is any closed planar figure that is made entirely of line segments that intersect at their endpoints. Polygons can have any number of sides and angles, but the sides can never be curved.

The segments are called the **sides** of the polygons, and the points where the segments intersect are called **vertices**. Note that the singular of vertices is **vertex**.

The easiest way to identify a polygon is to look for a closed figure with no curved sides. If there is any curvature in a shape, it cannot be a polygon. Also, the points of a polygon must all lie within the same plane (or it wouldn't be two-dimensional).

Example 1

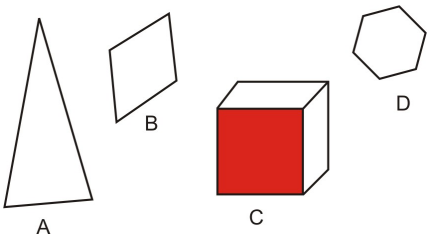
Which of the figures below is a polygon?



The easiest way to identify the polygon is to identify which shapes are not polygons. Choices B and C each have at least one curved side. So they cannot be polygons. Choice D has all straight sides, but one of the vertices is not at the endpoints of the two adjacent sides, so it is not a polygon. Choice A is composed entirely of line segments that intersect at their endpoints. So, it is a polygon. The correct answer is A.

Example 2

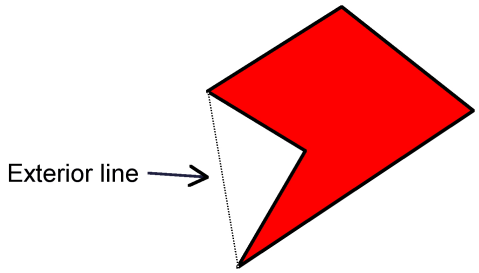
Which of the figures below is not a polygon?



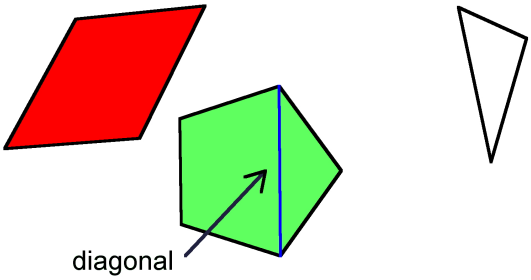
All four of the shapes are composed of line segments, so you cannot eliminate any choices based on that criteria alone. Notice that choices A, B, and D have points that all lie within the same plane. Choice C is a three-dimensional shape, so it does not lie within one plane. So it is not a polygon. The correct answer is C.

Convex and Concave Polygons

Now that you know how to identify polygons, you can begin to practice classifying them. The first type of classification to learn is whether a polygon is **convex** or **concave**. Think of the term concave as referring to a cave, or an interior space. A concave polygon has a section that "points inward" toward the middle of the shape. In any concave polygon, there are at least two vertices that can be connected without passing through the interior of the shape. The polygon below is concave and demonstrates this property.

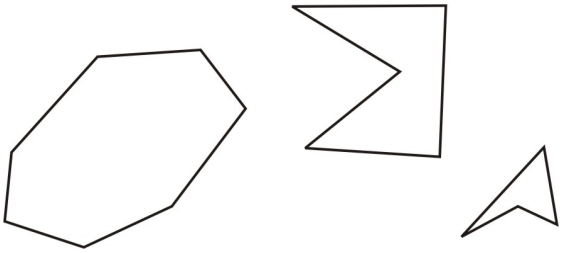


A convex polygon does not share this property. Any time you connect the vertices of a convex polygon, the segments between nonadjacent vertices will travel through the interior of the shape. Lines segments that connect to vertices traveling only on the interior of the shape are called **diagonals**.



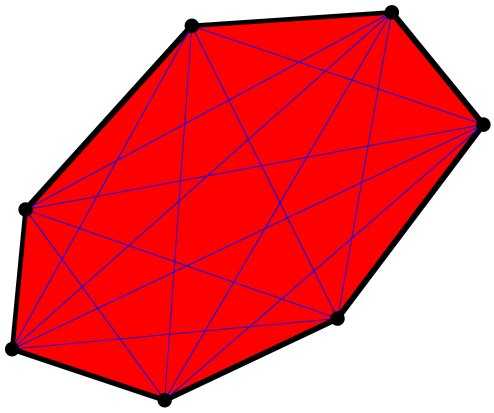
Example 3

Identify whether the shapes below are convex or concave.



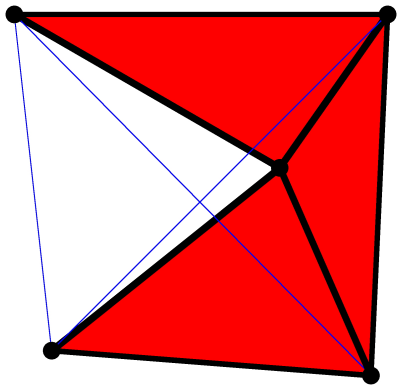
To solve this problem, connect the vertices to see if the segments pass through the interior or exterior of the shape.

A. The segments go through the interior.



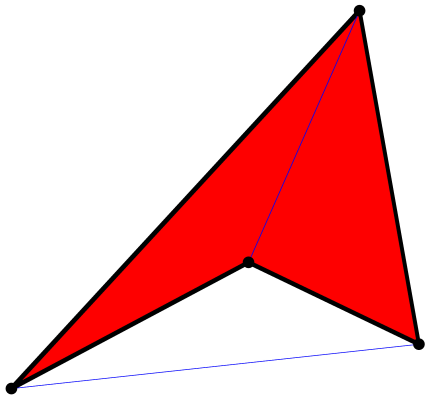
Therefore, the polygon is convex.

B. The segments go through the exterior.



Therefore, the polygon is concave.

C. One of the segments goes through the exterior.



Thus, the polygon is concave.

Classifying Polygons

The most common way to classify a polygon is by the number of sides. Regardless of whether the polygon is convex or concave, it can be named by the number of sides. The prefix in each name reveals the number of sides. The chart below shows names and samples of polygons.

Table 1.1:

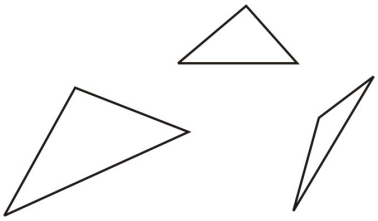
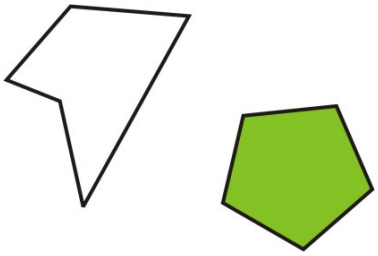
Polygon Name	Number of Sides	Sample Drawings
Triangle	3	
Quadrilateral	4	
Pentagon	5	

Table 1.1: (continued)

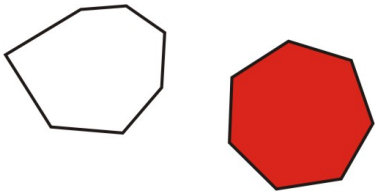
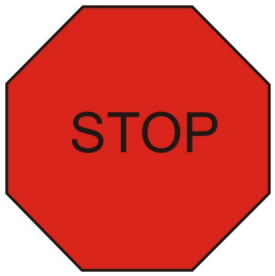
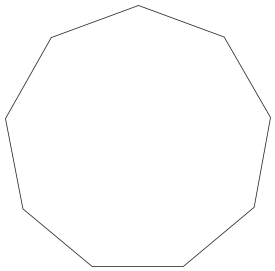
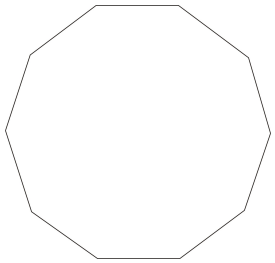
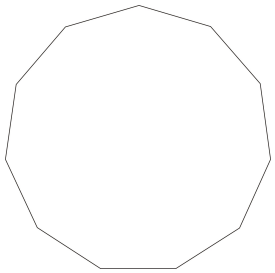
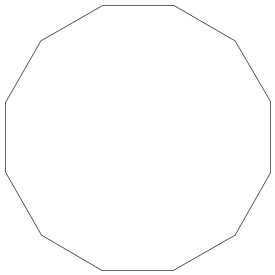
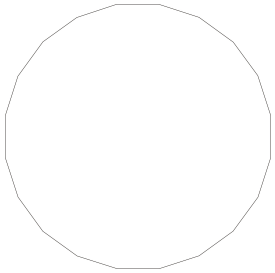
Polygon Name	Number of Sides	Sample Drawings
Hexagon	6	
Heptagon	7	
Octagon	8	
Nonagon	9	
Decagon	10	

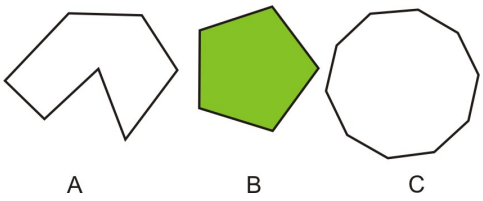
Table 1.1: (continued)

Polygon Name	Number of Sides	Sample Drawings
Undecagon or hendecagon (there is some debate!)	11	
Dodecagon	12	
$n-$**gon**	n (where $n > 12$)	

Practice using these polygon names with the appropriate prefixes. The more you practice, the more you will remember.

Example 4

Name the three polygons below by their number of sides.



A. This shape has seven sides, so it is a heptagon.

B. This shape has five sides, so it is a pentagon.

C. This shape has ten sides, so it is a decagon.

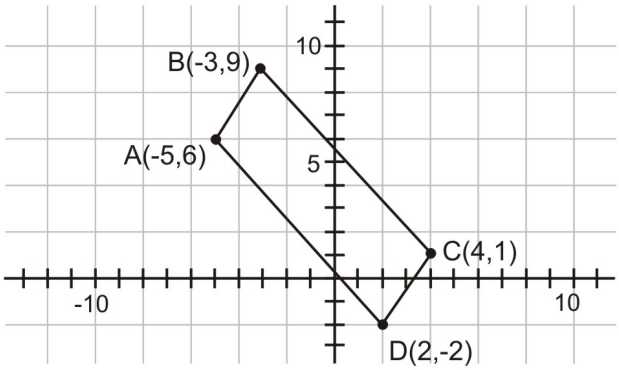
Using the Distance Formula on Polygons

You can use the distance formula to find the lengths of sides of polygons if they are on a coordinate grid. Remember to carefully assign the values to the variables to ensure accuracy. Recall from algebra that you can find the distance between points (x_1, y_1) and (x_2, y_2) using the following formula.

Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 5

A quadrilateral has been drawn on the coordinate grid below.



What is the length of segment BC?

Use the distance formula to solve this problem. The endpoints of $\overline{BC}$ are $(-3, 9)$ and $(4, 1)$. Substitute -3 for x_1 , 9 for y_1, 4 for x_2, and 1 for y_2. Then we have:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$D = \sqrt{(4 - (-3))^2 + (1 - 9)^2}$$
$$D = \sqrt{(7)^2 + (-8)^2}$$
$$D = \sqrt{49 + 64}$$
$$D = \sqrt{113}$$

So the distance between points B and C is $\sqrt{113}$, or about 10.63 units.

Lesson Summary

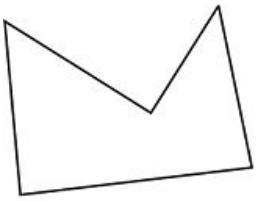
In this lesson, we explored polygons. Specifically, we have learned:

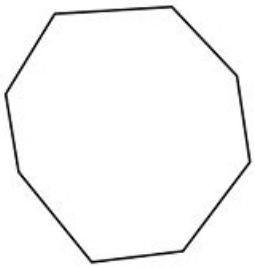
- How to define polygons.
- How to understand the difference between convex and concave polygons.
- How to classify polygons by number of sides.
- How to use the distance formula to find side lengths on a coordinate grid.

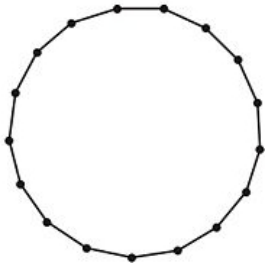
Polygons are important geometric shapes, and there are many different types of questions that involve them. Polygons are important aspects of architecture and design and appear constantly in nature. Notice the polygons you see every day when you look at buildings, chopped vegetables, and even bookshelves. Make sure you practice the classifications of different polygons so that you can name them easily.

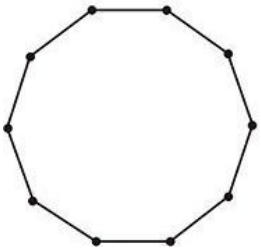
Review Questions

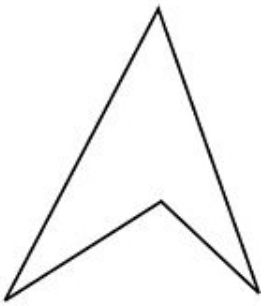
For exercises 1-5, name each polygon in as much detail as possible.

1. 

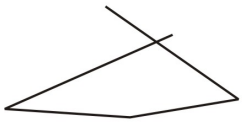
2. 

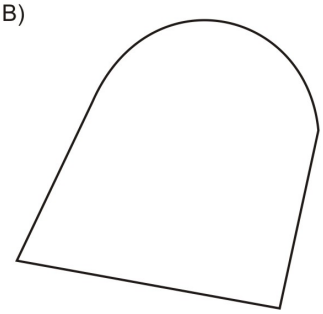
3. 

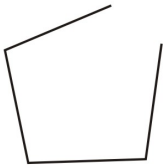
4. 

5. 

6. Explain why the following figures are NOT polygons:

A) 

B) 

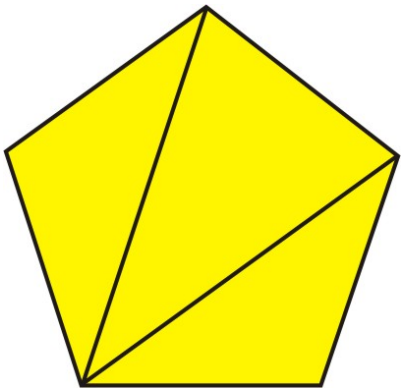
C) 

7. How many diagonals can you draw from one vertex of a pentagon? Draw a sketch of your answer.
8. How many diagonals can you draw from one vertex of an octagon? Draw a sketch of your answer.
9. How many diagonals can you draw from one vertex of a dodecagon?
10. Use your answers to 7, 8, and 9 and try more examples if necessary to answer the question: How many diagonals can you draw from one vertex of an $n-$gon?

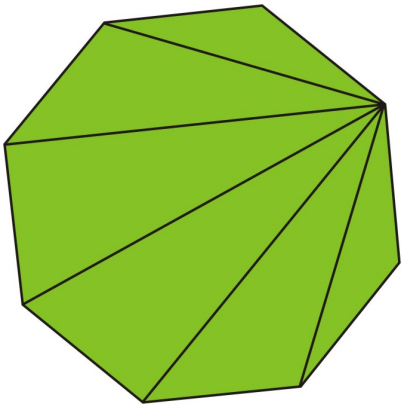
Review Answers

1. This is a convex pentagon.
2. Concave octagon.
3. Concave 17−gon (note that the number of sides is equal to the number of vertices, so it may be easier to count the points [vertices] instead of the sides).
4. Concave decagon.

5. Convex quadrilateral.
6. A is not a polygon since the two sides do not meet at a vertex; B is not a polygon since one side is curved; C is not a polygon since it is not enclosed.
7. The answer is 2.



8. The answer is 5.



9. A dodecagon has twelve sides, so you can draw nine diagonals from one vertex.
10. Use this table to answer question 10.

Table 1.2:

Sides	Diagonals from One Vertex
3	0
4	1
5	2
6	3
7	4
8	5
9	6
10	7
11	8
12	9

Table 1.2: (continued)

Sides	Diagonals from One Vertex
...	...
n	$n-3$

To see the pattern, try adding a "process" column that takes you from the left column to the right side.

Table 1.3:

Sides	Process	Diagonals from One Vertex
3	$(3)-3=0$	0
4	$(4)-3=1$	1
5	$(5)-3=2$	2
6	$(6)-3=3$	3
7	$(7)-3=4$	4
8	$(8)-3=5$	5
...		...
n	$(n)-3=$	$n-3$

Notice that we subtract 3 from each number on the left to arrive at the number in the right column. So, if the number in the left column is n (standing for some unknown number), then the number in the right column is $n-3$.

1.8 Problem Solving in Geometry

Learning Objectives

- Read and understand given problem situations.
- Use multiple representations to restate problem situations.
- Identify problem-solving plans.
- Solve real-world problems using planning strategies.

Introduction

One of the most important things we hope you will learn in school is how to solve problems. In real life, problem solving is not usually as clear as it is in school. Often, performing a calculation or measurement can be a simple task. Knowing what to measure or solve for

can be the greatest challenge in solving problems. This lesson helps you develop the skills needed to become a good problem solver.

Understanding Problem Situations

The first step whenever you approach a complicated problem is to simplify the problem. That means identifying the necessary information, and finding the desired value. Begin by asking yourself the simple question: What is this problem asking for?

If the problem had to ask you only one question, what would it be? This helps you identify how you should respond in the end.

Next, you have to find the information you need to solve the problem. Ask yourself another question: What do I need to know to find the answer?

This question will help you sift through information that may be helpful with this problem.

Use these basic questions to simplify the following problem. Don't try to solve it yet, just begin this process with questioning.

Example 1

Ehab drew a rectangle $PQRS$ on the chalkboard. PQ was 8 cm *and QR was* 6 cm. *If Ehab draws in the diagonal $\overline{QS}$, what will be its length?*

Begin to understand this problem by asking yourself two questions:

1. What is the problem asking for?

The question asks for the length of diagonal $\overline{QS}$.

2. What do I need to know to find the answer?

You need to know three things:

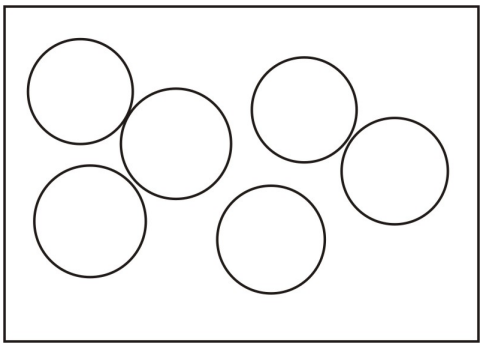
- The angles of a rectangle are all equal to 90°.
- The lengths of the sides of the rectangle are 8 cm and 6 cm.
- The Pythagorean Theorem can be used to find the third side of a right triangle.

Answering these questions is the first step to success with this problem.

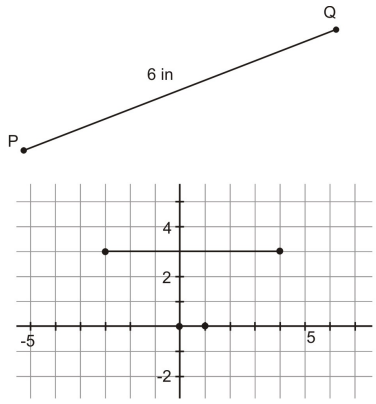
Drawing Representations

Up to this point, the analysis of the sample problem has dealt with words alone. It is important to distill the basic information from the problem, but there are different ways to proceed from here. Often, visual representations can be very helpful in understanding

problems. Make a simple drawing that represents what is being discussed. For example, a tray with six cookies could be represented by the diagram below.



The drawing takes only seconds to create, but it could help you visualize important information. Remember that there are many different ways to display information. Look at the way a line segment six inches long is displayed below.



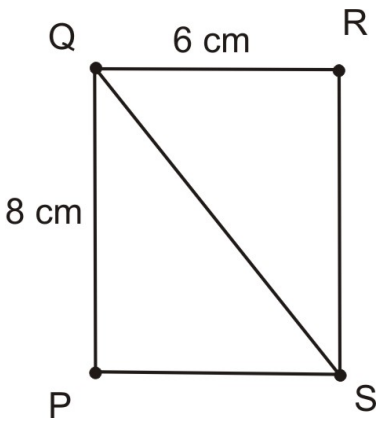
When you approach a problem, think about how you can represent the information in the most useful way. Continue your work on the sample problem by making drawings.

Let's return to that example.

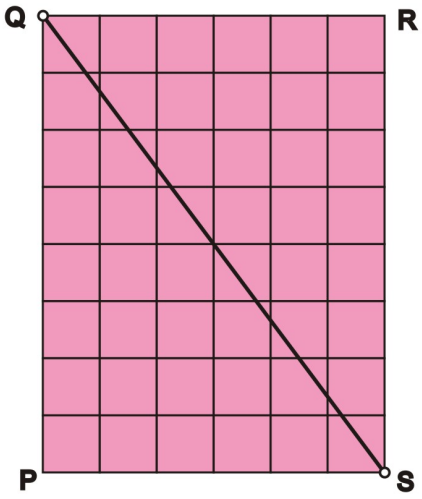
Example 1 (Repeated)

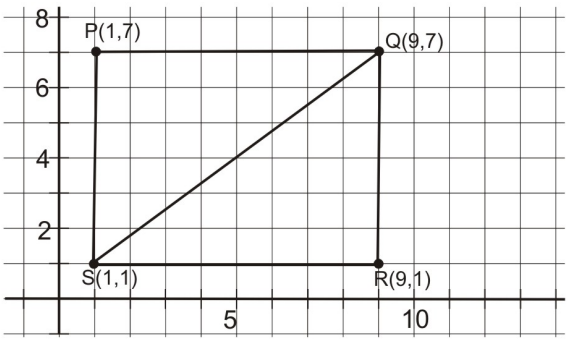
Ehab drew a rectangle $PQRS$ on the chalkboard. PQ was 8 cm *and QR was* 6 cm. *If Ehab draws in the diagonal $\overline{QS}$, what will be its length?*

Think about the different ways in which you could draw the information in this problem. The simplest idea is to draw a labeled rectangle. Be sure to label your drawing with information from the problem. This includes the names of the vertices as well as the side lengths.



As in most situations that you will encounter, there is more than one correct way to draw this shape. Two more possibilities follow.





The first example above shows the internal structure of the rectangle, as it is divided into square centimeters. The second example shows the rectangle situated on a coordinate grid. Notice that we rotated the figure by 90° in the second picture. This is fine as long as it was

drawn maintaining side lengths. One implication of putting the figure on the coordinate grid is that one square unit on the grid is equivalent to one square centimeter.

Identifying Your Strategy

At this point, you have simplified the problem by asking yourself questions about it, and created different representations of the important information. The time has come to establish a formal plan of attack. This is a crucial step in the problem-solving process, as it lays the groundwork for your solution.

To organize your thoughts, think of your geometric knowledge as a toolbox. Each time you learn a new strategy, technique, or concept, add it to your toolbox. Then, when you need to solve a problem, you can select the appropriate tool to use.

For now, take a quick look at the representations drawn for the example problem to identify what tools you might need. You can use this section to clearly identify your strategy.

Example

Ehab drew a rectangle $PQRS$ on the chalkboard. PQ was 8 cm *and QR was* 6 cm. *If Ehab draws in the diagonal $\overline{QS}$, what will be its length?*

In the first representation, there is simply a rectangle with a diagonal. Though there is a way to solve this problem using this diagram, it will not be covered until later in this book. For now, you do not have the tools to solve it.

The second diagram shows the building blocks that comprise the rectangle. The diagonal cuts through the blocks but presents the same challenges as the first diagram. You do not yet have the tools to solve the problem using this diagram either.

The third diagram shows a coordinate grid with the rectangle drawn in. The diagonal has two endpoints with specific coordinate pairs. In this chapter, you learned the distance formula to find lengths on a coordinate grid. This is the tool you need to solve the problem.

Your strategy for this problem is to identify the two endpoints of QS on the grid as (x_1, y_1) and (x_2, y_2). Use the distance formula to find the length. The result will be the solution to the problem.

Making Calculations

The last step in any problem-solving situation is employing your strategy to find the answer. Be sure that you use the correct values as identified in the relevant information. When you perform calculations, use a pencil and paper to keep track of your work. Many careless mistakes result from mental calculations. Keep track of each step along the way.

Finally, when you have found the answer, there are two more questions to ask yourself:

1. Did I provide the information the problem requested?

Go back to the first stages of the problem. Verify that you answered all parts of the question.

2. Does my answer make sense?

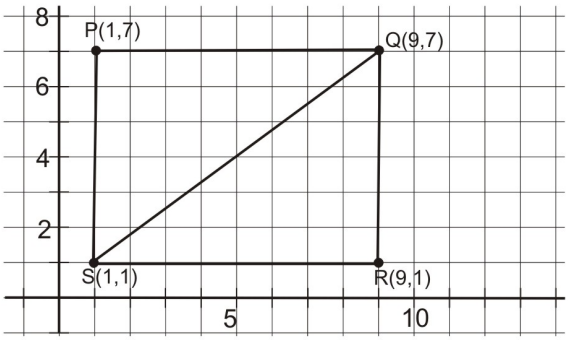
Your answer should make sense in the context of the problem. If your number is abnormally large or small in value, check your work.

Example

Ehab drew a rectangle $PQRS$ on the chalkboard. PQ was 8 cm *and WR was* 6 cm. *If Ehab draws in the diagonal $\overline{QS}$, what will be its length?*

At this point, we have distilled the problem, created multiple representations of the scenario, and identified the desired strategy. It is time to solve the problem.

The diagram below shows the rectangle on the coordinate grid.



To find the length of $\overline{QS}$, you must identify its endpoints on the grid. They are $(1, 1)$ and $(9, 7)$. Use the distance formula and substitute 1 for x_1, 1 for y_1, 9 for x_2, and 7 for y_2.

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\text{distance} = \sqrt{(9 - 1)^2 + (7 - 1)^2}$$
$$\text{distance} = \sqrt{(8)^2 + (6)^2}$$
$$\text{distance} = \sqrt{64 + 36}$$
$$\text{distance} = \sqrt{100}$$
$$\text{distance} = 10$$

QS is 10 cm.

Finally, make sure to ask yourself two more questions to verify your answer.

1. Did I provide the information the problem requested?

The problem asked you to identify the length of $\overline{QS}$. That is the information provided with our solution.

2. Does my answer make sense?

The value of 10 cm is slightly larger than 6 cm or 8 cm, but that is to be expected in this scenario. It is certainly within reason. A response of 80 cm or 0.08 cm would have been unreasonable.

Your work on this problem is now complete. The final answer is 10 cm.

Lesson Summary

In this lesson, we explored problem-solving strategies. Specifically, we have learned:

- How to read and understand given problem situations.
- How to use multiple representations to restate problem situations.
- How to identify problem-solving plans.
- How to solve real-world problems using planning strategies.

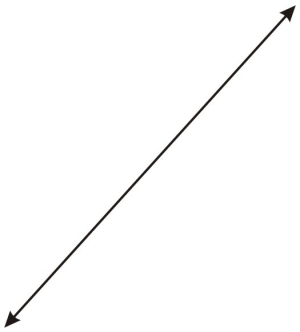
These skills are important for any type of problem, whether or not it is about geometry. Practice breaking down different problems in other parts of your life using these techniques. Forming plans and using strategies will help you in a number of different ways.

Points to Consider

This chapter focused on the basic postulates of geometry and the most common vocabulary and notations used throughout geometry. The following chapters focus on the skills of logic, reasoning, and proof. Review the material in this chapter whenever necessary to maintain your understanding of the basic geometric principles. They will be necessary as you continue in your studies.

Review Questions

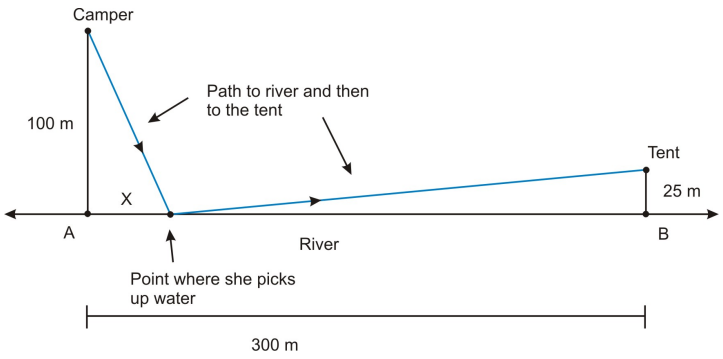
1. Suppose one line is drawn in a plane. How many regions of the plane are created?



2. Suppose two lines intersect in a plane. How many regions is the plane divided into? Draw a diagram of your answer.
3. Now suppose three coplanar lines intersect at the same point in a plane. How many regions is the plane divided into? Draw a diagram of your answer.
4. Make a table for the case of $4, 5, 6$, and 7 coplanar lines intersecting at one point.
5. Generalize your answer for number 4. If n coplanar lines intersect at one point, the plane is divided into _____ regions.
6. Bindi lives twelve miles south of Cindy. Mari lives five miles east of Bindi. What is the distance between Cindy's house and Mari's house?

   (a) Model this problem by drawing it on a coordinate grid. Let Bindi's house bet at the origin, $(0, 0)$. Use the labels B for Bindi's house, M for Mari's house, and C for Cindy's house.
   (b) What are the coordinates of Cindy's and Mari's house?
   (c) Use the distance formula to find the distance between

7. Suppose a camper is standing 100 meters north of a river that runs east-west in a perfectly straight line (we have to make some assumptions for geometric modeling!). Her tent is 25 meters north of the river, but 300 meters downstream. See the diagram below).

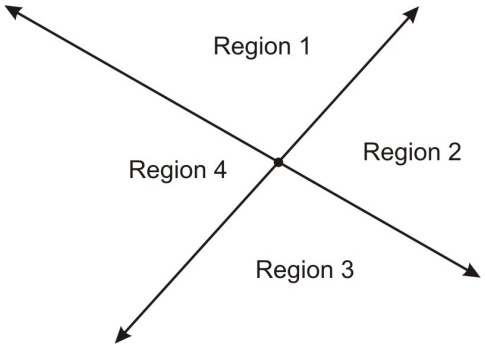


The camper sees that her tent has caught fire! Luckily she is carrying a bucket so she can get water from the river to douse the flames. The camper will run from her current position to the river, pick up a bucket of water, and then run to her tent to douse the flames (see the blue line in the diagram). But how far along the river should she run (distance x in the diagram) to pick up the bucket of water if she wants to minimize the total distance she runs? Solve this by any means you see fit—use a scale model, the distance formula, or some other geometric method.

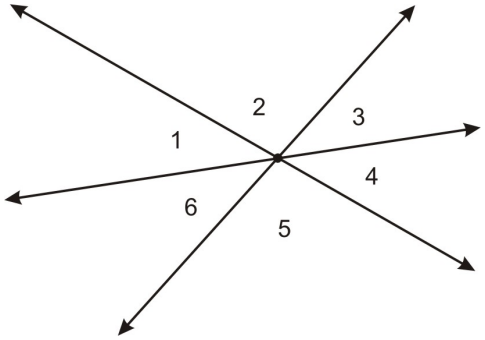
8. Does it make sense for the camper in problem 7 to want to minimize the total distance she runs? Make an argument for or against this assumption. (Note that in real-life problem solving finding the "best" answer is not always simple!).

Review Answers

1. 2
2. 4



3. 6

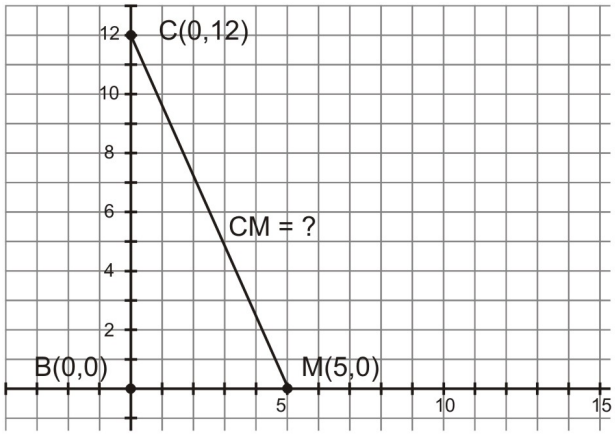


4. See the table below

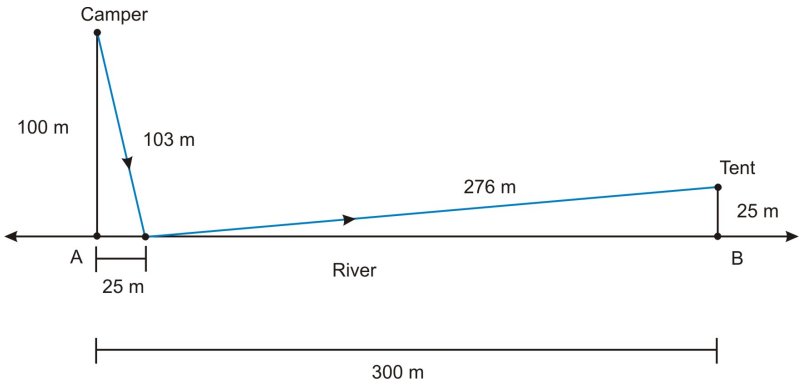
Table 1.4:

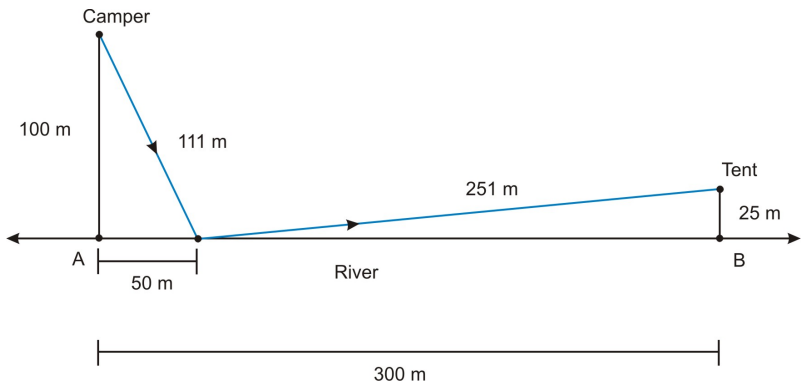
Number of Coplanar Lines Intersecting at One Point	Number of Regions Plane is Divided Into
1	2
2	4
3	6
4	8
5	10
6	12
7	14

5. Every number in the right-hand column is two times the number in the left-hand column, so the general statement is: "If n coplanar lines intersect at one point, the plane is divided into $2n$ regions."
6. (a)



   (b) Cindy's House: $(0, 12)$; Mari's house: $(5, 0)$
   (c) 13 miles
7. One way to solve this is to use a scale model and a ruler. Let 1 cm = 100 m. Then you can draw a picture and measure the distance the camper has to run for various locations of the point where she gets water. Be careful using the scale!





Now make a table for all measurements to find the best, shortest total distance.

Table 1.5:

x (meters)	Distance to Water (m)	Distance from Water to Tent (m)	Total Distance (m)
0	100	301	401
25	103	276	379
50	112	251	363
100	141	202	343
125	160	177	337
150	180	152	332
175	202	127	329
200	224	103	327
225	246	79	325
250	269	56	325
275	293	35	328
300	316	25	341

It looks like the best place to stop is between 225 and 250 meters. Based on other methods (which you will learn in calculus and some you will learn later in geometry), we can prove that the best distance is when she runs 240 meters downstream to pick up the bucket of water.

8. Answers will vary. One argument for why it is not best to minimize total distance is that she may run slower with the full bucket of water, so she should take the distance she must run with a full bucket into account.

Image Sources

(1) http://commons.wikimedia.org/wiki/File:Protractor.jpg. GNU Free Documentation.

(2) *Photograph of a tennis racket and two balls*. GNU Free Documentation.

(3) *Soccer field*. CCSA.

(4) http://commons.wikimedia.org/wiki/File:Dubia-cockroach-female-near-ruler.jpg. Public Domain.

(5) *Airscreen auf dem James Dean Festival in Marion, USA*. GNU Free Documentation.

Chapter 2

Reasoning and Proof

2.1 Inductive Reasoning

Learning Objectives

- Recognize visual patterns and number patterns.
- Extend and generalize patterns.
- Write a counterexample to a pattern rule.

Introduction

You learned about some of the basic building blocks of geometry in Chapter 1. Some of these are points, lines, planes, rays, and angles. In this section we will begin to study ways we can *reason* about these building blocks.

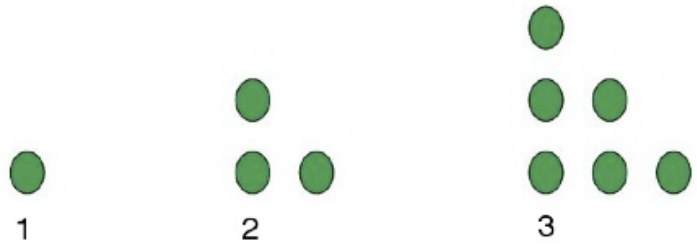
One method of reasoning is called **inductive reasoning**. This means drawing conclusions based on examples.

Visual Patterns

Some people say that mathematics is the study of patterns. Let's look at some *visual* patterns. These are patterns made up of shapes.

Example 1

A dot pattern is shown below.



A. How many dots would there be in the bottom row of a fourth pattern?

There will be 4 dots. There is one more dot in the bottom row of each figure than in the previous figure. Also, the number of dots in the bottom row is the same as the figure number.

B. What would the total number of dots be in the bottom row if there were 6 *patterns?*

There would be a total of 21 dots. The rows would contain $1, 2, 3, 4, 5$, and 6 dots.

The total number of dots is $1 + 2 + 3 + 4 + 5 + 6 = 21$.

Example 2

Next we have a pattern of squares and triangles.

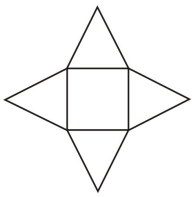


Figure 1

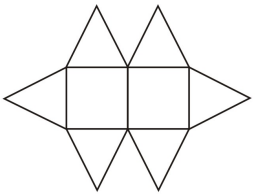


Figure 2

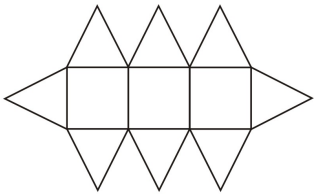


Figure 3

A. How many triangles would be in a tenth illustration?

There will be 22 triangles. There are 10 squares, with a triangle above and below each square. There is also a triangle on each end of the figure. That makes $10 + 10 + 1 + 1 = 22$ triangles in all.

B. One of the figures would contain 34 *triangles. How many squares would be in that figure?*

There will be 16 squares. Take off one triangle from each end. This leaves 32 triangles. Half of these 32 triangles, or 16 triangles, are above and 16 triangles are below the squares. This means there are 16 squares.

To check: With 16 squares, there is a triangle above and below each square, making $2 \times 16 =$ 32 squares. Add one triangle for each end and we have $32 + 1 + 1 = 34$ triangles in all.

C. How can we find the number of triangles if we know the figure number?

Let n be the figure number. This is also the number of squares. $2n$ is the number of triangles above and below the squares. Add 2 for the triangles on the ends.

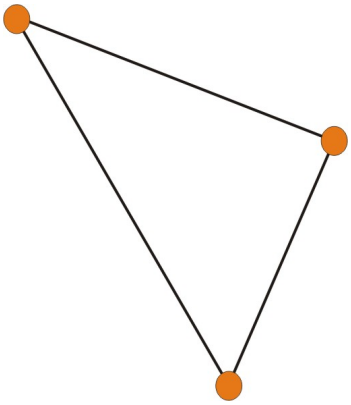
If the figure number is n, then there are $2n + 2$ triangles in all.

Example 3

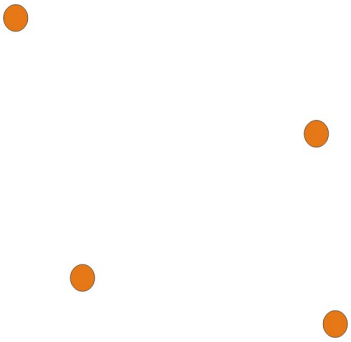
Now look at a pattern of points and line segments.

For two points, there is one line segment with those points as endpoints.

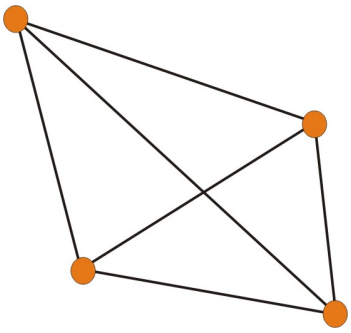
For three noncollinear points (points that do not lie on a single line), there are three line segments with those points as endpoints.



A. For four points, no three points being collinear, how many line segments with those points as endpoints are there?

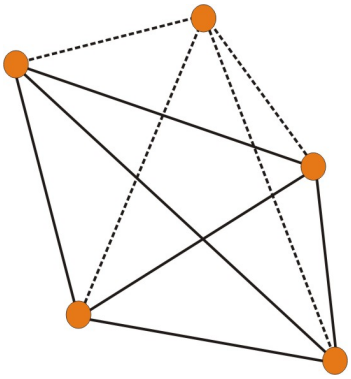


6. The segments are shown below.



B. For five points, no three points being collinear, how many line segments with those points as endpoints are there?

10. When we add a 5th point, there is a new segment from that point to each of the other four points. We can draw the four new dashed segments shown below. Together with the six segments for the four points in part A, this makes $6 + 4 = 10$ segments.



Number Patterns

You are already familiar with many number patterns. Here are a few examples.

Example 4 – Positive Even Numbers

The positive even numbers form the pattern $2, 4, 6, 8, 10, 12, \ldots$.

What is the 19^{th} *positive even number?*

The answer is 38. Each positive even number is 2 more than the preceding one. You could start with 2, then add 2, 18 times, to get the 19^{th} number. But there is an easier way, using more advanced mathematical thinking. Notice that the 3^{rd} even number is 2×3 , the 4^{th} even number is 2×4 , and so on. So the 19^{th} even number is $2 \times 19 = 38$.

Example 5 – Odd Numbers

Odd numbers form the pattern $1, 3, 5, 7, 9, 11, \ldots$.

A. *What is the* 34^{th} *odd number?*

The answer is 67. We can start with 1 and add 2, 33 times. $1+2\times 33=1+66=67$. Or, we notice that each odd number is 1 less than the corresponding even number. The 34^{th} even number is $2\times 34=68$ (example 4), so the 34^{th} odd number is $68-1=67$.

B. *What is the* n^{th} *odd number?*

$2n-1$. The n^{th} even number is $2n$ (example 4), so the n^{th} odd number is $2n-1$.

Example 6 – Square Numbers

Square numbers form the pattern $1, 4, 9, 16, 25, \ldots$.

These are called square numbers because $1=1^2, 4=2^2, 9=3^2, 16=4^2, 25=5^2, \ldots$.

A. *What is the* 10^{th} *square number?*

The answer is 100. The 10^{th} square number is $10^2=100$.

B. *The* n^{th} *square number is* 441. *What is the value of* n*?*

The answer is 21. The 21^{st} square number is $21^2=441$.

Conjectures and Counterexamples

A **conjecture** is an "educated guess" that is often based on examples in a pattern. Examples suggest a relationship, which can be stated as a possible rule, or conjecture, for the pattern.

Numerous examples may make you strongly believe the conjecture. However, no number of examples can *prove* the conjecture. It is always possible that the next example would show that the conjecture does not work.

Example 7

Here's an algebraic equation.

$$t=(n-1)(n-2)(n-3)$$

Let's evaluate this expression for some values of n.

$$n=1; t=(n-1)(n-2)(n-3)=0\times(-1)\times(-2)=0$$
$$n=2; t=(n-1)(n-2)(n-3)=1\times 0\times(-1)=0$$
$$n=3; t=(n-1)(n-2)(n-3)=2\times 1\times 0=0$$

These results can be put into a table.

n	1	2	3
t	0	0	0

After looking at the table, we might make this conjecture:

The value of $(n-1)(n-2)(n-3)$ *is* 0 *for any whole number value of* n.

However, if we try other values of n, such as $n = 4$, we have

$$(n-1)(n-2)(n-3) = 3 \times 2 \times 1 = 6$$

Obviously, our conjecture is wrong. For this conjecture, $n = 4$ is called a **counterexample**, meaning that this value makes the conjecture false. (Of course, it was a pretty poor conjecture to begin with!)

Example 8

Ramona studied positive even numbers. She broke some positive even numbers down as follows:

$$8 = 3 + 5 \qquad 14 = 5 + 9 \qquad 36 = 17 + 19 \qquad 82 = 39 + 43$$

What conjecture might be suggested by Ramona's results?

Ramona made this conjecture:

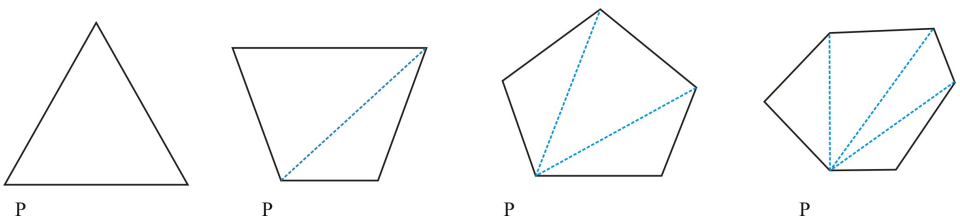
"Every positive even number is the sum of two different positive odd numbers."

Is Ramona's conjecture correct? Can you find a counterexample to the conjecture?

The conjecture is not correct. A counterexample is 2. The only way to make a sum of two odd numbers that is equal to 2 is: $2 = 1 + 1$, which is not the sum of *different* odd numbers.

Example 9

Artur is making figures for a graphic art project. He drew polygons and some of their diagonals.



Based on these examples, Artur made this conjecture:

If a convex polygon has n sides, then there are $n - 3$ diagonals from any given vertex of the polygon.

Is Artur's conjecture correct? Can you find a counterexample to the conjecture?

The conjecture appears to be correct. If Artur draws other polygons, in every case he will be able to draw $n - 3$ diagonals if the polygon has n sides.

Notice that we have not *proved* Artur's conjecture. Many examples have (almost) convinced us that it is true.

Lesson Summary

In this lesson you worked with visual and number patterns. You extended patterns to beyond the given items and used rules for patterns. You also learned to make conjectures and to test them by looking for counterexamples, which is how inductive reasoning works.

Points to Consider

Inductive reasoning about patterns is a natural way to study new material. But we saw that there is a serious limitation to inductive reasoning: No matter how many examples we have, examples alone do not prove anything. To *prove* relationships, we will learn to use **deductive** reasoning, also known as logic.

Review Questions

How many dots would there be in the fourth pattern of each figure below?

1.

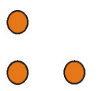


Figure 1

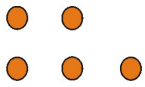


Figure 2

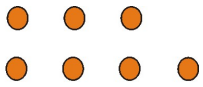


Figure 3

2.

Figure 1

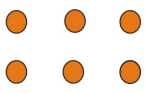


Figure 2

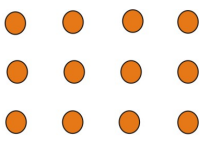


Figure 3

3.

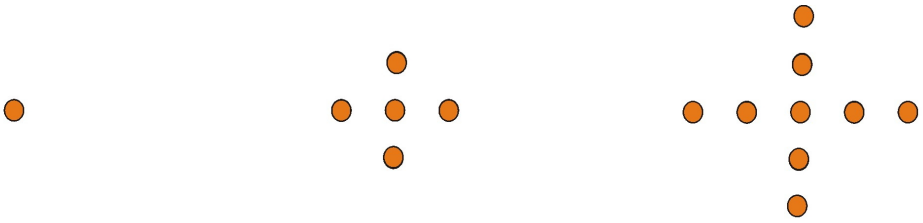


Figure 1 Figure 2 Figure 3

4. What is the next number in the following number pattern? $5, 8, 11, 14$
5. What is the tenth number in this number pattern? $3, 6, 11, 18$

The table below shows a number pattern.

n	1	2	3	4	5
t	3	8	15	24	35

6. What is the value of t when $n = 6$?
7. What is the value of n when $t = 99$?
8. Is 145 a value of t in this pattern? Explain your answer.

Give a counterexample for each of the following statements.

9. If n is a whole number, then $n^2 > n$.
10. Every prime number is an odd number.
11. If $AB = 5$ and $BC = 2$, then $AC = 7$.

Review Answers

1. 9
2. 20
3. 13
4. 17
5. 102
6. 48
7. 9
8. No. Values of t are $3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, \ldots$ or, $t = n^2 + 2n$; there is no value of n that makes $t = 145$.
9. 1, because $1^2 = 1$.
10. 2, because 2 is prime but not odd.
11. Any set of points where A, B, and C are not collinear.

2.2 Conditional Statements

Learning Objectives

- Recognize if-then statements.
- Identify the hypothesis and conclusion of an if-then statement.
- Write the converse, inverse, and contrapositive of an if-then statement.
- Understand a biconditional statement.

Introduction

In geometry we reason from known facts and relationships to create new ones. You saw earlier that inductive reasoning can help, but it does not *prove* anything. For that we need another kind of reasoning. Now you will begin to learn about **deductive reasoning,** the kind of reasoning used throughout mathematics and science.

If-Then Statements

In geometry, and in ordinary life, we often make **conditional**, or **if-then**, statements.

- Statement 1: If the weather is nice, I'll wash the car. ("Then" is implied even if not stated.)
- Statement 2: If you work overtime, then you'll be paid time-and-a-half.
- Statement 3: If 2 divides evenly into x, then x is an even number.
- Statement 4: If a triangle has three congruent sides, it is an equilateral triangle. ("Then" is implied; this is a definition.)
- Statement 5: All *equiangular* triangles are *equilateral*. ("If" and "then" are both implied.)

An if-then statement has two parts.

- The "if" part is called the **hypothesis.**
- The "then" part is called the **conclusion.**

For example, in statement 2 above, the hypothesis is "you work overtime." The conclusion is "you'll be paid time-and-a-half."

Look at statement 1 above. Even though the word "then" is not actually present, the statement could be rewritten as: If the weather is nice, *then* I'll wash the car. This is the

meaning of statement 1. The hypothesis is "the weather is nice." The conclusion is "I'll wash the car."

Statement 5 is a little more complicated. "If" and "then" are both implied without being stated. Statement 5 can be rewritten as: If a triangle is *equiangular,* then it is *equilateral.*

What is meant by an if-then statement? Suppose your friend makes the statement in statement 2 above, and adds another fact.

- If you work overtime, then you'll be paid time-and-a-half.
- You worked overtime this week.

If we accept these statements, what other fact *must* be true? Combining these two statements, we can state with no doubt:

You'll be paid time-and-a-half this week.

Let's analyze statement 1, which was rewritten as: If the weather is nice, then I'll wash the car. Suppose we accept statement 1 and another fact: I'll wash the car.

Can we conclude anything further from these two statements? No. Even if the weather is *not* nice, I might wash the car. We *do* know that if the weather is nice I'll wash the car. We *don't* know whether or not I might wash the car even if the weather is *not* nice.

Converse, Inverse, and Contrapositive of an If-Then Statement

Look at statement 1 above again.

If the weather is nice, then I'll wash the car.

This can be represented in a diagram as:

If p then q.

$p =$ the weather is nice $\qquad$ $q =$ I'll wash the car

"If p then q" is also written as

$$p \rightarrow q$$

Notice that conditional statements, hypotheses, and conclusions may be true or false. p, q, and the statement "If p, then q" may be true or false.

In deductive reasoning we sometimes study statements related to a given if-then statement. These are formed by using p, q, and their opposites, or **negations** ("not"). Note that "not p" is written in symbols as $\neg p$.

p, q, $\neg p$, and $\neg q$ can be combined to produce new if-then statements.

- The **converse** of $p \rightarrow q$ is $q \rightarrow p$.
- The **inverse** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

Now let's go back to statement 1: If the weather is nice, then I'll wash the car.

$p \rightarrow q$

$p =$ the weather is nice
$q =$ I'll wash the car
$\neg p =$ the weather is not nice
$q =$ I'll wash the car (or I wash the car)
$\neg q =$ I won't wash the car (or I don't wash the car)

Converse $q \rightarrow p$ If I wash the car, then the weather was nice.

Inverse $\neg p \rightarrow \neg q$ If the weather is not nice, then I won't wash the car.

Contrapositive $\neg q \rightarrow \neg p$ If I don't wash the car, then the weather is not nice.

Notice that if we accept statement 1 as true, then the converse and inverse may, or may not, be true. But the contrapositive is true. Another way to say this is: *The contrapositive is logically equivalent to the original if-then statement.* In future work you may be asked to prove an if-then statement. If it's easier to prove the contrapositive, then you can do this since the statement and its contrapositive are equivalent.

Example 1

Statement:

If $n > 2$, then $n^2 > 4$. True.

Converse:

If $n^2 > 4$, then $n > 2$. False.

A counterexample is $n = -3$, where $n^2 = 9 > 4$ but $n = -3$ is not > 2

Inverse:

If n is not > 2 then n^2 is not > 4. False.

A counterexample is $n = -3$, where n is not > 2 but $n^2 = 9 > 4$

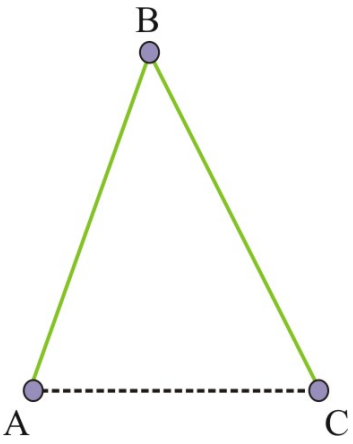
Contrapositive:

If n^2 is not > 4, then n is not > 2. True.

If n^2 is not > 4 , then $-2 < n < 2$ and n is not > 2

Example 2

Statement: If $AB = BC$, then B is the midpoint of AC. False (as shown below).



Needs $AB = BC$

Converse: If B is the midpoint of $\overline{AC}$, then $AB = BC$. True.

Inverse: If $AB \neq BC$, then B is not the midpoint of $\overline{AC}$. True.

Contrapositive: If B is not the midpoint of $\overline{AC}$, then $AB \neq BC$ False (see the diagram above).

Biconditional Statements

You recall that the converse of "If p then q" is "If q then p." When these two are combined, we have a **biconditional** statement.

Biconditional: $p \rightarrow q$ and $q \rightarrow p$

In symbols, this is written as: $p \leftrightarrow q$

We read $p \leftrightarrow q$ as: "p if and only if q"

Example 3

True statement: $m\angle ABC > 90^\circ$ if and only if $\angle ABC$ is an obtuse angle.

You can break this down to say:

If $m\angle ABC > 90°$ then $\angle ABC$ is an obtuse angle and if $\angle ABC$ is an obtuse angle then $m\angle ABC > 90°$.

Notice that both parts of this biconditional are true; the biconditional itself is true.

You most likely recognize this as the *definition* of an obtuse angle.

Geometric definitions are biconditional statements that are true.

Example 4

Let p be $x < 10$

Let q be $2x < 50$

a. *Is $p \rightarrow q$ true?*

Yes.

$p \rightarrow q$ is if $x < 10$ then $2x < 50$.

From algebra we know that if $x < 10$ then $2x < 2(10)$ and $2x < 20$. If $2x < 20$, then we know that $2x < 50$.

So if $x < 10$ then $2x < 50$, or $p \rightarrow q$, is true.

b. *Is $q \rightarrow p$ true?*

No.

$q \rightarrow p$ is if $2x < 50$, then $x < 10$.

From algebra we know that if $2x < 50$, then $x < 25$.

However, $x < 25$ does not guarantee that $x < 10$.

x can be less than 25 but still not less than 10, for example if x is 20.

So if $2x < 50$, then $x < 10$, or $q \rightarrow p$, is false.

c. *Is $p \leftrightarrow q$ true?*

No.

$p \leftrightarrow q$ is $x < 10$ if and only if $2x < 50$.

We saw above that the if part of this statement, which is

If $2x < 50$ then $x < 10$.

This statement is false. One counterexample is $x = 20$.

Note that if either $p \rightarrow q$ or $q \rightarrow p$ is false, then $p \leftrightarrow q$ is false.

Lesson Summary

In this lesson you have learned how to express mathematical and other statements in if-then form. You also learned that each if-then statement is linked to variations on the basic theme of "If p then q." These variations are the converse, inverse, and contrapositive of the if-then statement. Biconditional statements combine the statement and its converse into a single "if and only if" statement. Definitions are an important type of biconditional, or if-and-only-if, statement.

Points to Consider

We called points, lines, and planes the building blocks of geometry. We will soon see that hypothesis, conclusion, as well as if-then and if-and-only-if statements are the building blocks that deductive reasoning, or logic, is built on. This type of reasoning will be used throughout your study of geometry. In fact, once you understand logical reasoning you will find that you apply it to other studies and to information you encounter all your life.

Review Questions

Write the hypothesis and the conclusion for each statement.

1. If 2 divides evenly into x, then x is an even number.
2. If a triangle has three congruent sides, it is an equilateral triangle.
3. All *equiangular* triangles are *equilateral*.
4. What is the converse of the statement in exercise 1 above? Is the converse true?
5. What is the inverse of the statement in exercise 2 above? Is the inverse true?
6. What is the contrapositive of the statement in exercise 3? Is the contrapositive true?
7. The converse of a statement about collinear points A, B, and C is: If $AB = 5$ and $BC = 5$, then B is the midpoint of $\overline{AC}$.
   - What is the statement?
   - Is it true?
8. What is the inverse of the inverse of if p then q?
9. What is the one-word name for the converse of the inverse of an if-then statement?
10. What is the one-word name for the inverse of the converse of an if-then statement?

For each of the following biconditional statements:

- Write p in words.
- Write q in words.
- Is $p \rightarrow q$ true?

- Is $q \rightarrow p$ true?
- Is $p \leftrightarrow q$ true?

Note that in these questions, p and q could be reversed and the answers would be correct.

11. A U.S. citizen can vote if and only if he or she is 18 or more years old.
12. A whole number is prime if and only if it is an odd number.
13. Points are collinear if and only if there is a line that contains the points.
14. $x + y = 17$ if only if $x = 8$ and $y = 9$

Review Answers

1. Hypothesis: 2 divides evenly into x; conclusion: x is an even number.
2. Hypothesis: A triangle has three congruent sides; conclusion: it is an equilateral triangle.
3. Hypothesis: A triangle is equiangular; conclusion: the triangle is equilateral.
4. If x is an even number, then 2 divides evenly into x. True.
5. If a triangle does not have three congruent sides, then it is not an equilateral triangle. True.
6. If a triangle is not equilateral, then it is not equiangular. True.
7. If B is the midpoint of $\overline{AC}$, then $AB = 5$ and $BC = 5$. False (AB and BC could both be 6, 7, etc.).
8. If p then q.
9. Contrapositive
10. Contrapositive
11. p = he or she is 18 or more years old; q = a U. S. citizen can vote; $p \rightarrow q$ is true; $q \rightarrow p$ is true; $p \leftrightarrow q$ is true.
12. p = a whole number is an odd number; q = a whole number is prime; $p \rightarrow q$ is false; $q \rightarrow p$ is false; $p \leftrightarrow q$ is false.
13. p = a line contains the points; q = the points are collinear; is $p \rightarrow q$ is true; $q \rightarrow p$ is true; $p \leftrightarrow q$ is true.
14. $p = x = 8$ and $y = 9$; $q = x + y = 17$; $p \rightarrow q$ is true; $q \rightarrow p$ is false; $p \leftrightarrow q$ is false.

2.3 Deductive Reasoning

Learning Objectives

- Recognize and apply some basic rules of logic.
- Understand the different parts that inductive reasoning and deductive reasoning play in logical reasoning.
- Use truth tables to analyze patterns of reasoning.

Introduction

You began to study deductive reasoning, or *logic,* in the last section, when you learned about if-then statements. Now we will see that logic, like other fields of knowledge, has its own rules. When we follow those rules, we will expand our base of facts and relationships about points, lines, and planes. We will learn two of the most useful rules of logic in this section.

Direct Reasoning

We all use logic—whether we call it that or not—in our daily lives. And as adults we use logic in our work as well as in making the many decisions a person makes every day.

- Which product should you buy?
- Who should you vote for?
- Will this steel beam support the weight you place on it?
- What will be your company's profit next year?

Let's see how common sense leads to the two most basic rules of logic.

Example 1

Suppose Bea makes the following statements, which are known to be true.

If Central High School wins today, they will go to the regional tournament.

Central High School does win today.

Common sense tells us that there is an obvious logical conclusion if these two statements are true:

Central High School will go to the regional tournament.

Example 2

Here are two true statements.

5 is an odd number.

Every odd number is the *sum* of an even and an odd number.

Based on only these two true statements, there is an obvious further conclusion:

5 is the sum of an even and an odd number.

(This is true, since $5 = 2 + 3$).

Example 3

Suppose the following two statements are true.

1. *If you love me let me know, if you don't then let me go.* (A country music classic. Lyrics by John Rostill.)
2. *You don't love me.*

What is the logical conclusion?

Let me go.

There are two statements in the first line. The second one is:

If you don't (love me) then let me go.

You don't love me is stated to be true in the second line.

Based on these true statements,"Let me go" is the logical conclusion.

Now let's look at the structure of all of these examples, using the p and q symbols that we used earlier.

Each of the examples has the same form.

$$p \rightarrow q$$
$$p$$

conclusion: q

A more compact form of this argument, (logical pattern) is:

$$p \rightarrow q$$
$$p$$
$$_____$$
$$q$$

To state this differently, we could say that the true statement q follows automatically from the true statements $p \rightarrow q$ and p.

This reasoning pattern is one of the basic rules of logic. It's called the **law of detachment.**

Law of Detachment

Suppose p and q are statements. Then given

$p \rightarrow q$ and p

You can conclude

q

Practice saying the law of detachment like this: "If $p \rightarrow q$ is true, and p is true, *then* q is true."

Example 4

Here are two true statements.

If $\angle A$ and $\angle B$ are a linear pair, then $m\angle A + m\angle B = 180°$.

$\angle A$ and $\angle B$ are a linear pair.

What conclusion do we draw from these two statements?

$m\angle A + m\angle B = 180°$.

The next example is a warning not to turn the law of detachment around.

Example 5

Here are two true statements.

If $\angle A$ and $\angle B$ are a linear pair, then $m\angle A + m\angle B = 180°$.

$m\angle A = 90°$ and $m\angle B = 90°$.

What conclusion can we draw from these two statements?

None! These statements are in the form

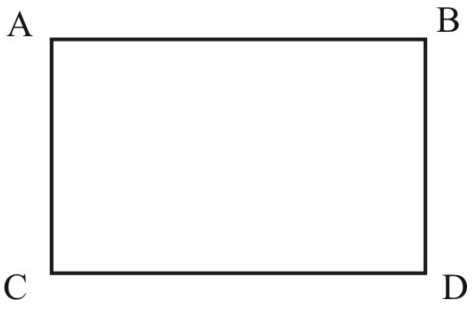
$p \rightarrow q$

q

Note that since $m\angle A = 90°$ and $m\angle B = 90°$, we also know that $m\angle A + m\angle B = 180°$, but this does not mean that they are a linear pair.

The law of detachment does not apply. No further conclusion is justified.

You might be tempted to conclude that $\angle A$ and $\angle B$ are a linear pair, but if you think about it you will realize that would not be justified. For example, in the rectangle below $m\angle A = 90°$ and $m\angle B = 90°$ (and $m\angle A + m\angle B = 180°$, but $\angle A$, and $\angle B$ are definitely NOT a linear pair.



Now let's look ahead. We will be doing some more complex deductive reasoning as we move ahead in geometry. In many cases we will build chains of connected if-then statements, leading to a desired conclusion. Start with a simplified example.

Example 6

Suppose the following statements are true.

1. If Pete is late, Mark will be late.
2. If Mark is late, Wen will be late.
3. If Wen is late, Karl will be late.

To these, add one more true statement.

4. Pete is late.

One clear consequence is: Mark will be late. But make sure you can see that Wen and Karl will also be late.

Here's a symbolic form of the statements.

1. $p \rightarrow q$
2. $q \rightarrow r$
3. $r \rightarrow s$
4. p

---

s

Our statements form a "chain reaction." Each "then" becomes the next "if" in a chain of statements. The chain can consist of any number of connected statements. Once we add the true p statement as above, we know that the conclusion (the then part) of the last statement is justified.

Another way to look at this is to imagine a chain of dominoes. The dominoes are the linked if-then statements. Once the first domino falls, each domino knocks the next one over, and the last domino falls. p is the tipping over of the first domino. The final conclusion of the last if-then statement is the last domino.

This is called **the law of syllogism**. A formal statement of this rule of logic is given below.

Law of Syllogism

Suppose $a_1, a_2, \ldots, a_{n-1}$, and a_n are statements. Then given that a_1 is true and that you have the following relationship:

$$\begin{aligned} a_1 &\rightarrow a_2 \\ a_2 &\rightarrow a_3 \\ &\vdots \\ a_{n-1} &\rightarrow a_n \end{aligned}$$

Then, you can conclude

$$a_1 \rightarrow a_n$$

Inductive vs. Deductive Reasoning

You have now worked with both *inductive* and *deductive* reasoning. They are different but not opposites. In fact, they will work together as we study geometry and other mathematics.

How do these two kinds of reasoning complement (strengthen) each other? Think about the examples you saw earlier in this chapter.

Inductive reasoning means reasoning from examples. You may look at a few examples, or many. Enough examples might make you *suspect* that a relationship is true always, or might even make you *sure* of this. But until you go beyond the inductive stage, you can't be absolutely sure that it is always true.

That's where *deductive reasoning* enters and takes over. We have a suggestion arrived at inductively. We then apply rules of logic to prove, beyond any doubt, that the relationship is true always. We will use the *law of detachment* and the *law of syllogism,* and other logic rules, to build these proofs.

Symbolic Notation and Truth Tables

Logic has its own rules and symbols. We have already used letters like p and q to represent statements: for the negation ("not"), and the arrow $\rightarrow$ to indicate if-then. Here are two more symbols we can use.

$$\wedge = \text{and}$$
$$\vee = \text{or}$$

Truth tables are a way to analyze statements in logic. Let's look at a few simple truth tables.

Example 1

How is $\neg p$ related to p logically? We make a truth table to find out. Begin with all the possible truth values of p. This is very simple; p can be either true (T), or false (F).

Table 2.1:

p
T
F

Next we write the corresponding truth values for $\neg p$. $\neg p$ has the opposite truth value as p.

If p is true, then $\neg p$ is false, and vice versa. Complete the truth table by filling in the $\neg p$ column.

Table 2.2:

p	$\neg p$
T	F
F	T

Now we construct truth tables for slightly more complex logic.

Example 2

Draw a truth table for p and q written $p \wedge q$.

Begin by filling in all the T/F combinations possible for p and q.

Table 2.3:

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

How can p and q be true? Common sense tells us that p and q is false whenever either p or q is false. We complete the last column accordingly.

Table 2.4:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Another way to state the meaning of the truth table is that $p \wedge q$ is true only when p is true *and* q is true.

Let's do the same for p or q. Before we do that, we need to clarify which "or" we mean in mathematics. In ordinary speech, or is sometimes used to mean, "this or that, but not both." This is called the **exclusive or** (it excludes or keeps out both). In mathematics, or

means "this, that, or both this and that." This is called the **inclusive or**. Knowing that or is inclusive makes the truth table an easy job.

Example 2

$5 = 2 + 3$ or $5 > 6$ is true, because $5 = 2 + 3$ is true.

$5 < 6$ or $6 < 5$ is true, because $5 < 6$ is true.

$5 = 2 + 3$ or $5 < 6$ is true because $5 = 2 + 3$ is true and $5 < 6$ is true.

$5 = 2 + 4$ or $5 > 6$ is false because $5 = 2 + 4$ is false and $5 > 6$ is false.

Example 3

Draw a truth table for p or q, which is written $p \vee q$.

Begin by filling in all the T/F combinations possible for p and q. Keeping in mind the definition of or above (inclusive), fill in the third column. p or q will only be *false* when both p and q are false; it is true otherwise.

Table 2.5:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Lesson Summary

Do we all have our own version of what is logical? Let's hope not—we wouldn't be able to agree on what is or isn't logical! To avoid this, there are agreed-on rules for logic, just like there are rules for games. The two most basic rules of logic that we will be using throughout our studies are the **law of detachment** and the **law of syllogism.**

Points to Consider

Rules of logic are universal; they apply to all fields of knowledge. For us, the rules give a powerful method for proving new facts that are suggested by our explorations of points, lines, planes, and so on. We will structure a specific format, the two-column proof, for proving these new facts. In upcoming lessons you will write two-column proofs. The facts or relationships that we prove are called **theorems**.

Review Questions

Must the third sentence be true if the first two sentences are true? Explain your answer.

1. People who vote for Jane Wannabe are smart people.

I am a smart person.

I will vote for Jane Wannabe.

2. If Rae is the driver today then Maria is the driver tomorrow.

Ann is the driver today.

Maria is not the driver tomorrow.

3. All equiangular triangles are equilateral.

$\triangle ABC$ is equiangular.

$\triangle ABC$ is equilateral.

What additional statement must be true if the given sentences are true?

4. If West wins, then East loses. If North wins, then West wins.
5. If $x > 5$ then $x > 3$. If $x > 3$ then $y > 7$.
   $x = 6$.

Fill in the truth tables.

Table 2.6:

p	$\neg p$	$p \wedge \neg p$
T		
F		

Table 2.7:

p	$\neg p$	$p \vee \neg p$
T		
F		

Table 2.8:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T			
T	F			
F	T			
F	F			

Table 2.9:

p	q	$\neg q$	$q \vee \neg q$	$p \wedge (q \vee \neg q)$
T	T			
T	F			
F	T			
F	F			

10. When is $p \vee \ q \vee \ r$ true?
11. For what values of x is the following statement true?

$x \geq 2$ or $x^2 < 4$

12. For what values of x is the following statement true?

$x \geq 2$ or $x^2 < 4$

Review Answers

1. No (converse error).
2. No (inverse error).
3. Yes.
4. If North wins, then East loses.
5. $y > 7$. (also $x > 3$)

Table 2.10:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

6. Note that $p \wedge \neg p$ is *never* true.

Table 2.11:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

7. Note that $p \vee \neg p$ is *always* true.

Table 2.12:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

8. Note that $\neg p \wedge \neg q$ is true only when p and q are *both* false.

Table 2.13:

p	q	$\neg q$	$q \vee \neg q$	$p \wedge (q \vee \neg q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

9.
10. $p \vee q \vee r$ is always true except when p, q, and r are all false.
11. $x > -2$
12. none, $\emptyset$.

2.4 Algebraic Properties

Learning Objectives

- Identify and apply properties of equality.
- Recognize properties of congruence "inherited" from the properties of equality.
- Solve equations and cite properties that justify the steps in the solution.

- Solve problems using properties of equality and congruence.

Introduction

We have begun to assemble a toolbox of building blocks of geometry (points, lines, planes) and rules of logic that govern deductive reasoning. Now we start to expand our geometric knowledge by applying logic to the geometric building blocks. We'll make a smooth transition as some fundamental principles of algebra take on new life when expressed in the context of geometry.

Properties of Equality

All things being equal, in mathematics the word "equal" means "the same as." To be precise, the equal sign = means that the expression on the left of the equal sign and the expression on the right represent the same number. So equality is specifically about *numbers*—numbers that may be expressed differently but are in fact the same.

Some examples:

- $12 - 5 = 7$
- $\frac{372+372+372+372}{4} = 300 + 70 + 2$
- $1.5(40 + 60) = 150$

Basic properties of equality are quite simple and you are probably familiar with them already. They are listed here in formal language and then translated to common sense terms.

Properties of Equality

For all real numbers a, b, and c:

- **Reflexive Property**: $a = a$

That is, any number is *equal to* itself, or *the same as* itself.

Example: $25 = 25$

- **Symmetric Property**: If $a = b$ then $b = a$.

You can read an equality left to right, or right to left.

Example: If $8a = 32$ then $32 = 8a$

Example: If $m\angle P + m\angle Q = 180$, then $180 = m\angle P + m\angle Q$.

Sometimes it is more convenient to write $b = a$ than $a = b$. The symmetric property allows this.

- **Transitive Property**: If $a = b$ and $b = c$ then $a = c$.

Translation: If there is a "chain" of linked equations, then the first number is equal to the last number. (You can prove that this applies to more than two equalities in the review questions.)

Example: If $a + 4 = 10$ and $10 = 6 + 4$, then $a + 4 = 6 + 4$.

As a reminder, here are some properties of equality that you used heavily when you learned to solve equations in algebra.

- **Substitution Property**: If $a = b$, then b can be put in place of a anywhere or everywhere.

Example: Given that $a = 9$ and that $a - c = 5$. Then $9 - c = 5$.

- **Addition Property of Equality**: If $a = b$, then $a + c = b + c$.

Translation: You can add the same number to both sides of an equation and retain the equivalency.

Example: If $m\angle A + 30 = 90$, then $m\angle A + 30 + -30 = 90 + -30$.

- **Multiplication Property of Equality**: If $a = b$, then $ac = bc$.

Translation: You can multiply the same number on both sides of an equation and retain the equivalency.

Example: If $3x = 18$, then $\frac{1}{3}(3x) = \frac{1}{3}(18)$.

Keep in mind that these are properties about *numbers*. As you go further into geometry, you can apply the properties of equality to anything that is a number: lengths of segments and angle measures, for example.

Properties of Congruence

Let's review the definitions of congruent segments and angles.

Congruent Segments: $\overline{MN} \cong \overline{PQ}$ if and only if $MN = PQ$.

Remember that, although $\overline{MN}$ and $\overline{PQ}$ are *segments*, MN and PQ are *lengths* of those segments, meaning that MN and PQ are *numbers*. The properties of equality apply to MN and PQ.

Congruent Angles: $\angle F \cong \angle G$ if and only if $m\angle F = m\angle G$

The comment above about segment lengths also applies to angle measures. The properties of equality apply to $m\angle F$ and $m\angle G$.

Any statement about congruent segments or congruent angles can be translated directly into a statement about numbers. This means that each property of equality has a corresponding property of congruent segments and a corresponding property of congruent angles.

Here are some of the basic properties of equality and the corresponding congruence properties.

Given that x, y, and z are real numbers.

Reflexive Property of Equality: $x = x$

Reflexive Property of Congruence of Segments: $\overline{MN} \cong \overline{MN}$

Reflexive Property of Congruence of Angles: $\angle P \cong \angle P$

Symmetric Property of Equality: If $x = y$, then $y = x$.

Symmetric Property of Congruence of Segments: If $\overline{MN} \cong \overline{PQ}$, then $\overline{PQ} \cong \overline{MN}$.

Symmetric Property of Congruence of Angles: If $\angle P \cong \angle Q$, then $\angle Q \cong \angle P$.

Transitive Property of Equality: If $x = y$ and $y = z$, then $x = z$

Transitive Property of Congruence of Segments

If $\overline{MN} \cong \overline{PQ}$ and $\overline{PQ} \cong \overline{ST}$ then $\overline{MN} \cong \overline{ST}$

Transitive Property of Congruence of Angles

If $\angle P \cong \angle Q$, and $\angle Q \cong \angle R$ then $\angle P \cong \angle R$.

Using Congruence Properties in Equations

When you solve equations in algebra you use properties of *equality*. You might not write out the logical justification for each step in your solution, but you know that there is an equality property that justifies that step.

Let's see how we can use the properties of *congruence* to justify statements in deductive reasoning. Abbreviated names of the properties can be used.

Example 1

Given points A, B, and C, with $AB = 8, BC = 17$, and $AC = 20$.

Are A, B, and C collinear?

$AB + BC = AB + BC$(reflexive).	Why do we want this?
	So that we can bring in the numbers that are AB and BC.
$AB + BC = 8 + 17$	Justification is substitution of 8 for AB and 17 for BC
$AB + BC = 25$	$8 + 17 = 25$; This is arithmetic.
	No justification is needed as long as the arithmetic is correct.
$25 \neq 20$	More arithmetic.
$AB + BC \neq AC$	Substituting $AB + BC$ for 25 and AC for 20.
$A, B,$ and C are not collinear.	Segment addition postulate.
	$A, B,$ and C are collinear if and only if $AB + BC = AC$.

Example 2

Given that $m\angle A + m\angle B = 100^\circ$ and $\angle B = 40^\circ$.

Prove that $\angle A$ is an acute angle.

$m\angle A + m\angle B = 100,\ m\angle B = 40.$	These are the given facts.
$m\angle A + 40 = 100.$	Substitute 40 for $m\angle B$ using the transitive property.
$m\angle A + 40 + (-40) = 100 + (-40).$	Addition property of equality; add -40 to both sides.
$m\angle A = 60.$	Arithmetic.
$60 < 90.$	More arithmetic.
$m\angle A < 90^\circ.$	Substitute $m\angle A$ for 60.
$\angle A$ is an acute angle.	Definition. An angle is acute if and only if its measure is betw

The deductive reasoning scheme in example 2 is called a **proof**. The final statement *must* be true if the given information is true.

Lesson Summary

We built on our previous knowledge of properties of equality to derive corresponding properties of congruence. This enabled us to test statements about congruence, and to create new properties and relationships about congruence. We had our first introduction, in informal terms, to logical *proof*.

Points to Consider

In the examples and review questions, terms like given, prove, and reason were used. In upcoming lessons we'll see how to identify the given facts, how to draw a diagram to represent a statement that we need to prove, and how to organize proofs more formally. As we move ahead we'll prove many important geometric relationships called theorems. We have now established the framework of logic that we'll use repeatedly in future work.

Review Questions

Given: x, y, and z are real numbers.

Use the given property or properties of equality to fill in the blank in each of the following questions.

1. Symmetric: If $x = 3$, then _____.
2. Reflexive: If $x + 2 = 9$, then _____.
3. Transitive: If $y = 12$ and $x = y$ then _____.
4. Symmetric: If $x + y = y + z$, then _____.
5. Reflexive: If $x + y = y + z$, then _____.
6. Substitution: If $x = y - 7$ and $x = z + 4$, then _____-_____.
7. Use the transitive property of equality to write a convincing logical argument (a proof) that the statement below is true.

If $a = b$ and $b = c$ and $c = d$ and $d = e$, then $a = e$.

Note that this chain could be extended with additional links.

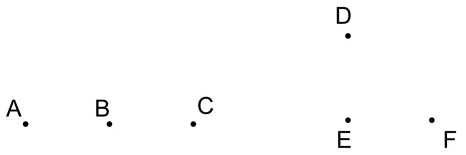
Let M be the relation "is the mother of." Let B be the relation "is the brother of."

8. Is M symmetric? Explain your answer.
9. Is B symmetric? Explain your answer.
10. Is M transitive? Explain your answer.
11. Is B transitive? Explain your answer.
12. Let w, x, y, and z be real numbers. Prove: If $w = y$ and $x = z$, then $w + x = y + z$.
13. The following statement is **not** true. "Let A, B, C, D, E, and F be points. If $AB = DE$ and $BC = EF$, then $AC = DF$." Draw a diagram with these points shown to provide a counterexample.

Review Answers

1. $3 = x$.

2. $x + 2 = 9$.
3. $x = 12$.
4. $y + z = x + y$.
5. $x + y = y + z$.
6. $z + 4 = y + 7$ (or $y - 7 = z + 4$).
7. If $a = b$ and $b = c$ then $a = c$ (transitive property). If $a = c$ and $c = d$ then $a = d$ (transitive property). If $a = d$ and $d = e$ then $a = e$ (transitive property).
8. No. If Maria is the mother of Juan, it does NOT follow that Juan is the mother of Maria!
9. Yes. For example, if Bill is Frank's brother, then Frank is Bill's brother.
10. No. If M were transitive, then "Maria is Fern's mother and Fern is Gina's mother" would lead to "Maria is Gina's mother." However, Maria would actually have to be Gina's **grandmother!**
11. Yes. If Bill is Frank's brother and Frank is Greg's brother, then Bill is Greg's brother. You might say the brother of my brother is (also) my brother.
12. $w = y$ and $x = z$ (given); $w + x = w + x$ (reflexive); $w + x = y + z$ (substitute y for w and z for x).
13. Below is an example:



A correct response is a diagram showing:

- $AB = DE$.
- $BC = EF$.
- $AC \neq DF$.

If A, B, and C are collinear and D, E, and F are not collinear then the conditions are satisfied.

2.5 Diagrams

Learning Objectives

- Provide the diagram that goes with a problem or proof.
- Interpret a given diagram.
- Recognize what can be assumed from a diagram and what can not be.
- Use standard marks for segments and angles in diagrams.

Introduction

Geometry is about objects such as points, lines, segments, rays, planes, and angles. If we are to solve problems about these objects, our work is made much easier if we can represent these objects in diagrams. In fact, for most of us, diagrams are absolutely essential for problem solving in geometry.

Basic Postulates—Another Look

Just as undefined terms are building blocks that other definitions are built on, postulates are the building blocks of logic. We're now ready to restate some of the basic postulates in slightly more formal terms, and to use diagrams.

Postulate 1 Through any two distinct points, there is exactly one line.

Comment: Any two points are collinear.

Postulate 2 There is exactly one plane that contains any three noncollinear points.

Comment: Sometimes this is expressed as: "Three noncollinear points determine a plane."

Postulate 3 If two points are in a plane, then the whole line through those two points is in the plane.

Postulate 4 If two distinct lines intersect, then the intersection is exactly one point.

Comments: Some lines intersect, some do not. If lines do intersect, it is in only one point, otherwise one or both "lines" would have to curve, which lines do not do.

Postulate 5 If two distinct planes intersect, then the intersection is exactly one line.

Comments: Some planes intersect, some do not. Think of a floor and a ceiling as models for planes that do ***not*** *intersect. If planes do intersect, it is in a line. Think of the edge of a box (a line) formed where two sides of the box (planes) meet.*

Postulate 6 **The Ruler Postulate**: The points on a line can be assigned real numbers, so that for any two points, one corresponds to 0 and the other corresponds to a nonzero real number.

Comments: This is how a number line and a ruler work. This also means we can measure any segment.

Postulate 7 **The Segment Addition Postulate**: Points P, Q, and R are collinear if and only if $PQ + QR = PR$.

Comment: If P, Q, and R are not collinear, then $PQ + QR > PR$. We saw examples of this fact in earlier sections of this chapter.

Postulate 8 **The Protractor Postulate**: If rays in a plane have a common endpoint, 0 can

be assigned to one ray and a number between 0 and 180 can be assigned to each of the other rays.

Comment: This means that any angle has a (degree) measure.

Postulate 9 **The Angle Addition Postulate**: Let P, Q, R, and S be points in a plane. S is in the interior of $\angle PQR$ if and only if $m\angle PQR + m\angle SQR = m\angle PQR$.

Comment: If an angle is made up of other angles, the measures of the component angles can be added to get the measure of the "big" angle.

Postulate 10 **The Midpoint Postulate**: Every line segment has exactly one midpoint.

*Comments: If M is a point on $\overline{AB}$ and $AM = MB$ there is not another point on $\overline{AB}$, let's say point N,with $AN = NB$. The midpoint of a segment is **unique**.*

Postulate 11 **The Angle Bisector Postulate**: Every angle has exactly one bisector.

*Comments: The bisector of an angle is a ray. If $\overrightarrow{BP}$ bisects , $\angle ABC$ there is not another ray that bisects the angle. The bisector (ray) of an angle is **unique**.*

Using Diagrams

Now we apply our definitions and postulates to a geometric figure. When measures are given on a figure, we can assume that the measurements on the figure are correct. We can also assume that:

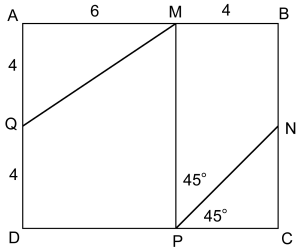
- Points that appear to be collinear *are* collinear.
- Lines, rays, or segments that appear to intersect *do* intersect.
- A ray that appears to be in the interior of an angle *is* in the interior of the angle.

We **cannot** assume the following from a diagram:

- That lines, segments, rays, or planes are parallel or perpendicular.
- That segments or angles are congruent.

These must be stated or indicated in the diagram.

The diagram below shows some segment and angle measures.



Example 1

A. *Is M the midpoint of $\overline{AB}$? Explain your answer.*

No. M is on $\overline{AB}$, but $AM \neq MB$.

B. *Is Q the midpoint of $\overline{AD}$? Explain your answer.*

Yes. Q is on $\overline{AD}$, and $AQ = QD$.

C. *Name an angle bisector and the angle that it bisects.*

$\overrightarrow{PN}$ bisects $\angle MPC$.

D. *Fill in the blank: $m\angle AMP = m\angle AMQ + m\angle$ _____.*

QMP

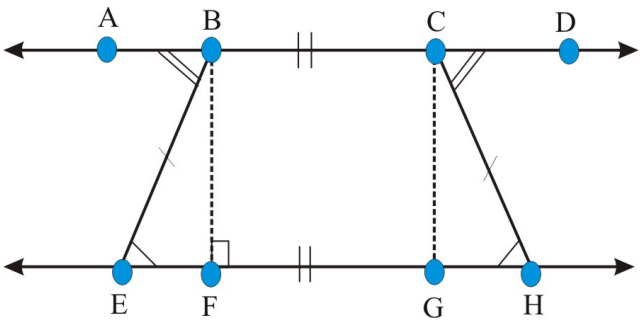
E. *Is $\overrightarrow{MQ}$ the bisector of $\angle AMP$? Explain your answer.*

No. If $\overrightarrow{MQ}$ bisected $\angle AMP$, then $m\angle AMQ$ would be $45°$. That would make $AQ = AM$, but $AQ \neq AM$.

Sometimes we use special marks in diagrams. Tick marks show congruent segments. Arc marks show congruent angles. Right angle marks show right angles and perpendicular lines and segments.

When these signs are used, the relationships they represent become part of the given information for a problem.

Example 2



Blue blobs are dots for points and single and double arc marks to show equal angles.

Based on the marks on the diagram, we know that:

- $BE = CH$ (single tick marks).
- $BC = FG$ (double tick marks).
- $m\angle BEF = m\angle CHG$ (single arc marks).
- $m\angle ABE = m\angle DCH$ (double arc marks).
- $\overline{BF} \perp \overline{EF}$.

Lesson Summary

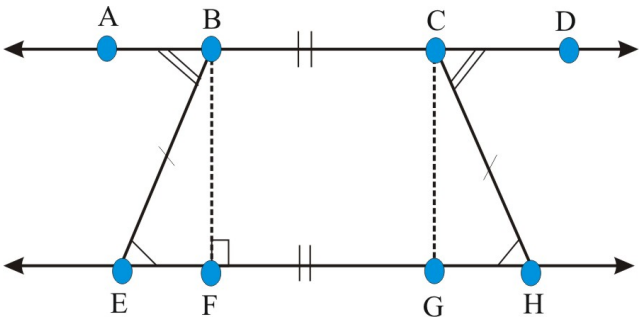
As we move forward toward more formal reasoning, we have reviewed the basic postulates and expressed them more formally. We saw that most geometric situations involve diagrams. In diagrams we can assume some facts, and we cannot assume others.

Points to Consider

In upcoming lessons you will organize your reasoning pattern into the two-column proof. This is a traditional pattern that still works very well today. It gives us a clear, direct format, and uses the basic rules of logic that we saw in earlier lessons. We will prove many important geometric relationships called theorems throughout the rest of this geometry course.

Review Questions

Use the diagram to answer questions 1-8.



1. Name a right angle.
2. Name two perpendicular lines (not segments).
3. Given that $EF = GH$, is $EG = FH$ true? Explain your answer completely.
4. Given that $\overleftrightarrow{BC} \| \overleftrightarrow{FG}$, is $BCGF$ a rectangle? Explain your answer informally. (Note: This is a new question. Do not assume that the given from a previous question is included in this question.)
5. Fill in the blanks: $m\angle ABF = m\angle ABE + m\angle___$. Why?
   $m\angle DCG = m\angle DCH + m\angle___$. Why?
6. Fill in the blanks:

$$AB + ___ = AC$$
$$___ + CD = BD$$

7. Given that $\angle EBF \cong \angle HCG$, prove $\angle ABF \cong \angle DCG$.
8. Given that $AB = CD$, prove: $AC = BD$.

What geometric objects does the real-world model suggest?

9. Model: two railroad tracks
10. Model: a floor and a ceiling
11. Model: two lines on a piece of graph paper
12. Model: referee's arms when signaling a touchdown
13. Model: capital letter L
14. Model: the spine of a book where the front and back covers join

Review Answers

1. $\angle BFG$
2. $\overleftrightarrow{BF}$ and $\overleftrightarrow{EH}$
3. Yes

$$\begin{aligned} EF &= GH \text{ Given} \\ EF + FG &= EF + FG \text{ Reflexive} \\ EF + FG &= GH + FG \text{ Substitution} \\ EF + FG &= EG, GH + FG = FH \text{ Segment addition postulate} \\ EG &= FH \text{ Substitution} \end{aligned}$$

4. Yes. It's given that $\overline{BC} \cong \overline{FG}$ (so $BC = FG$). Since $\overleftrightarrow{BC} \parallel \overleftrightarrow{FG}$ and $\overline{FG} \perp \overline{BF}$, then $\overline{BC} \perp \overline{BF}$ CG must be equal to BF, and this would make $BCGF$ a rectangle.
5. EBF.
   HCG.
6. BC.
   BC.
7. $\angle EBF \cong \angle HCG$ Given
   $\angle ABE \cong \angle DCH$ Given
   $m\angle ABE = m\angle ABE + m\angle EBF$ Angle Addition Postulate
   $m\angle DCG = m\angle DCH + m\angle HCG$ Angle Addition Postulate
   $m\angle ABF = m\angle DCH + m\angle HCG$ Substitution
   $m\angle ABF = m\angle DCG$ Substitution
   $\angle ABF \cong \angle DCG$ Definition of congruent angles
8. $AB = CD$ Given
   $AB + BC = AB + BC$ Reflexive
   $AB + BC = CD + BC$ Substitution
   $AB + BC = AC, CD + BC = BD$ Segment addition postulate
   $AC = BD$ Substitution
9. Parallel lines
10. Parallel planes

11. Parallel or perpendicular lines
12. Parallel lines or segments
13. Perpendicular segments
14. Intersecting planes

2.6 Two-Column Proof

Learning Objectives

- Draw a diagram to help set up a two-column proof.
- Identify the given information and statement to be proved in a two-column proof.
- Write a two-column proof.

Introduction

You have done some informal proofs in earlier sections. Now we raise the level of formality higher. In this section you will learn to write formal two-column proofs. You'll need to draw a diagram, identify the given and prove, and write a logical chain of statements. Each statement will have a **reason**, such as a definition, postulate, or previously proven theorem, that justifies it.

Given, Prove, and Diagram

Example 1

Write a two-column proof for the following:

If A, B, C, and D are points on a line, in the given order, and $AB = CD$, then $AC = BD$.

Comments: The if part of the statement contains the given. The then part is the section that you must prove. A diagram should show the given facts.

We start with the given, prove, and a diagram.

- Given: A, B, C, and D are points on a line in the order given. $AB = CD$.
- Prove: $AC = BD$.

4 points on the line; $AB = CD$

Now it's time to start with the given. Then we use logical reasoning to reach the statement we want to prove. Often (not always) the proof starts with the given information.

In the two column format, **Statements** go on the left side, and **Reasons** for each statement on the right. Reasons are generally definitions, postulates, and previously proved statements (called theorems).

Table 2.14:

Statement	Reason
1. $AB = CD$	Given
2. A, B, C, and D are collinear in that order	Given
3. $BC = BC$	Reflexive
4. $AC = AB + BC$ and $BD = CD + BC$	Segment Addition Postulate
5. $AB + BC = CD + BC$	Addition Property of Equality
6. $AC = BD$	Substitution

$AC = BD$ is what we were given to prove, and we've done it.

Example 2

Write a two-column proof of the following:

- Given: $\overrightarrow{BF}$ bisects $\angle ABC; \angle ABD \cong \angle CBE$
- Prove: $\angle DBF \cong \angle EBF$

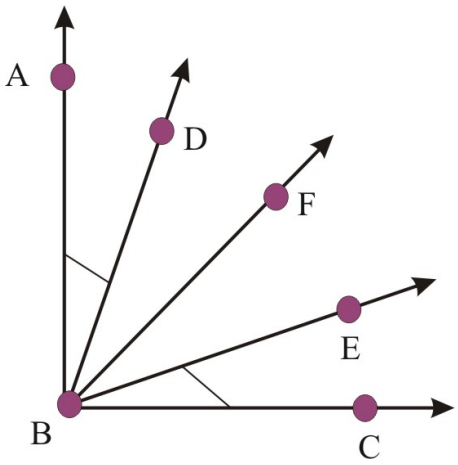


Table 2.15:

Statement	Reason
1. $\overrightarrow{BF}$ bisects $\angle ABC$	Given
2. $m\angle ABE = m\angle CBF$	Definition of Angle Bisector
3. $m\angle ABF = m\angle ABD + m\angle DBF$	Angle Addition Postulate
4. $m\angle CBF = m\angle CBE + m\angle EBF$	Angle Addition Postulate
5. $m\angle ABD + m\angle DBF = m\angle CBE + m\angle EBF$	Substitution
6. $\angle ABD \cong \angle CBE$	Given
7. $m\angle CBE + m\angle DBF = m\angle CBE + m\angle EBF$	Substitution
8. $m\angle DBF = m\angle EBF$	Subtract $m\angle CBE$ from both sides (*Reminder: Angle measures are all real numbers, so properties of equality apply.*)
9. $\angle DBF \cong \angle EBF$	Definition of congruent angles

This is the end of the proof. The last statement is the requirement made in the proof above. This is the signal that the proof is completed.

Lesson Summary

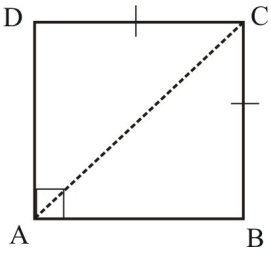
In this section you have seen two examples illustrating the format of two-column proofs. The format of two-column proofs is the same regardless of the specific details. Geometry originated many centuries ago using this same kind of deductive reasoning proof.

Points to Consider

You will see and write many two-column proofs in future lessons. The framework will stay the same, but the details will be different. Some of the statements that we prove are important enough that they are identified by their names. You will learn about many theorems and use them in proofs and problem solving.

Review Questions

Use the diagram below to answer questions 1-10.

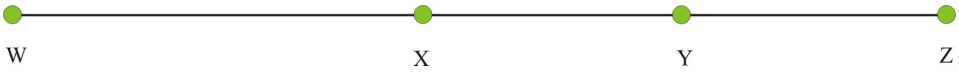


Which of the following can be assumed to be true from the diagram? Answer yes or no.

1. $\overline{AD} \cong \overline{BC}$
2. $\overline{AB} \cong \overline{CD}$
3. $\overline{CD} \cong \overline{BC}$
4. $\overline{AB} \parallel \overline{CD}$
5. $\overline{AB} \perp \overline{AD}$
6. $\overrightarrow{AC}$ bisects $\angle DAB$
7. $m\angle CAB = 45°$
8. $m\angle DCA = 45°$
9. $ABCD$ is a square
10. $ABCD$ is a rectangle

Use the diagram below to answer questions 11-14.

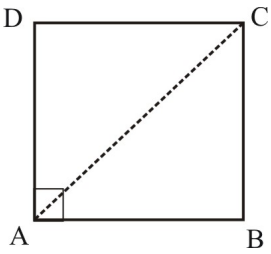
Given: X bisects $\overline{WZ}$, Y is the midpoint of $\overline{XZ}$, and $WZ = 12$.



11. How many segments have two of the given points as endpoints? What is the value of each of the following?
12. WY
13. XZ
14. ZW
15. Write a two-column proof for the following:

Given: $\overrightarrow{AC}$ bisects $\angle DAB$

Prove: $m\angle BAC = 45$



Review Answers

1. No
2. No
3. Yes
4. No
5. Yes
6. No
7. No
8. No
9. No
10. No
11. 6
12. 9
13. 6
14. 12

Table 2.16:

Statement	Reason
1. $\overrightarrow{AC}$ bisects $\angle DAB$	Given
2. $m\angle DAC = m\angle BAC$	Definition of angle bisector
3. $m\angle DAC + m\angle BAC = m\angle DAB$	Angle Addition Postulate
4. $\overline{AD} \perp \overline{AB}$	Given
5. $m\angle DAB = 90$	Definition of perpendicular segments
6. $m\angle BAC + m\angle BAC = 90$	Substitution
7. $2m\angle BAC = 90$	Algebra (Distributive Property)
8. $m\angle BAC = 45$	Multiplication Property of Equality

15.

2.7 Segment and Angle Congruence Theorems

Learning Objectives

- Understand basic congruence properties.
- Prove theorems about congruence.

Introduction

In an earlier lesson you reviewed many of the basic properties of equality. Properties of equality are about numbers. Angles and segments are not numbers, but their measures are numbers. Congruence of angles and segments is defined in terms of these numbers. To prove congruence properties, we immediately turn congruence statements into number statements, and use the properties of equality.

Equality Properties

Reminder: Here are some of the basic properties of equality. These are postulates—no proof needed. For each of these there is a corresponding property of congruence for segments, and one for angles. These are theorems—we'll prove them.

Properties of Equality for real numbers x, y, and z.

- Reflexive $x = x$
- Symmetric If $x = y$ then $y = x$
- Transitive If $x = y$ and $y = z$, then $x = z$

These properties are convertibles; we can convert them quickly and easily into congruence theorems.

Note that diagrams are needed to prove the congruence theorems. They are about angles and segments...ALL angles and segments, wherever and whenever they are found. No special setting (diagram) is needed.

Segment Congruence Properties

In this section we'll prove a series of segment theorems.

Reflexive: $\overline{AB} \cong \overline{AB}$

Table 2.17:

Statement	Reason
1. $AB = AB$	Reflexive Property of Equality
2. $\overline{AB} \cong \overline{AB}$	Definition of congruent segments

Symmetric: If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$

Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{CD} \cong \overline{AB}$

Table 2.18:

Statement	Reason
1. $\overline{AB} \cong \overline{CD}$	Given
2. $AB = CD$	Definition of congruent segments
3. $CD = AB$	Symmetric Property of Equality
4. $\overline{CD} \cong \overline{AB}$	Definition of congruent segments

Transitive: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$

Given: $\overline{AB} \cong \overline{CD}; \overline{CD} \cong \overline{EF}$

Prove: $\overline{AB} \cong \overline{EF}$

Table 2.19:

Statement	Reason
1. $\overline{AB} \cong \overline{CD}; \overline{CD} \cong \overline{EF}$	Given
2. $AB = CD; CD = EF$	Definition of congruent segments
3. $AB = EF$	Transitive property of equality
4. $\overline{AB} \cong \overline{EF}$	Definition of congruent segments

Angle Congruence Properties

Watch for proofs of the Angle Congruence Properties in the Lesson Exercises.

Reflexive: $\angle A \cong \angle A$

Symmetric: *If* $\angle A \cong \angle A$, then $\angle B \cong \angle A$

Transitive: *If* $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$

Lesson Summary

In this lesson we looked at old information in a new light. We saw that the properties of equality—reflexive, symmetric, transitive—convert easily into theorems about congruent segments and angles. In the next section we'll move ahead into new ground. There we'll get to use all the tools in our geometry toolbox to solve problems and to create new theorems.

Points to Consider

We are about to transition from introductory concepts that are necessary but not too "geometric" to the real heart of geometry. We needed a certain amount of foundation material before we could begin to get into more unfamiliar, challenging concepts and relationships. We have the definitions and postulates, and analogs of the equality properties, as the foundation. From here on out, we will be able to experience geometry on a richer and deeper level.

Review Questions

Prove the Segment Congruence Properties, in questions 1-3.

1. Reflexive: $\angle A \cong \angle A$.
2. Symmetric: *If* $\angle A \cong \angle B$, then $\angle B \cong \angle A$.
3. Transitive: *If* $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.
4. Is the following statement true? If it's not, give a counterexample. If it is, prove it.

If $\angle A \cong \angle B$ and $\angle C \cong \angle D$, then $m\angle A + m\angle C = m\angle B + m\angle D$.

5. Give a reason for each statement in the proof below.

If A, B, C, and D are collinear, and $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

Given: A, B, C, and D are collinear, and $\overline{AB} \cong \overline{CD}$.

Prove: $\overline{AC} \cong \overline{BD}$.

6. Is the following statement true? Explain your answer. (A formal two-column proof is not required.)

Let P, Q, R, S, and T be points in a single plane. If $\overrightarrow{QS}$ is in the interior of $\angle PQR$, and $\overrightarrow{QT}$ is in the interior of $\angle PQS$, then $\overrightarrow{QT}$ is in the interior of $\angle PQR$.

Note that this is a bit like a Transitive Property for a ray in the interior of an angle.

Review Answers

Table 2.20:

Statement	Reason
A. $m\angle A = m\angle A$	Reflexive Property of Equality
B. $\angle A \cong \angle A$	Definition of Congruent Angles

1.
2. Given: $\angle A \cong \angle B$
   Prove:
   $\angle B \cong \angle A$

Table 2.21:

Statement	Reason
A. $\angle A \cong \angle B$	Given
B. $m\angle A = m\angle B$	Definition of Congruent Angles
C. $m\angle B = m\angle A$	Symmetric Property of Equality
D. $\angle B = \angle A$	Definition of Congruent Angles

3. Given: $\angle A \cong \angle B$ and $\angle B \cong \angle C$
   Prove:
   $\angle A \cong \angle C$

Table 2.22:

Statement	Reason
A. $\angle A \cong \angle B$ and $\angle B \cong \angle C$	Given
B. $m\angle A = m\angle B$ and $m\angle B = m\angle C$	Definition of Congruent Angles
C. $m\angle A = m\angle C$	Transitive Property of Equality
D. $\angle A \cong \angle C$	Definition of Congruent Angles

4. Yes Given:
   $\angle A \cong \angle B$ and $\angle C \cong \angle D$
   Prove:
   $m\angle A + m\angle C = m\angle B + m\angle D$

Table 2.23:

Statement	Reason
A. $\angle A \cong \angle B$ and $\angle C \cong \angle D$	Given
B. $m\angle A = m\angle B, m\angle C = m\angle D$	Definition of Congruent Angles
C. $m\angle A + m\angle C = m\angle B + m\angle C$	Addition Property of Equality

Table 2.23: (continued)

Statement	Reason
D. $m\angle A + m\angle C = m\angle B + m\angle D$	Substitution

Table 2.24:

Statement	Reason
$A, B, C,$ *and* D are collinear	A._____ Given
$\overline{AB} \cong \overline{CD}$	B._____ Given
$AB = CD$	C._____ Definition of Congruent Segments
$AB + BC = CD + BC$	D._____ Addition Property of Equality
$AB + BC = BC + CD$	E._____ Commutative Property of Equality
$AB + BC = AC$	F._____ Definition of Collinear Points
$BC + CD = BD$	G._____ Definition of Collinear Points
$AC = BD$	H._____ Substitution Property of Equality
$\overline{AC} \cong \overline{BD}$	I._____ Definition of Congruent Segments

5.

6. True. Since $\overrightarrow{QS}$ is in the interior of $\angle PQR$, $m\angle PQS + m\angle SQR = m\angle PQR$ Since $\overrightarrow{QT}$ is in the interior of $\angle PQS$, then $m\angle PQT + m\angle TQS = m\angle PQS$. So

$$\begin{aligned}(m\angle PQT + m\angle TQS) + m\angle SQR &= m\angle PQR \\ m\angle PQT + (m\angle TQS + m\angle SQR) &= m\angle PQR \\ m\angle PQT + m\angle TQR &= m\angle PQR\end{aligned}$$

$\overrightarrow{QT}$ is in the interior of $\angle PQR$ by the angle addition property.

2.8 Proofs About Angle Pairs

Learning Objectives

- State theorems about special pairs of angles.
- Understand proofs of the theorems about special pairs of angles.
- Apply the theorems in problem solving.

Introduction

So far most of the things we have proven have been fairly straightforward. Now we have the tools to prove some more in-depth theorems that may not be so obvious. We'll start with theorems about special pairs of angles. They are:

- right angles
- supplementary angles
- complementary angles
- vertical angles

Right Angle Theorem

If two angles are right angles, then the angles are congruent.

Given: $\angle A$ and $\angle B$ are right angles.

Prove:$\angle A \cong \angle B$

Table 2.25:

Statement	Reason
1. $\angle A$ and $\angle B$ are right angles.	Given
2. $m\angle A = 90,\ m\angle B = 90$	Definition of right angle
3. $m\angle A = m\angle B$	Substitution
4. $\angle A \cong \angle B$	Definition of congruent angles

Supplements of the Same Angle Theorem

If two angles are both supplementary to the same angle (or congruent angles) then the angles are congruent.

Comments: As an example, we know that if $\angle A$ is supplementary to a 30° angle, then $m\angle A = 150^\circ$. If $\angle B$ is also supplementary to a 30° angle, then $m\angle B = 150^\circ$ too, and $m\angle A = m\angle B$.

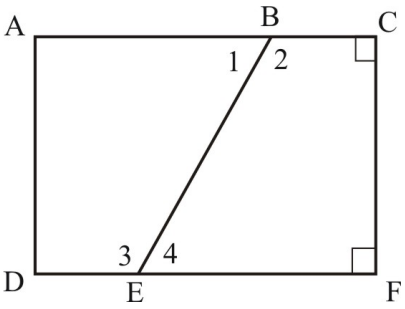
Given: $\angle A$ and $\angle B$ are supplementary angles. $\angle A$ and $\angle C$ are supplementary angles.

Prove: $\angle B \cong \angle C$

Table 2.26:

Statement	Reason
1. $\angle A$ and $\angle B$ are supplementary angles.	Given
2. $\angle A$ and $\angle C$ are supplementary angles.	Given
3. $m\angle A+m\angle B = 180$, $m\angle A+m\angle C = 180$	Definition of Supplementary Angles
4. $m\angle A + m\angle B = m\angle A + m\angle C$	Substitution
5. $m\angle B = m\angle C$	Addition Property of Equality
6. $\angle B \cong \angle C$	Definition of Congruent Angles

Example 1



Given that $\angle 1 \cong \angle 4$, what other angles must be congruent?

Answer:

$\angle C \cong \angle F$ by the Right Angle Theorem, because they're both right angles.

$\angle 2 \cong \angle 3$ by the Supplements of the Same Angle Theorem and the Linear Pair Postulate: $\angle 1$ and $\angle 2$ are a linear pair, which makes them supplementary. $\angle 3$ and $\angle 4$ are also a linear pair, which makes them supplementary too. Then by Supplements of the Same Angle Theorem, $\angle 2 \cong \angle 3$ because they're supplementary to congruent angles $\angle 1$ and $\angle 4$.

Complements of the Same Angle Theorem

If two angles are both complementary to the same angle (or congruent angles) then the angles are congruent.

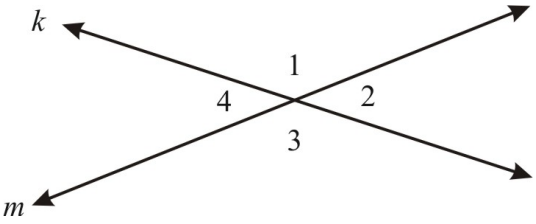
Comments: Only one word is different in this theorem compared to the Supplements of the Same Angle Theorem. Here we have angles that are *complementary*, rather than *supplementary*, to the same angle.

The proof of the Complements of the Same Angle Theorem is in the Lesson Exercises, and it is very similar to the proof above.

Vertical Angles Theorem

Vertical Angles Theorem: Vertical angles are congruent.

Vertical angles are common in geometry problems, and in real life wherever lines intersect: cables, fence lines, highways, roof beams, etc. A theorem about them will be useful. The vertical angle theorem is one of the world's briefest theorems. Its proof draws on the new theorems just proved earlier in this section.



Given: Lines k and m intersect.

Prove: $\angle 1 \cong \angle 3$, and $\angle 2 \cong \angle 4$

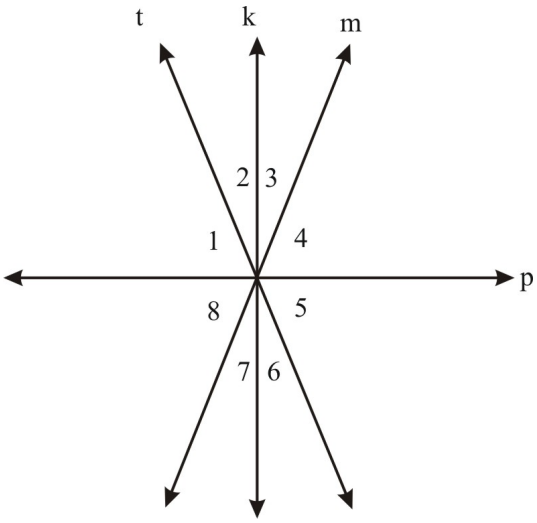
Table 2.27:

Statement	Reason
1. Lines k and m intersect.	Given
2. $\angle 1$ and $\angle 2$, $\angle 2$ and $\angle 3$ are linear pairs.	Definition of Linear Pairs
3. $\angle 1$ and $\angle 2$ are supplementary, and $\angle 2$ and $\angle 3$ are supplementary.	Linear Pair Postulate
4. $\angle 1 \cong \angle 3$	Supplements of the Same Angle Theorem

This shows that $\angle 1 \cong \angle 3$. The same proof can be used to show that $\angle 2 \cong \angle 4$.

Example 2

Given: $\angle 2 \cong \angle 3$, $k \perp p$



Each of the following pairs of angles are congruent. Give a reason.

$\angle 1$ and $\angle 5$ answer: Vertical Angles Theorem

$\angle 1$ and $\angle 4$ answer: Complements of Congruent Angles Theorem

$\angle 2$ and $\angle 6$ answer: Vertical Angles Theorem

$\angle 3$ and $\angle 7$ answer: Vertical Angles Theorem

$\angle 6$ and $\angle 7$ answer: Vertical Angles Theorem and Transitive Property

$\angle 3$ and $\angle 6$ answer: Vertical Angles Theorem and Transitive Property

$\angle 4$ and $\angle 5$ answer: Complements of Congruent Angles Theorem

Example 3

- *Given:* $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4$
- *Prove:* $\angle 1 \cong \angle 4$

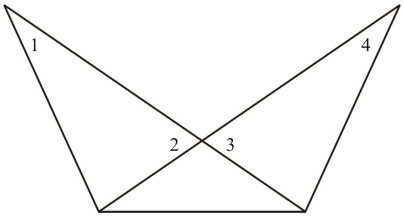


Table 2.28:

Statement	Reason
1.$\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4$	Given

Table 2.28: (continued)

Statement	Reason
2.$\angle 2 \cong \angle 3$	Vertical Angles Theorem
3.$\angle 1 \cong \angle 4$	Transitive Property of Congruence

Lesson Summary

In this lesson we proved theorems about angle pairs.

- Right angles are congruent.
- Supplements of the same, or congruent, angles are congruent.
- Complements of the same, or congruent, angles are congruent.
- Vertical angles are congruent.

We saw how these theorems can be applied in simple or complex figures.

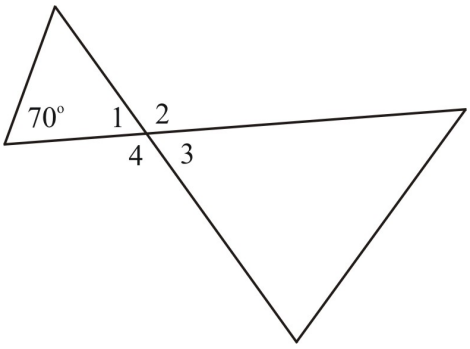
Points to Consider

No matter how complicated or abstract the model of a real-world situation may seem, in the final analysis it can often be expressed in terms of simple lines, segments, and angles. We'll be able to use the theorems of this section when we encounter complicated relationships in future figures.

Review Questions

Use the diagram to answer questions 1-3.

Given: $m\angle 1 = 60°$



$m\angle 1 = m\angle 3 = 60°$

Fill in the blanks.

1. $m\angle 2 =$ _____
2. $m\angle 3 =$ _____
3. $m\angle 4 =$ _____
4. Fill in the reasons in the following proof. Given: $\overline{AE} \perp \overline{EC}$ and $\overline{BE} \perp \overline{ED}$ Prove: $\angle 1 \cong \angle 3$

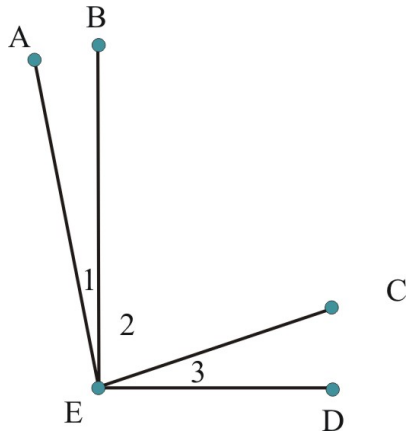
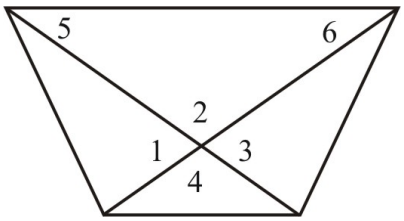


Table 2.29:

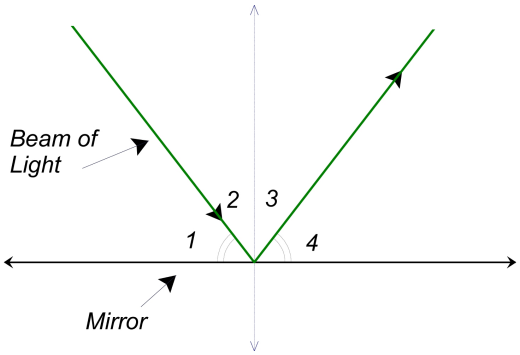
Statement	Reason
$\overline{AE} \perp \overline{EC}$ and $\overline{BE} \perp \overline{ED}$	a._____
$\angle AEC$ and $\angle BED$ are right angles	b._____
$m\angle AEC = m\angle 1 + m\angle 2$ and $m\angle BED = m\angle 2 + m\angle 3$	c._____
$m\angle AEC = m\angle BED = 90$	d._____
$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3 = 90°$	e._____
$\angle 1$ and $\angle 2$ are complementary , $\angle 2$ and $\angle 3$ are complementary	f._____
$\angle 1 \cong \angle 3$	g._____

5.
6. Which of the following statements **must** be true? Answer Yes or No.



(a) $\angle 1 \cong \angle 2$
(b) $\angle 2 \cong \angle 4$
(c) $\angle 5 \cong \angle 6$

7. The following diagram shows a ray of light that is reflected from a mirror. The dashed segment is perpendicular to the mirror. $\angle 2 \cong \angle 3$.



$\angle 1$ is called the angle of incidence; $\angle 4$ is called the angle of reflection. Explain how you know that the angle of incidence is congruent to the angle of reflection.

Review Answers

1. 120°
2. 60°
3. 120°
4. (a) Given
   (b) Definition of Perpendicular Segments
   (c) Angle Addition Postulate
   (d) Definition of Right Angle
   (e) Substitution (Transitive Property of Equality)
   (f) Definition of Complementary Angles
   (g) Complements of the Same Angle are Congruent
5. (a) No
   (b) Yes
   (c) No
6. $\angle 1$ and $\angle 2$ are complementary; $\angle 3$ and $\angle 4$ are complementary . $\angle 1 \cong \angle 4$ because they are complements of congruent angles 2 and 3.

Chapter 3

Parallel and Perpendicular Lines

3.1 Lines and Angles

Learning Objectives

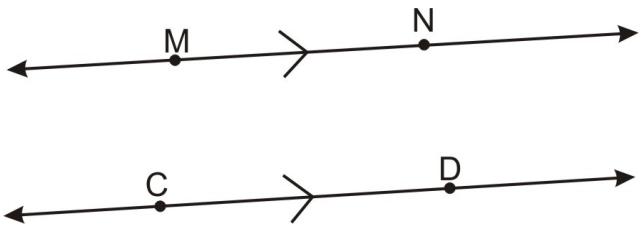
- Identify parallel lines, skew lines, and parallel planes.
- Know the statement of and use the Parallel Line Postulate.
- Know the statement of and use the Perpendicular Line Postulate.
- Identify angles made by transversals.

Introduction

In this chapter, you will explore the different types of relationships formed with parallel and perpendicular lines and planes. There are many different ways to understand the angles formed, and a number of tricks to find missing values and measurements. Though the concepts of parallel and perpendicular lines might seem complicated, they are present in our everyday life. Roads are often parallel or perpendicular, as are crucial elements in construction, such as the walls of a room. Remember that every theorem and postulate in this chapter can be useful in practical applications.

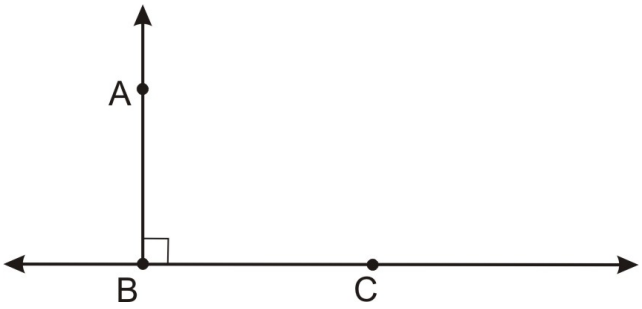
Parallel and Perpendicular Lines and Planes, and Skew Lines

Parallel lines are two or more lines that lie in the same **plane** and never intersect.



We use the symbol $\|$ for parallel, so to describe the figure above we would write $\overleftrightarrow{MN} \| \overleftrightarrow{CD}$. When we draw a pair of parallel lines, we use an arrow mark (>) to show that the lines are parallel. Just like with congruent segments, if there are two (or more) pairs of parallel lines, we use one arrow (>) for one pair and two (or more) arrows (>>) for the other pair.

Perpendicular lines intersect at a right angle. They form a 90° angle. This intersection is usually shown by a small square box in the 90° angle.



The symbol $\perp$ is used to show that two lines, segments, or rays are perpendicular. In the preceding picture, we could write $\overrightarrow{BA} \perp \overleftrightarrow{BC}$. (Note that $\overrightarrow{BA}$ is a ray while $\overleftrightarrow{BC}$ is a line.)

Note that although "parallel" and "perpendicular" are defined in terms of lines, the same definitions apply to rays and segments with the minor adjustment that two segments or rays are parallel (perpendicular) if the lines that contain the segments or rays are parallel (perpendicular).

Example 1

Which roads are parallel and which are perpendicular on the map below?

The first step is to remember the definitions or parallel and perpendicular lines. Parallel lines lie in the same plane but will never intersect. Perpendicular lines intersect at a right angle. All of the roads on this map lie in the same plane, and Rose Avenue and George Street never intersect. So, they are parallel roads. Henry Street intersects both Rose Avenue and George Street at a right angle, so it is perpendicular to those roads.

Planes can be parallel and perpendicular just like lines. Remember that a plane is a two-dimensional surface that extends infinitely in all directions. If planes are parallel, they will never intersect. If they are perpendicular, they will intersect at a right angle.

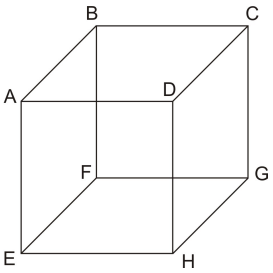
Figure 3.1

If you think about a table, the top of the table and the floor below it are usually in parallel planes.

The other of relationship you need to understand is **skew lines.** Skew lines are lines that are in different planes, and never intersect. Segments and rays can also be skew. In the cube

Figure 3.2

shown below segment $\overline{AB}$ and segment $\overline{CG}$ are skew. Can you name other pairs of skew segments in this diagram? (How many pairs of skew segments are there in all?)



Example 2

What is the relationship between the front and side of the building in the picture below?

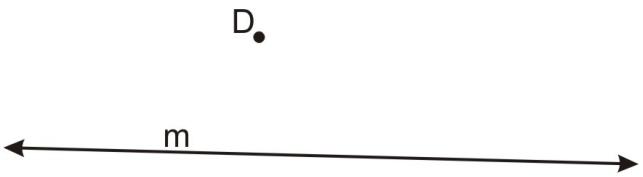
Figure 3.3

The planes that are represented by the front and side of the building above intersect at the corner. The corner appears to be a right angle (90°), so the planes are perpendicular.

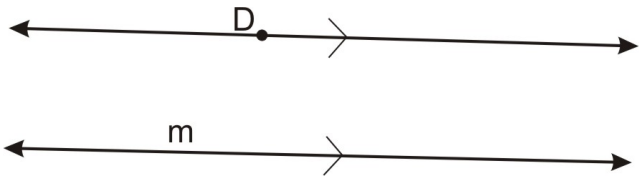
Parallel Line Postulate

As you already know, there are many different postulates and theorems relating to geometry. It is important for you to maintain a list of these ideas as they are presented throughout these chapters. One of the postulates that involves lines and planes is called the **Parallel Line Postulate.**

Parallel Postulate: Given a line and a point not on the line, there is exactly one line parallel to the given line that goes through that point. Look at the following diagram to see this illustrated.

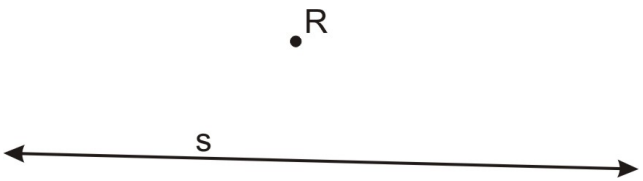


Line m in the diagram above is near point D. If you want to draw a line that is parallel to m that goes through point D there is only one option. Think of lines that are parallel to m as different latitude, like on a map. They can be drawn anywhere above and below line m, but only one will travel through point D.

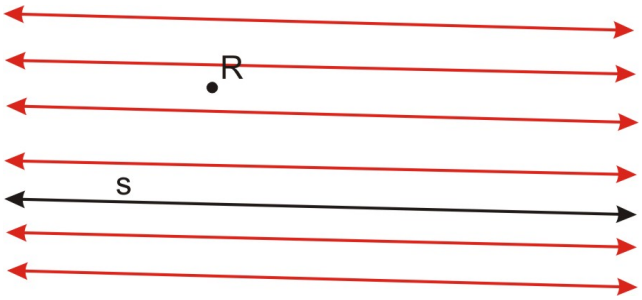


Example 3

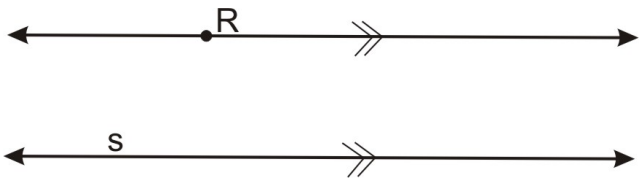
Draw a line through point R that is parallel to line s.



Remember that there are many different lines that could be parallel to line s.



There can only be one line parallel to s that travels through point R. This line is drawn below.

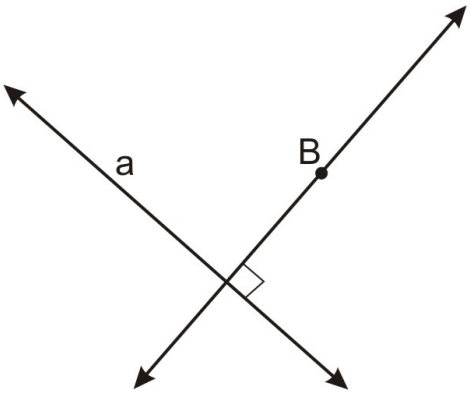


Perpendicular Line Postulate

Another postulate that is relevant to these scenarios is the **Perpendicular Line Postulate.**

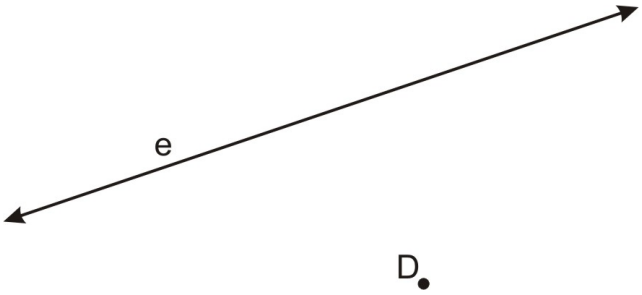
Perpendicular Line Postulate: Given a line and a point not on the line, there is exactly one line perpendicular to the given line that passes through the given point.

This postulate is very similar to the Parallel Line Postulate, but deals with perpendicular lines. Remember that perpendicular lines intersect at a right (90°) angle. So, as in the diagram below, there is only one line that can pass through point B while being perpendicular to line a.

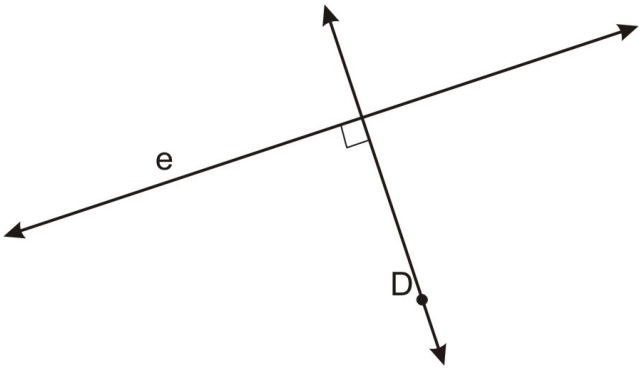


Example 4

Draw a line through point D that is perpendicular to line e.

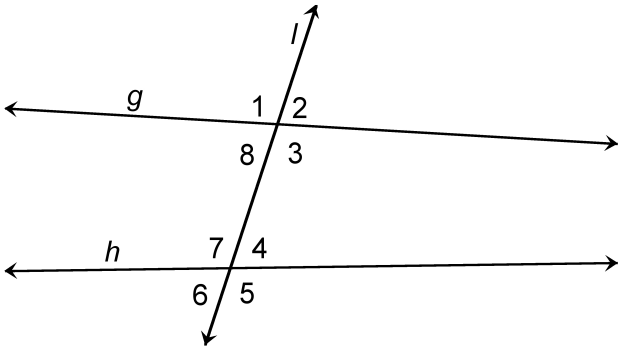


Remember that there can only be one line perpendicular to e that travels through point D. This line is drawn below.



Angles and Transversals

Many math problems involve the intersection of three or more lines. Examine the diagram below.



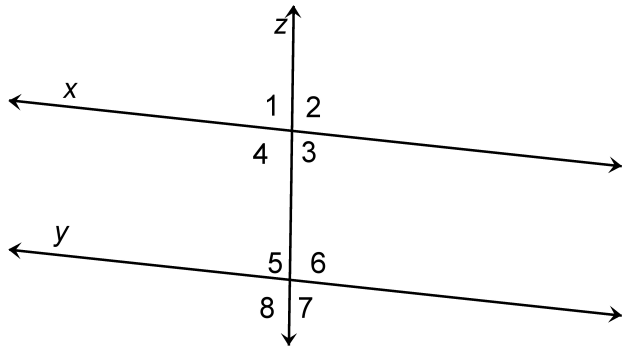
In the diagram, lines g and h are crossed by line l. We have quite a bit of vocabulary to describe this situation:

- Line l is called a **transversal** because it intersects two other lines (g and h). The intersection of line l with g and h forms eight angles as shown.

- The area between lines g and h is called the **interior** of the two lines. The area not between lines g and h is called the **exterior**.
- Angles $\angle 1$ and $\angle 2$ are called **adjacent angles** because they share a side and do not overlap. There are many pairs of adjacent angles in this diagram, including $\angle 2$ and $\angle 3, \angle 4$ and $\angle 7$, and $\angle 8$ and $\angle 1$.
- $\angle 1$ and $\angle 3$ are **vertical angles.** They are nonadjacent angles made by the intersection of two lines. Other pairs of vertical angles in this diagram are $\angle 2$ and $\angle 8, \angle 4$ and $\angle 6$, and $\angle 5$ and $\angle 7$.
- **Corresponding angles** are in the same position relative to both lines crossed by the transversal. $\angle 1$ is on the upper left corner of the intersection of lines g and l. $\angle 7$ is on the upper left corner of the intersection of lines h and l. So we say that $\angle 1$ and $\angle 7$ are corresponding angles.
- $\angle 3$ and $\angle 7$ are called **alternate interior angles.** They are in the interior region of the lines g and h and are on opposite sides of the transversal.
- Similarly, $\angle 2$ and $\angle 6$ are **alternate exterior angles** because they are on opposite sides of the transversal, and in the exterior of the region between g and h.
- Finally, $\angle 3$ and $\angle 4$ are **consecutive interior angles.** They are on the interior of the region between lines g and h and are next to each other. $\angle 8$ and $\angle 7$ are also consecutive interior angles.

Example 5

List all pairs of alternate angles in the diagram below.



There are two types of alternate angles—alternate interior angles and alternate exterior angles. As you need to list them both, begin with the alternate interior angles.

Alternate interior angles are on the interior region of the two lines crossed by the transversal, so that would include angles $3, 4, 5$, and 6. Alternate angles are on opposite sides of the transversal, z. So, the two pairs of alternate interior angles are $\angle 3$ & $\angle 5$, and $\angle 4$ and $\angle 6$.

Alternate exterior angles are on the exterior region of the two lines crossed by the transversal, so that would include angles $1, 2, 8$, and 7. Alternate angles are on opposite sides of the transversal, z. So, the two pairs of alternate exterior angles are $\angle 2$ & $\angle 8$, and $\angle 1$ and $\angle 7$.

Lesson Summary

In this lesson, we explored how to work with different types of lines, angles and planes. Specifically, we have learned:

- How to identify parallel lines, skew lines, and parallel planes.
- How to identify and use the Parallel Line Postulate.
- How to identify and use the Perpendicular Line Postulate.
- How to identify angles and transversals of many types.

These will help you solve many different types of problems. Always be on the lookout for new and interesting ways to examine the relationship between lines, planes, and angles.

Points to Consider

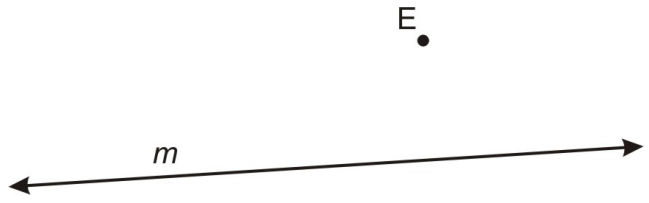
Parallel planes are two planes that do not intersect. Parallel lines must be in the same plane and they do not intersect. If more than two lines intersect at the same point and they are perpendicular, then they cannot be in same plane (e.g., the $x-$, $y-$, and $z-$ axes are all perpendicular). However, if just two lines are perpendicular, then there is a plane that contains those two lines.

As you move on in your studies of parallel and perpendicular lines you will usually be working in one plane. This is often assumed in geometry problems. However, you must be careful about instances where you are working with multiple planes in space. Generally in three-dimensional space parallel and perpendicular lines are more challenging to work with.

Review Questions

Solve each problem.

1. Imagine a line going through each branch of the tree below (see the red lines in the image). What term best describes the two branches with lines in the tree pictured below?
2. How many lines can be drawn through point E that will be parallel to line m?

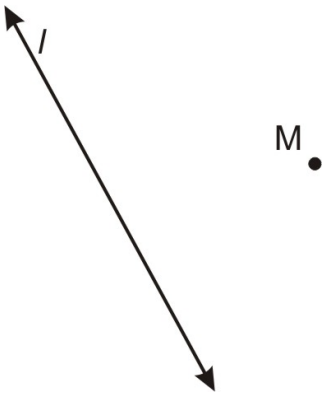


3. Which of the following best describes skew lines?

Figure 3.4

(a) They lie in the same plane but do not intersect.
(b) They intersect, but not at a right angle.
(c) They lie in different planes and never intersect.
(d) They intersect at a right angle.

4. Are the sides of the Transamerica Pyramid building in San Francisco parallel?
5. How many lines can be drawn through point M that will be perpendicular to line l?



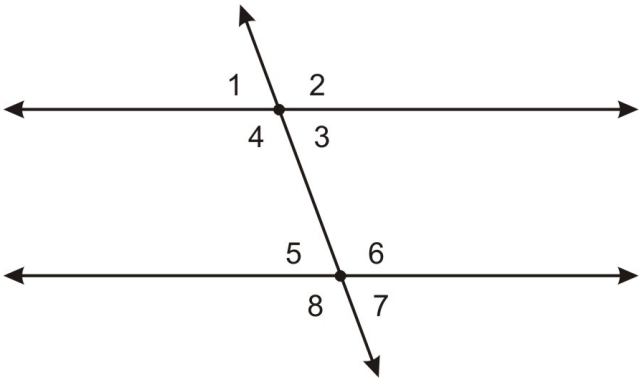
6. Which of the following best describes parallel lines?

(a) They lie in the same plane but do not intersect.
(b) They intersect, but not at a right angle.
(c) They lie in different planes and never intersect.
(d) They intersect at a right angle.

7. Draw five parallel lines in the plane. How many regions is the plane divided into by these five lines?

Figure 3.5

8. If you draw n parallel lines in the plane, how many regions will the plane be divided into?



The diagram below shows two lines cut by a transversal. Use this diagram to answer questions 9 and 10.

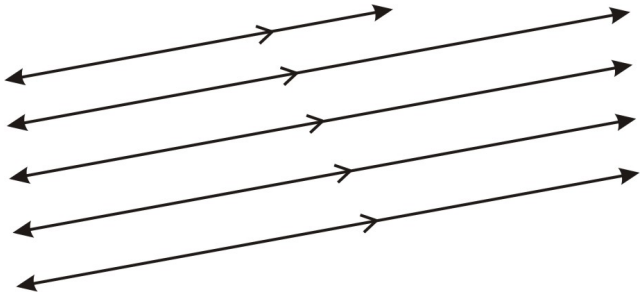
9. What term best describes the relationship between angles 1 and 5?
   (a) Consecutive interior
   (b) Alternate exterior
   (c) Alternate interior
   (d) Corresponding

10. What term best describes angles 7 and 8?

    (a) Linear pair
    (b) Alternate exterior
    (c) Alternate interior
    (d) Corresponding

Review Answers

1. Skew
2. One
3. C
4. No
5. One
6. A
7. Five parallel lines divide the plane into six regions



8. n parallel lines divide the plane into $n + 1$ regions
9. D
10. A

3.2 Parallel Lines and Transversals

Learning Objectives

- Identify angles formed by two parallel lines and a non-perpendicular transversal.
- Identify and use the Corresponding Angles Postulate.
- Identify and use the Alternate Interior Angles Theorem.
- Identify and use the Alternate Exterior Angles Theorem.
- Identify and use the Consecutive Interior Angles Theorem.

Introduction

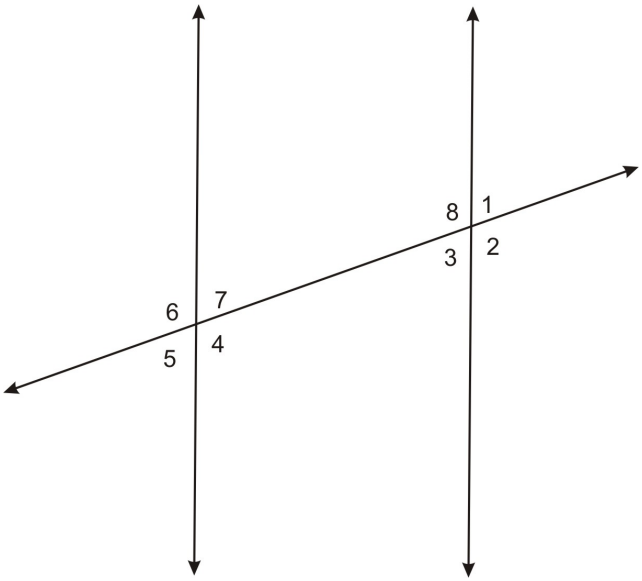
In the last lesson, you learned to identify different categories of angles formed by intersecting lines. This lesson builds on that knowledge by identifying the mathematical relationships inherent within these categories.

Parallel Lines with a Transversal—Review of Terms

As a quick review, it is helpful to practice identifying different categories of angles.

Example 1

In the diagram below, two vertical parallel lines are cut by a transversal.



Identify the pairs of corresponding angles, alternate interior angles, alternate exterior angles, and consecutive interior angles.

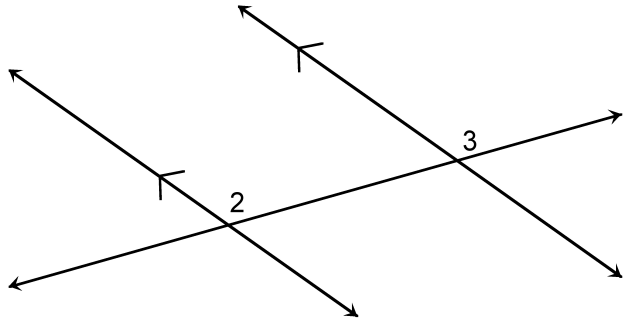
- Corresponding angles: Corresponding angles are formed on different lines, but in the same relative position to the transversal—in other words, they face the same direction. There are four pairs of corresponding angles in this diagram—$\angle 6$ and $\angle 8$, $\angle 7$ and $\angle 1$, $\angle 5$ and $\angle 3$, and $\angle 4$ and $\angle 2$.
- Alternate interior angles: These angles are on the interior of the lines crossed by the transversal and are on opposite sides of the transversal. There are two pairs of alternate interior angles in this diagram—$\angle 7$ and $\angle 3$, and $\angle 8$ and $\angle 4$.
- Alternate exterior angles: These are on the exterior of the lines crossed by the transversal and are on opposite sides of the transversal. There are two pairs of alternate exterior angles in this diagram—$\angle 1$ and $\angle 5$, and $\angle 2$ and $\angle 6$.

- Consecutive interior angles: Consecutive interior angles are in the interior region of the lines crossed by the transversal, and are on the same side of the transversal. There are two pairs of consecutive interior angles in this diagram—$\angle 7$ and $\angle 8$ and $\angle 3$ and $\angle 4$.

Corresponding Angles Postulate

By now you have had lots of practice and should be able to easily identify relationships between angles.

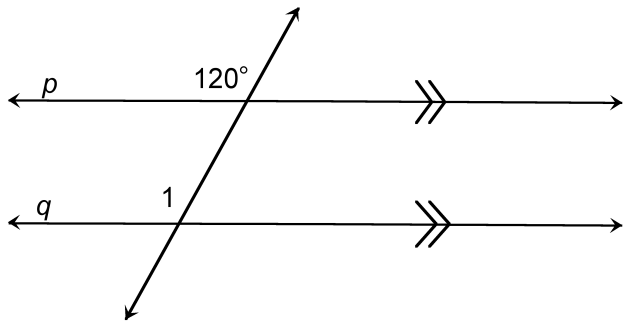
Corresponding Angles Postulate: If the lines crossed by a transversal are parallel, then corresponding angles will be congruent. Examine the following diagram.



You already know that $\angle 2$ and $\angle 3$ are corresponding angles because they are formed by two lines crossed by a transversal and have the same relative placement next to the transversal. The Corresponding Angles postulate says that *because the lines are parallel to each other,* the corresponding *angles will be congruent.*

Example 2

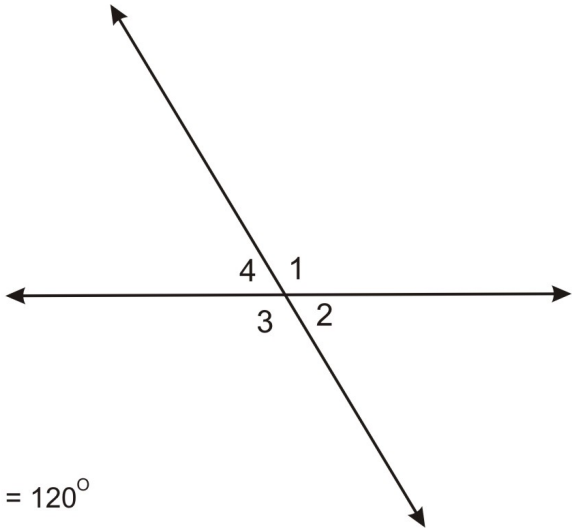
In the diagram below, lines p and q are parallel. What is the measure of $\angle 1$*?*



Because lines p and q are parallel, the 120° angle and $\angle 1$ are corresponding angles, we know by the Corresponding Angles Postulate that they are congruent. Therefore, $m\angle 1 = 120°$.

Alternate Interior Angles Theorem

Now that you know the Corresponding Angles Postulate, you can use it to derive the relationships between all other angles formed when two lines are crossed by a transversal. Examine the angles formed below.



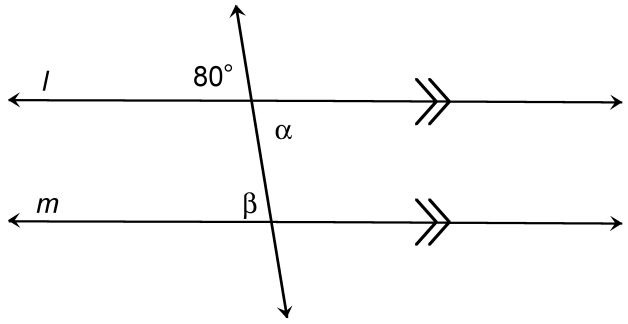
If you know that the measure of $\angle 1$ is 120°, you can find the measurement of all the other angles. For example, $\angle 1$ and $\angle 2$ must be **supplementary** (sum to 180°) because together they are a linear pair (we are using the Linear Pair Postulate here). So, to find $m\angle 2$, subtract 120° from 180°.

$$m\angle 2 = 180^\circ - 120^\circ$$
$$m\angle 2 = 60^\circ$$

So, $m\angle 2 = 60^\circ$. Knowing that $\angle 2$ and $\angle 3$ are also supplementary means that $m\angle 3 = 120^\circ$, since $120 + 60 = 180$. If $m\angle 3 = 120^\circ$, then $m\angle 4$ must be 60°, because $\angle 3$ and $\angle 4$ are also supplementary. Notice that $\angle 1 \cong \angle 3$ (they both measure 120°) and $\angle 2 \cong \angle 4$ (both measure 60°). These angles are called **vertical angles**. Vertical angles are on opposite sides of intersecting lines, and will always be congruent by the Vertical Angles Theorem, which we proved in an earlier chapter. Using this information, you can now deduce the relationship between alternate interior angles.

Example 3

Lines l and m in the diagram below are parallel. What are the measures of angles α and β?



In this problem, you need to find the angle measures of two alternate interior angles given an exterior angle. Use what you know. There is one angle that measures 80°. Angle β corresponds to the 80° angle. So by the Corresponding Angles Postulate, $m\angle\beta = 80°$.

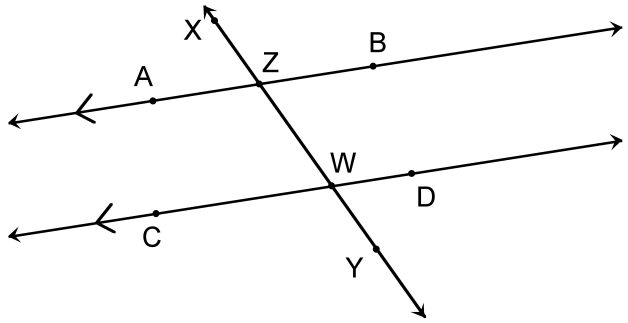
Now, because $\angle\alpha$ is made by the same intersecting lines and is opposite the 80° angle, these two angles are vertical angles. Since you already learned that vertical angles are congruent, we conclude $m\angle\alpha = 80°$. Finally, compare angles α and β. They both measure 80°, so they are congruent. This will be true any time two parallel lines are cut by a transversal.

We have shown that alternate interior angles are congruent in this example. Now we need to show that it is always true for any angles.

Alternate Interior Angles Theorem

Alternate interior angles formed by two parallel lines and a transversal will always be congruent.

- Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines crossed by transversal $\overleftrightarrow{XY}$
- Prove that Alternate Interior Angles are congruent



Note: It is sufficient to prove that one pair of alternate interior angles are congruent. Let's focus on proving $\angle DWZ \cong \angle WZA$.

Table 3.1:

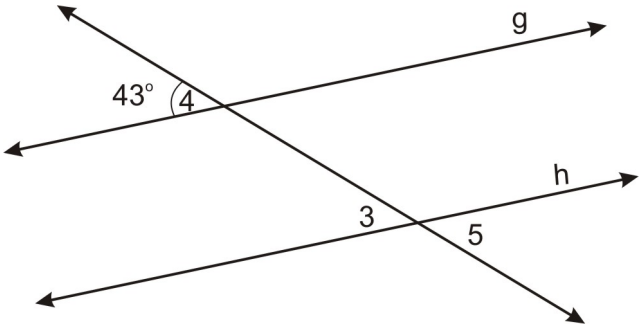
Statement	Reason
1. $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$	1.Given
2. $\angle DWZ \cong \angle BZX$	2. Corresponding Angles Postulate
3. $\angle BZX \cong \angle WZA$	3. Vertical Angles Theorem
4. $\angle DWZ \cong \angle WZA$	4. Transitive property of congruence

Alternate Exterior Angles Theorem

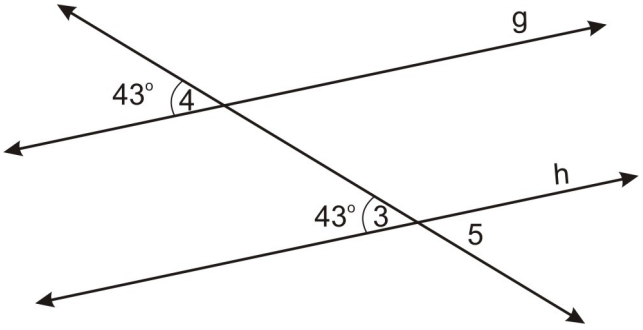
Now you know that pairs of corresponding, vertical, and alternate interior angles are congruent. We will use logic to show that Alternate Exterior Angles are congruent—when two parallel lines are crossed by a transversal, of course.

Example 4

Lines g and h in the diagram below are parallel. If $m\angle 4 = 43°$, what is the measure of $\angle 5$?



You know from the problem that $m\angle 4 = 43°$. That means that $\angle 4's$ corresponding angle, which is $\angle 3$, will measure 43° as well.



The corresponding angle you just filled in is also vertical to $\angle 5$. Since vertical angles are congruent, you can conclude $m\angle 5 = 43°$.

This example is very similar to the proof of the alternate exterior angles Theorem. Here we

write out the theorem in whole:

Alternate Exterior Angles Theorem

If two parallel lines are crossed by a transversal, then alternate exterior angles are congruent.

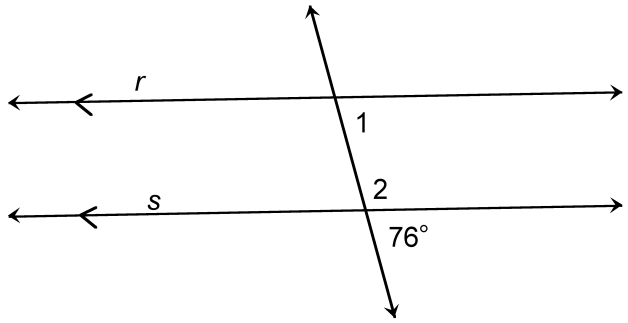
We omit the proof here, but note that you can prove alternate exterior angles are congruent by following the method of example 4, but not using any particular measures for the angles.

Consecutive Interior Angles Theorem

The last category of angles to explore in this lesson is consecutive interior angles. They fall on the interior of the parallel lines and are on the same side of the transversal. Use your knowledge of corresponding angles to identify their mathematical relationship.

Example 5

Lines r and s in the diagram below are parallel. If the angle corresponding to $\angle 1$ *measures* 76°, *what is* $m\angle 2$*?*



This process should now seem familiar. The given 76° angle is adjacent to $\angle 2$ and they form a linear pair. Therefore, the angles are supplementary. So, to find $m\angle 2$, subtract 76° from 180°.

$$m\angle 2 = 180 - 76$$
$$m\angle 2 = 104^\circ$$

This example shows that if two parallel lines are cut by a transversal, the consecutive interior angles are supplementary; they sum to 180°. This is called the Consecutive Interior Angles Theorem. We restate it here for clarity.

Consecutive Interior Angles Theorem

If two parallel lines are crossed by a transversal, then consecutive interior angles are supplementary.

Proof: You will prove this as part of your exercises.

Multimedia Link Now that you know all these theorems about parallel lines and transversals, it is time to practice. In the following game you use apply what you have learned to name and describe angles formed by a transversal. Interactive Angles Game.

Lesson Summary

In this lesson, we explored how to work with different angles created by two parallel lines and a transversal. Specifically, we have learned:

- How to identify angles formed by two parallel lines and a non-perpendicular transversal.
- How to identify and use the Corresponding Angles Postulate.
- How to identify and use the Alternate Interior Angles Theorem.
- How to identify and use the Alternate Exterior Angles Theorem.
- How to identify and use the Consecutive Interior Angles Theorem.

These will help you solve many different types of problems. Always be on the lookout for new and interesting ways to analyze lines and angles in mathematical situations.

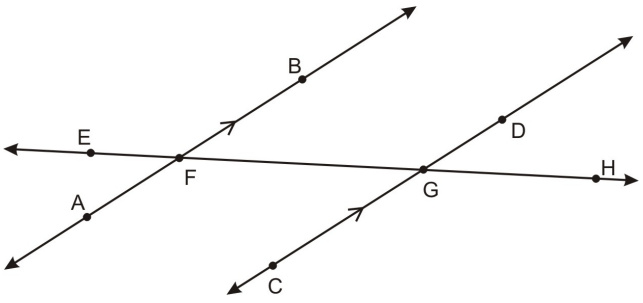
Points To Consider

You used logic to work through a number of different scenarios in this lesson. Always apply logic to mathematical situations to make sure that they are reasonable. Even if it doesn't help you solve the problem, it will help you notice careless errors or other mistakes.

Review Questions

Solve each problem.

Use the diagram below for Questions 1-4. In the diagram, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.



1. What term best describes the relationship between $\angle AFG$ and $\angle CGH$?

(a) alternate exterior angles
(b) consecutive interior angles
(c) corresponding angles
(d) alternate interior angles

2. What term best describes the mathematical relationship between $\angle BFG$ and $\angle DGF$?

(a) congruent
(b) supplementary
(c) complementary
(d) no relationship

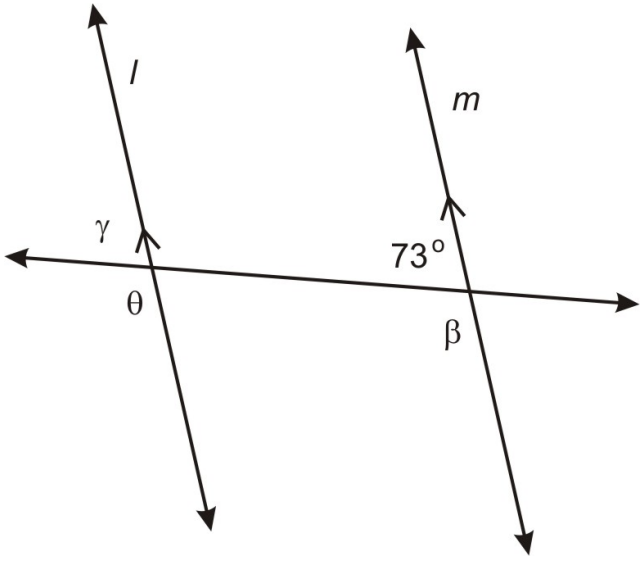
3. What term best describes the relationship between $\angle FGD$ and $\angle AFG$?

(a) alternate exterior angles
(b) consecutive interior angles
(c) complementary
(d) alternate interior angles

4. What term best describes the mathematical relationship between $\angle AFE$ and $\angle CGH$?

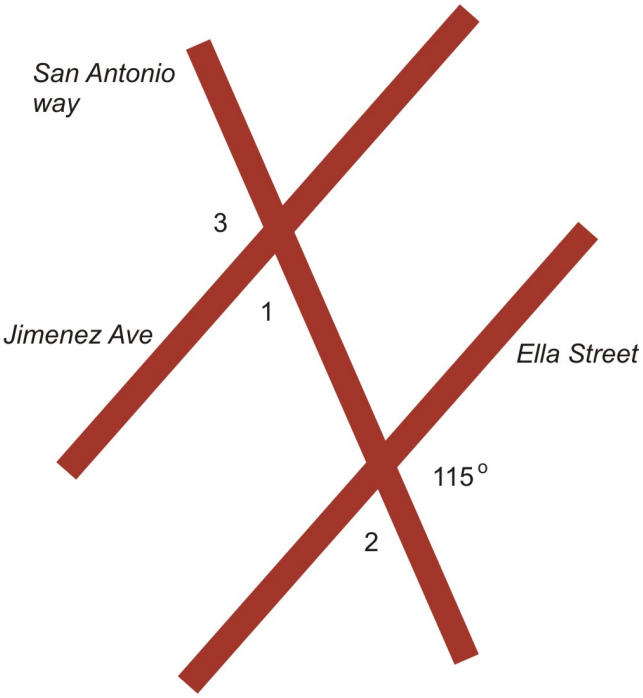
(a) congruent
(b) supplementary
(c) complementary
(d) no relationship

Use the diagram below for questions 5-7. In the diagram, lines l and m are parallel γ, β, θ represent the measures of the angles.



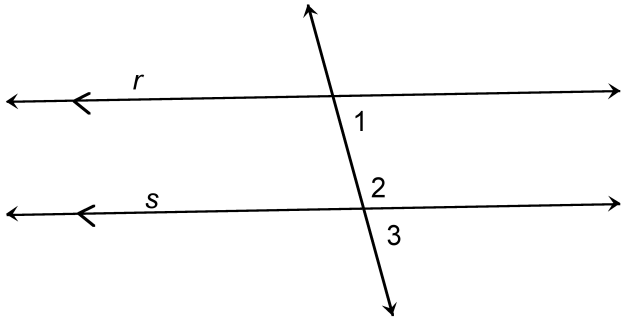
5. What is γ?
6. What is β?
7. What is θ?

The map below shows some of the streets in Ahmed's town.



Jimenez Ave and Ella Street are parallel. Use this map to answer questions 8-10.

8. What is the measure of angle 1?
9. What is the measure of angle 2?
10. What is the measure of angle 3?
11. Prove the Consecutive Interior Angle Theorem. Given $r||s$, prove $\angle 1$ and $\angle 2$ are supplementary.



Review Answers

1. c
2. a
3. d

4. b
5. 73°
6. 107°
7. 107°
8. 65°
9. 65°
10. 115°
11. Proof of Consecutive Interior Angle Theorem. Given $r||s$, prove $\angle 1$ and $\angle 2$ are supplementary.

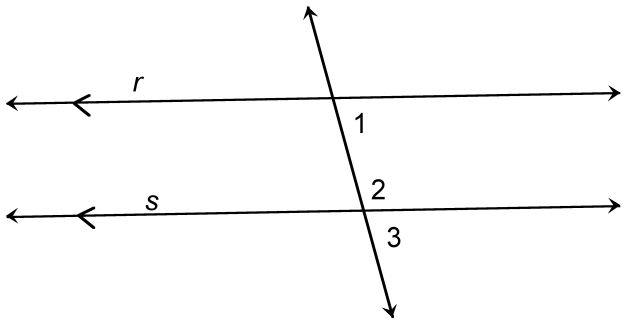


Table 3.2:

Statement	Reason
1. $r\|\|s$	1. Given
2. $\angle 1 \cong \angle 3$	2. Corresponding Angles Postulate
3. $\angle 2$ and $\angle 3$ are supplementary	3. Linear Pair Postulate
4. $m\angle 2 + m\angle 3 = 180°$	4. Definition of supplementary angles
5. $m\angle 2 + m\angle 1 = 180°$	5. Substitution ($\angle 1 \cong \angle 3$)
6. $\angle 2$ and $\angle 1$ are supplementary	6. Definition of supplementary angles

3.3 Proving Lines Parallel

Learning Objectives

- Identify and use the Converse of the Corresponding Angles Postulate.
- Identify and use the Converse of Alternate Interior Angles Theorem.
- Identify and use the Converse of Alternate Exterior Angles Theorem.
- Identify and use the Converse of Consecutive Interior Angles Theorem.
- Identify and use the Parallel Lines Property.

Introduction

If two angles are vertical angles, then they are congruent. You learned this as the Vertical Angles Theorem. Can you reverse this statement? Can you swap the "if" and "then" parts and will the statement still be true?

The **converse** of a logical statement is made by reversing the hypothesis and the conclusion in an if-then statement. With the Vertical Angles Theorem, the converse is "If two angles are congruent then they are vertical angles." Is that a true statement? In this case, no. The converse of the Vertical Angles Theorem is NOT true. There are many examples of congruent angles that are not vertical angles—for example the corners of a square.

Sometimes the converse of an if-then statement will also be true. Can you think of an example of a statement in which the converse is true? This lesson explores converses to the postulates and theorems about parallel lines and transversals.

Corresponding Angles Converse

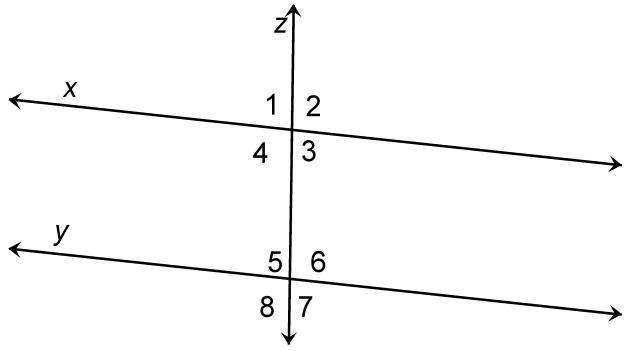
Let's apply the concept of a converse to the Corresponding Angles Postulate. Previously you learned that "if two parallel lines are cut by a transversal, the corresponding angles will be congruent." The converse of this statement is "if corresponding angles are congruent when two lines are cut by a transversal, then the two lines crossed by the transversal are parallel." This converse is **true**, and it is a postulate.

Converse of Corresponding Angles Postulate

If corresponding angles are congruent when two lines are crossed by a transversal, then the two lines crossed by the transversal are parallel.

Example 1

Suppose we know that $m\angle 8 = 110^\circ$ *and* $m\angle 4 = 110^\circ$. What can we conclude about lines x and y?



Notice that $\angle 8$ and $\angle 4$ are corresponding angles. Since $\angle 8 \cong \angle 4$, we can apply the Converse

of the Corresponding Angles Postulate and conclude that $x \| y$.

You can also use converse statements in combination with more complex logical reasoning to prove whether lines are parallel in real life contexts. The following example shows a use of the contrapositive of the Corresponding Angles Postulate.

Example 2

The three lines in the figure below represent metal bars and a cable supporting a water tower.

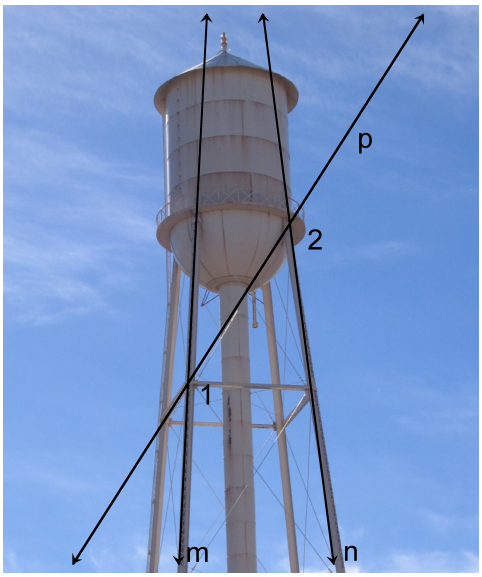


Figure 3.6

$m\angle 2 = 135^\circ$ and $m\angle 1 = 150^\circ$. Are the lines m and n parallel?

To find out whether lines m and n are parallel, you must identify the corresponding angles and see if they are congruent. In this diagram, $\angle 1$ and $\angle 2$ are corresponding angles because they are formed by the transversal and the two lines crossed by the transversal and they are in the same relative place.

The problem states that $m\angle 1 = 150^\circ$ and $m\angle 2 = 135^\circ$. Thus, they are not congruent. If those two angles are not congruent, the lines are not parallel. In this scenario, the lines m and n (and thus the support bars they represent) are NOT parallel.

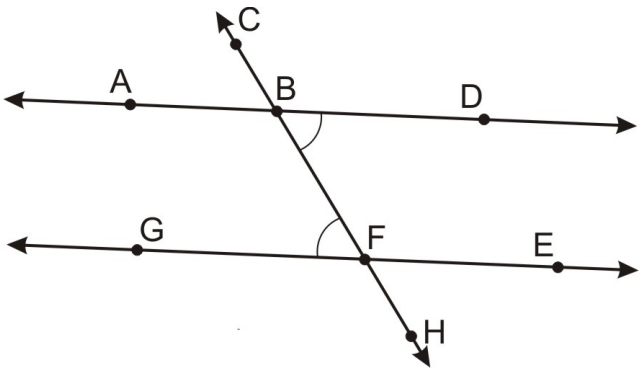
Note that just because two lines may look parallel in the picture that is not enough information to say that the lines are parallel. To prove two lines are parallel you need to look at the angles formed by a transversal.

Alternate Interior Angles Converse

Another important theorem you derived in the last lesson was that when parallel lines are cut by a transversal, the alternate interior angles formed will be congruent. The converse of this theorem is, "If alternate interior angles formed by two lines crossed by a transversal are congruent, then the lines are parallel." This statement is also true, and it can be proven using the Converse of the Corresponding Angles Postulate.

Converse of Alternate Interior Angles Theorem

If two lines are crossed by a transversal and alternate interior angles are congruent, then the lines are parallel.



Given $\overleftrightarrow{AD}$ and $\overleftrightarrow{GE}$ are crossed by $\overleftrightarrow{HC}$ and $\angle GFB \cong \angle DBF$.

Prove $\overleftrightarrow{AD} \| \overleftrightarrow{GE}$

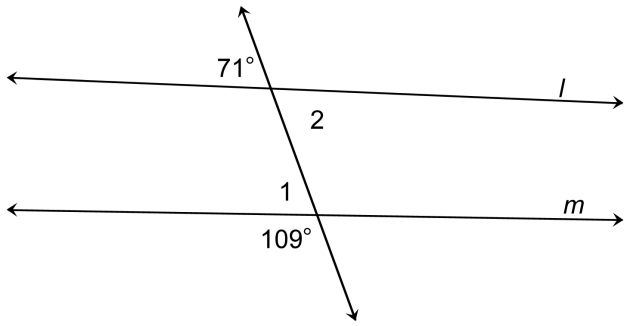
Table 3.3:

Statement	Reason
1. $\overleftrightarrow{AD}$ and $\overleftrightarrow{GE}$ are crossed by $\overleftrightarrow{HC}$ and $\angle GFB \cong \angle DBF$.	1. Given
2. $\angle DBF \cong \angle ABC$	2. Vertical Angles Theorem
3. $\angle ABC \cong \angle GFB$	3. Transitive Property of Angle Congruence
4. $\overleftrightarrow{AD} \| \overleftrightarrow{GE}$	4. Converse of the Corresponding Angles Postulate.

Notice in the proof that we had to show that the corresponding angles were congruent. Once we had done that, we satisfied the conditions of the Converse of the Corresponding Angles postulate, and we could use that in the final step to prove that the lines are parallel.

Example 3

Are the two lines in this figure parallel?



This figure shows two lines that are cut by a transversal. We don't know $m\angle 1$. However, if you look at its linear pair, that angle has a measure of 109°. By the Linear Pair Postulate, this angle is supplementary to $\angle 1$. In other words, the sum of 109° and $m\angle 1$ will be 180°. Use subtraction to find $m\angle 1$.

$$m\angle 1 = 180 - 109$$
$$m\angle 1 = 71^\circ$$

So, $m\angle 1 = 71^\circ$. Now look and $\angle 2$. $\angle 2$ is a vertical angle with the angle measuring 71°. By the Vertical Angles Theorem, $m\angle 2 = 71^\circ$.

Since $\angle 1 \cong \angle 2$ as can apply the converse of the Alternate Interior Angles Theorem to conclude that $l \| m$.

Notice in this example that you could have also used the Converse of the Corresponding Angles Postulate to prove the two lines are parallel. Also, This example highlights how, if a figure is not drawn to scale you cannot assume properties of the objects in the figure based on looks.

Converse of Alternate Exterior Angles

The more you practice using the converse of theorems to find solutions, the easier it will become. You have already probably guessed that the converse of the Alternate Exterior Angles Theorem is true.

Converse of the Alternate Exterior Angles Theorem

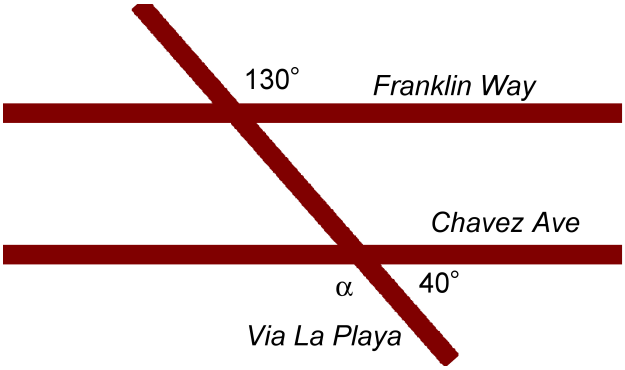
If two lines are crossed by a transversal and the alternate exterior angles are congruent, then the lines crossed by the transversal are parallel.

Putting together the alternate exterior angles theorem and its converse, we get the biconditional statement: Two lines crossed by a transversal are parallel if and only if alternate exterior angles are congruent.

Use the example below to apply this concept to a real-world situation.

Example 4

The map below shows three roads in Julio's town.



In Julio's town, Franklin Way and Chavez Avenue are both crossed by Via La Playa. Julio used a surveying tool to measure two angles in the intersections as shown and he drew the sketch above (NOT to scale). Julio wants to know if Franklin Way is parallel to Chavez Avenue. How can he solve this problem and what is the correct answer?

Notice that this question asks you not only to identify the answer, but also the process required to solve it. Make sure that your solution is step-by-step so that anyone reading it can follow your logic.

To begin, notice that the labeled 130° angle and $\angle\alpha$ are alternate exterior angles. If these two angles are congruent, then the lines are parallel. If they are not congruent, the lines are not parallel. To find the measure of angle $m\angle\alpha$, you can use the other angle labeled in this diagram, measuring 40°. This angle is supplementary to $\angle\alpha$ because they are a linear pair. Using the knowledge that a linear pair must be supplementary, find the value of $m\angle\alpha$.

$$m\angle\alpha = 180 - 40$$
$$m\angle\alpha = 140$$

Angle $m\angle\alpha$ is equal to 140°. This angle is 10° wider than the other alternate exterior angle, which measures 130° so the alternate exterior angles are not congruent. Therefore, Franklin Way and Chavez Avenue are not parallel streets.

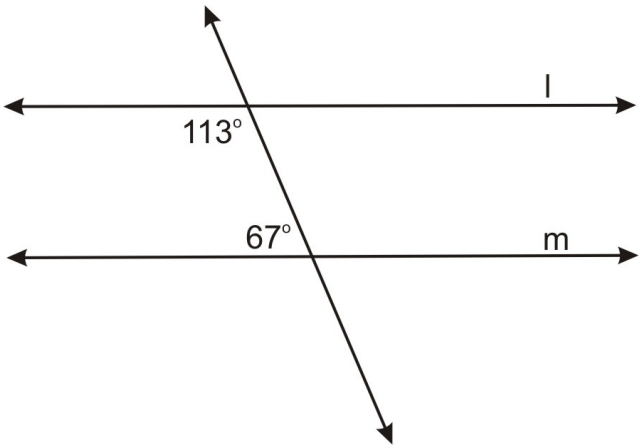
In this example, we used the contrapositive of the converse of the Alternate Exterior Angles Theorem to prove that the two lines were not parallel.

Converse of Consecutive Interior Angles

The final converse theorem to explore in this lesson addressed the Consecutive Interior Angles Theorem. Remember that these angles aren't congruent when lines are parallel, they are supplementary. In other words, if the two lines are parallel, the angles on the interior and on the same side of the transversal will sum to 180°. So, if two consecutive interior angles made by two lines and a transversal add up to 180°, the two lines that form the consecutive angles are parallel.

Example 5

Identify whether lines l and m in the diagram below are parallel.



Using the converse of the Consecutive Interior Angles Theorem, you should be able to identify that if the two angles in the figure are supplementary, then lines l and m are parallel. We add the two consecutive interior angles to find their sum.

$$113 + 67 = ?$$
$$113 + 67 = 180$$

The two angles in the figure sum to 180° so lines l and m are in fact parallel.

Parallel Lines Property

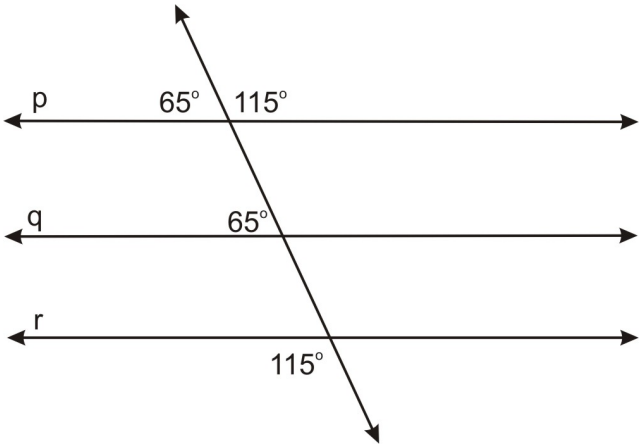
The last theorem to explore in this lesson is called the Parallel Lines Property. It is a transitive property. Does the phrase *transitive property* sound familiar? You have probably studied other transitive properties before, but usually talking about numbers. Examine the statement below.

If $a = b$ and $b = c$, then $a = c$

Notice that we used a property similar to the transitive property in a proof above. The Parallel Lines Property says that if line l is parallel to line m, and line m is parallel to line n, then lines l and n are also parallel. Use this information to solve the final practice problem in this lesson.

Example 6

Are lines p and q in the diagram below parallel?



Look at this diagram carefully to establish the relationship between lines p and r and lines q and r. Starting with line p, the angle shown measures 115°. This angle is an alternate exterior angle to the 115° angle labeled on line r. Since the alternate exterior angles are congruent, these two lines are parallel. Next look at the relationship between q and p. The angle shown on line q measures 65° and it corresponds to the 65° angle marked on line p. Since the corresponding angles on these two lines are congruent, lines p and q are also parallel.

Using the Parallel Lines Property, we can identify that lines p and q are parallel, because p is parallel to r and q is also parallel to r.

Note that there are many other ways to reason through this problem. Can you think of one or two alternative ways to show $p \| q \| r$?

Lesson Summary

In this lesson, we explored how to work with the converse of theorems we already knew. Specifically, we have learned:

- How to identify and use the Corresponding Angles Converse Postulate.
- How to identify and use the Converse of Alternate Interior Angles Theorem.
- How to identify and use the Converse of Alternate Exterior Angles Theorem.
- How to identify and use the Converse of Consecutive Interior Angles Theorem.

- How to identify and use the Parallel Lines Property.

These will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply theorems and postulates to mathematical situations.

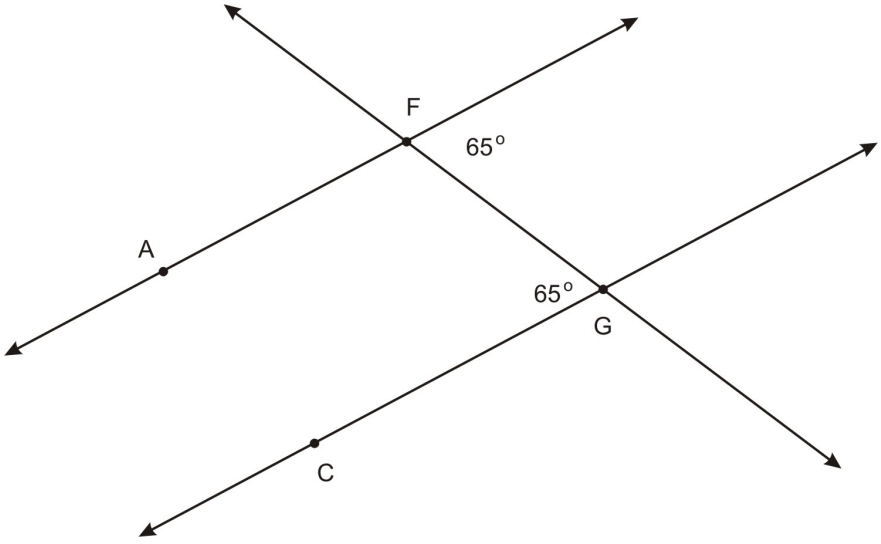
Points To Consider

You have now studied the many rules about parallel lines and the angles they form. In the next lesson, you will delve deeper into concepts of lines in the $xy-$plane. You will apply some of the geometric properties of lines to slopes and graphing in the coordinate plane.

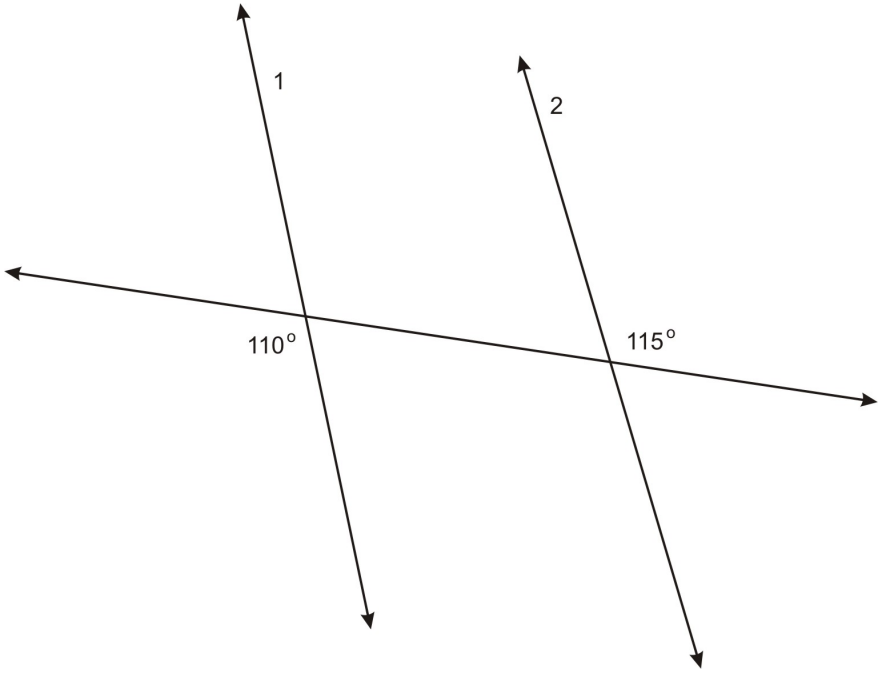
Review Questions

Solve each problem.

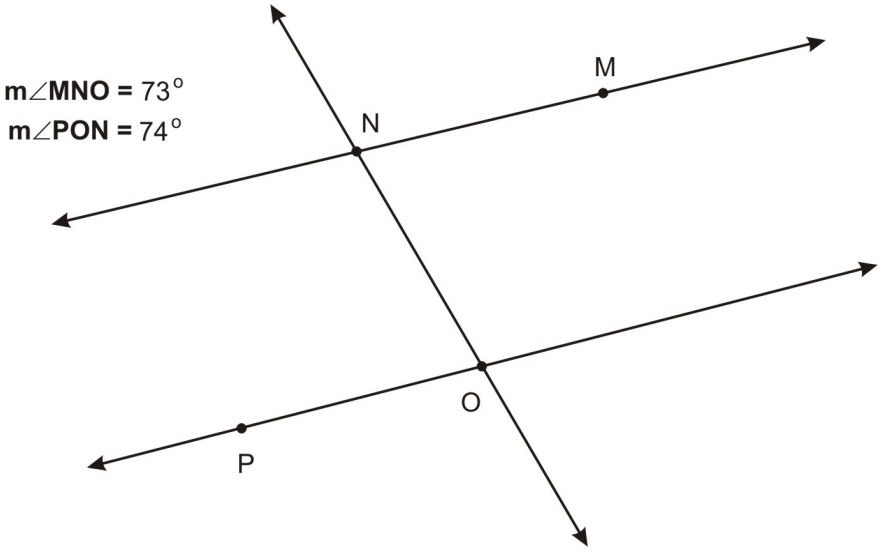
1. Are lines $\overleftrightarrow{AF}$ and $\overleftrightarrow{CG}$ parallel in the diagram below? If yes, how do you know?



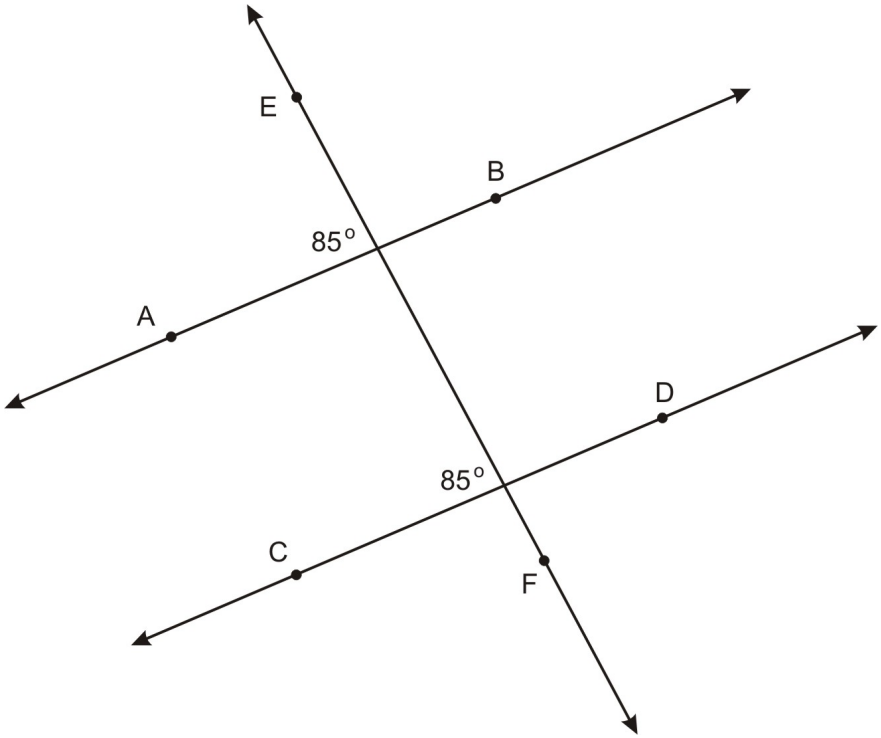
2. Are lines 1 and 2 parallel in the following diagram? Why or why not?



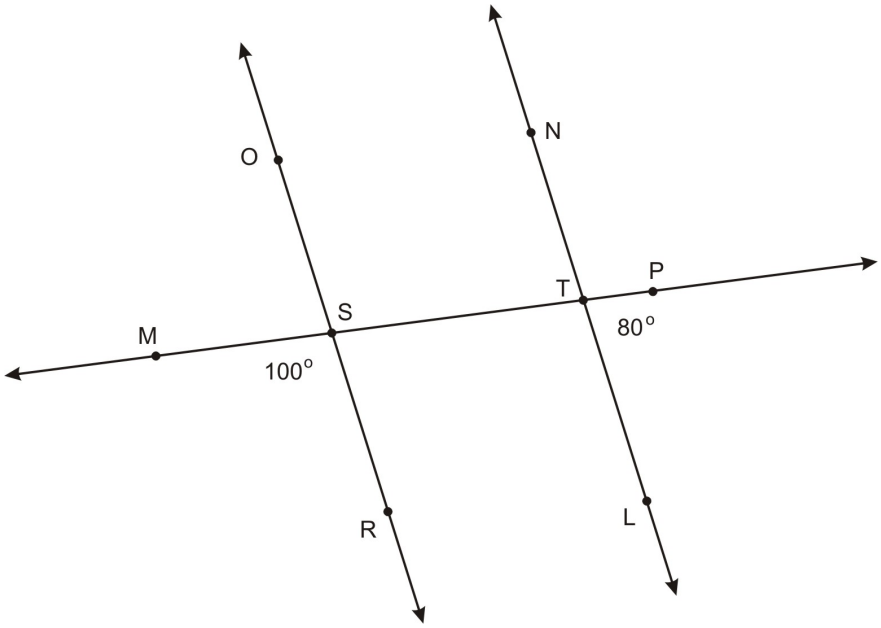
3. Are lines $\overleftrightarrow{MN}$ and $\overleftrightarrow{OP}$ parallel in the diagram below? Why or why not?



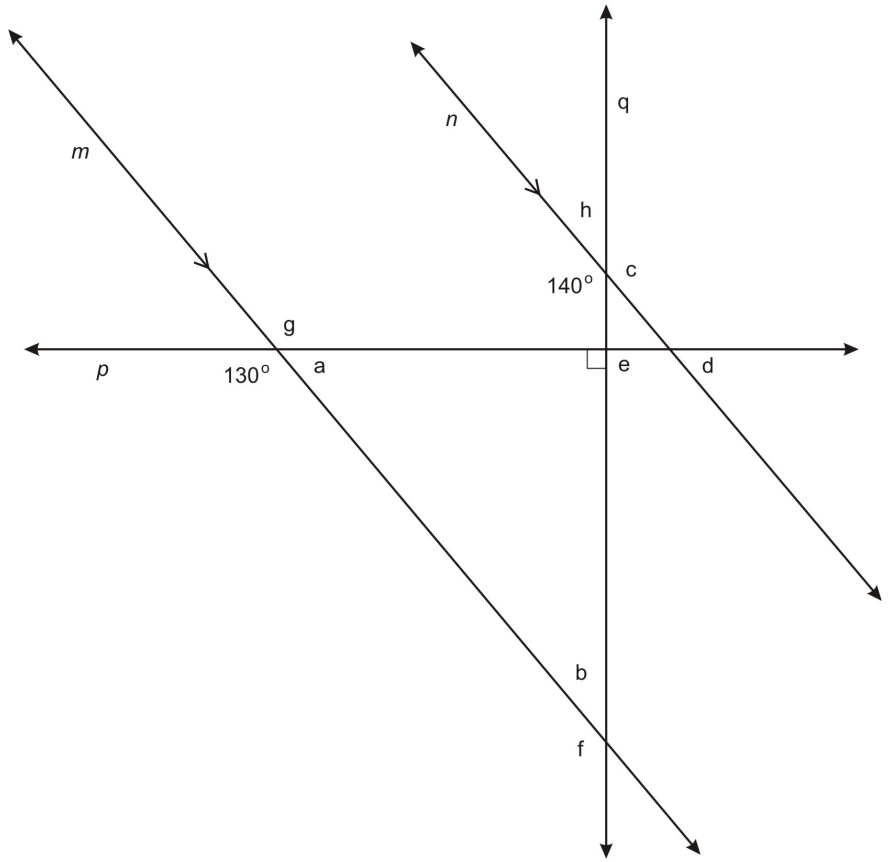
4. Are lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ parallel in the following diagram? Justify your answer.



5. Are lines $\overleftrightarrow{OR}$ and $\overleftrightarrow{LN}$ parallel in the diagram below? Justify your answer.



For exercises 6-13, use the following diagram. Line $m||n$ and $p \perp q$. Find each angle and give a justification for each of your answers.



6. $a =$ _____.
7. $b =$ _____.
8. $c =$ _____.
9. $d =$ _____.
10. $e =$ _____.
11. $f =$ _____.
12. $g =$ _____.
13. $h =$ _____.

Review Answers

1. Yes. If alternate interior angles are congruent, then the lines are parallel
2. No. Since alternate exterior angles are NOT congruent the lines are NOT parallel
3. No. Since alternate interior angles are NOT congruent, the lines are NOT parallel.
4. Yes. If corresponding angles are congruent, then the lines are parallel
5. Yes. If exterior angles on the same side of the transversal are supplementary, then the lines are parallel
6. $a = 50°$. Since a and $130°$ are a linear pair, they are supplementary
7. $b = 40°$. b is an interior angle on the same side of transversal q with the angle marked

$140°$. So b and the $140°$ angle are supplementary and $b = 40°$
8. $c = 140°$. c is a vertical angle to the angle marked $140°$.
9. $d = 50°$. It is a corresponding angle with angle a
10. $e = 90°$. It is a linear pair with a right angle
11. $f = 140°$. It is a corresponding angle with the angle marked $140°$
12. $g = 130°$. It is a vertical angle with $130°$
13. $h = 40°$. It is a linear pair with the angle marked $140°$

3.4 Slopes of Lines

Learning Objectives

- Identify and compute slope in the coordinate plane.
- Use the relationship between slopes of parallel lines.
- Use the relationship between slopes of perpendicular lines.
- Plot a line on a coordinate plane using different methods.

Introduction

You may recall from algebra that you spent a lot of time graphing lines in the $xy-$coordinate plane. How are those lines related to the lines we've studied in geometry? Lines on a graph can be studied for their **slope** (or rate of change), and how they intersect the $x-$ and $y-$axes.

Slope in the Coordinate Plane

If you look at a graph of a line, you can think of the slope as the steepness of the line (assuming that the $x-$ and $y-$ scales are equal. Mathematically, you can calculate the slope using two different points on a line. Given two points (x_1, y_1) and (x_2, y_2) the slope is computed as:

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

You may have also learned this as "slope equals rise over run." In other words, first calculate the distance that the line travels up (or down), and then divide that value by the distance the line travels left to right. The left to right distance in this scenario is referred to as the run.

A line that goes up from left to right has positive slope, and a line that goes down from left to right has negative slope.

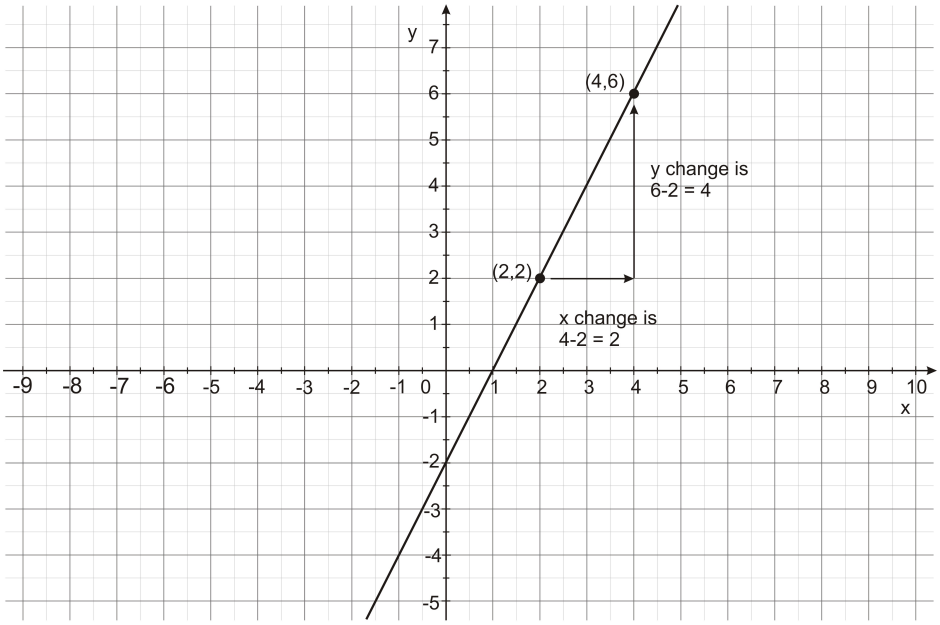
Example 1

What is the slope of a line that travels through the points $(2, 2)$ *and* $(4, 6)$*?*

You can use the previous formula to find the slope of this line. Let's say that (x_1, y_1) is $(2, 2)$ and (x_2, y_2) is $(4, 6)$. Then we find the slope as follows:

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(6 - 2)}{(4 - 2)}$$
$$\text{slope} = \frac{4}{2}$$
$$\text{slope} = 2$$

The slope of the line in Example 1 is 2. Let's look at what that means graphically.



These are the two points in question. You can see that the line rises 4 units as it travels 2 units to the right. So, the rise is 4 units and the run is 2 units. Since $4 \div 2 = 2$, the slope of this line is 2.

Notice that the slope of the line in example 1 was 2, a positive number. Any line with a positive slope will travel up from left to right. Any line with a negative slope will travel down from left to right. Check this fact in example 2.

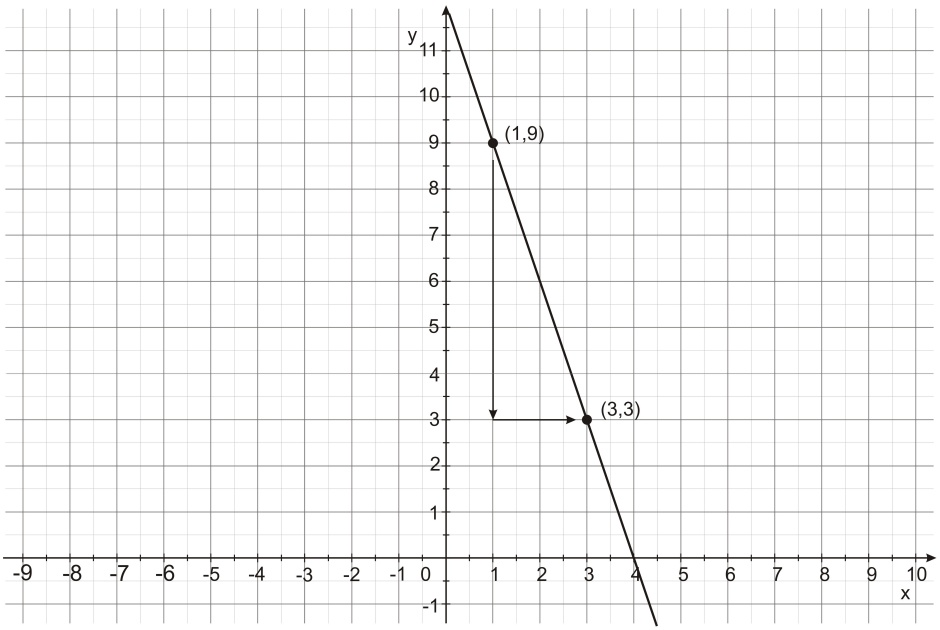
Example 2

What is the slope of the line that travels through $(1, 9)$ and $(3, 3)$?

Use the formula again to identify the slope of this line.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(3 - 9)}{(3 - 1)}$$
$$\text{slope} = \frac{-6}{2}$$
$$\text{slope} = -3$$

The slope of this line in Example 2 is -3. It will travel down to the right. The points and the line that connects them is shown below.



There are other types of lines with their own distinct slopes. Perform these calculations carefully to identify their slopes.

Example 3

What is the slope of a line that travels through $(4, 4)$ *and* $(8, 4)$*?*

Use the formula to find the slope of this line.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(4 - 4)}{(8 - 4)}$$
$$\text{slope} = \frac{0}{4}$$
$$\text{slope} = 0$$

This line, which is horizontal, has a slope of 0. Any horizontal line will have a slope of 0.

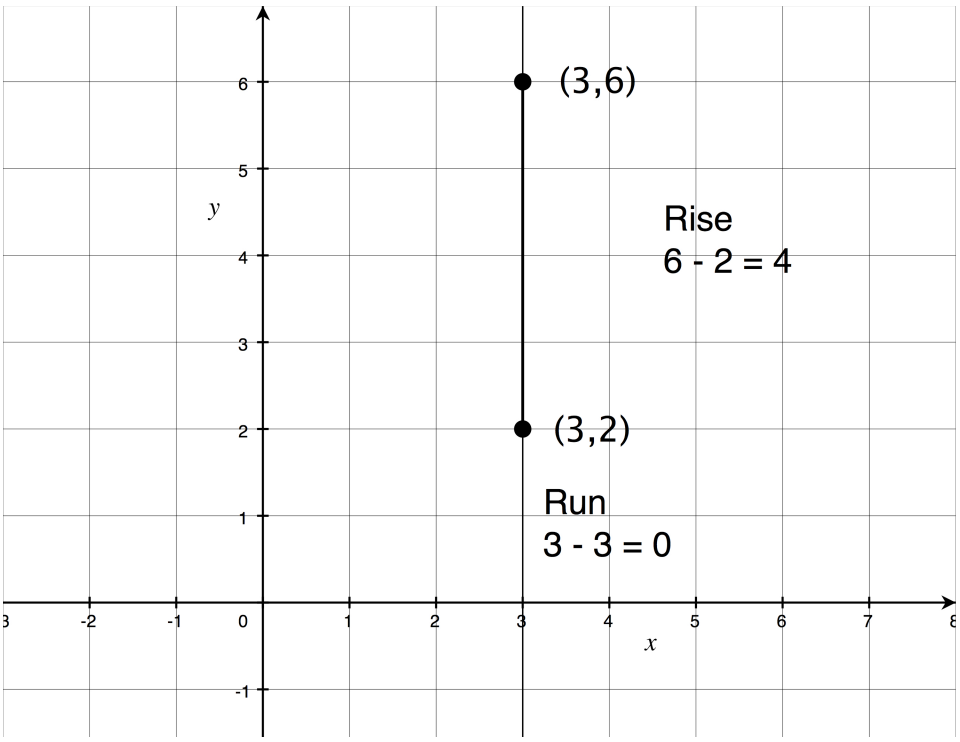
Example 4

What is the slope of a line through $(3, 2)$ *and* $(3, 6)$*?*

Use the formula to identify the slope of this line.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(6 - 2)}{(3 - 3)}$$
$$\text{slope} = \frac{4}{0}$$
$$\text{slope} = \text{undefined}$$

The line in this example is vertical and we found that the numerical value of the slope is **undefined**.



In review, if you scan a graph of a line from left to right, then,

- Lines with *positive* slopes point *up to the right*,
- Lines with *negative* slopes point *down to the right*,
- *Horizontal* lines have a slope of *zero*, and
- *Vertical* lines have *undefined* slope. You can use these general rules to check your work when working with slopes and lines.

Slopes of Parallel Lines

Now that you know how to find the slope of lines using coordinates, you can think about how lines and their slopes are related.

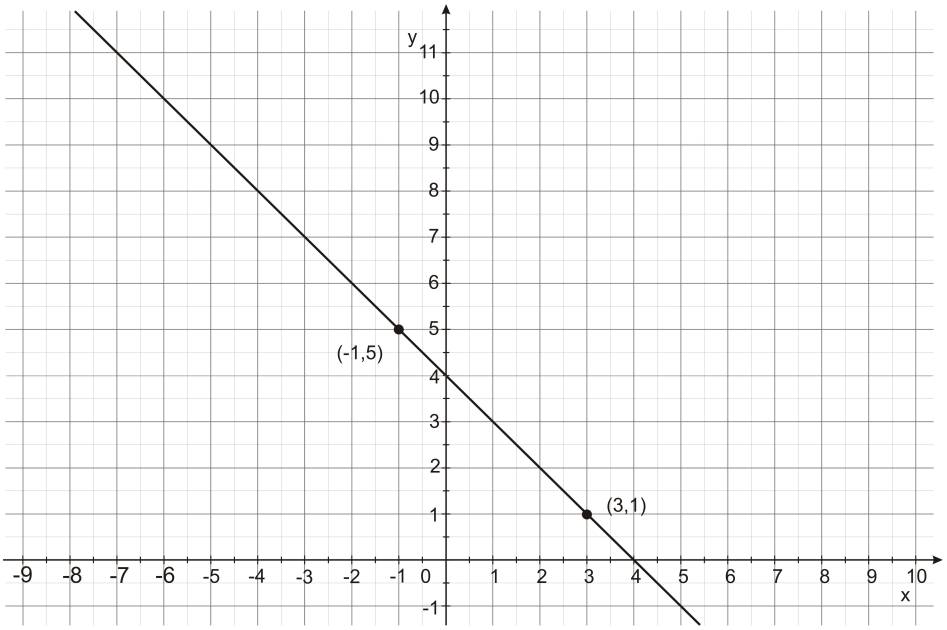
Slope of Parallel Lines Theorem

If two lines in the coordinate plane are parallel they will have the same slope, conversely, if two lines in the coordinate plane have the same slope, those lines are parallel.

Note the proof of this theorem will have to wait until you have more mathematical tools, but for now you can use it to solve problems.

Example 5

Which of the following could represent the slope of a line parallel to the one following?



A. -4
B. -1
C. $\frac{1}{4}$
D. 1

Since you are looking for the slope of a parallel line, it will have the same slope as the line in the diagram. First identify the slope of the line given, and select the answer with that slope. You can use the slope formula to find its value. Pick two points on the line. For example, $(-1, 5)$ and $(3, 1)$.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(1 - 5)}{(3 - (-1))}$$
$$\text{slope} = \frac{-4}{4}$$
$$\text{slope} = -1$$

The slope of the line in the diagram is -1. The answer is B.

Slopes of Perpendicular Lines

Parallel lines have the same slope. There is also a mathematical relationship for the slopes of perpendicular lines.

Perpendicular Line Slope Theorem

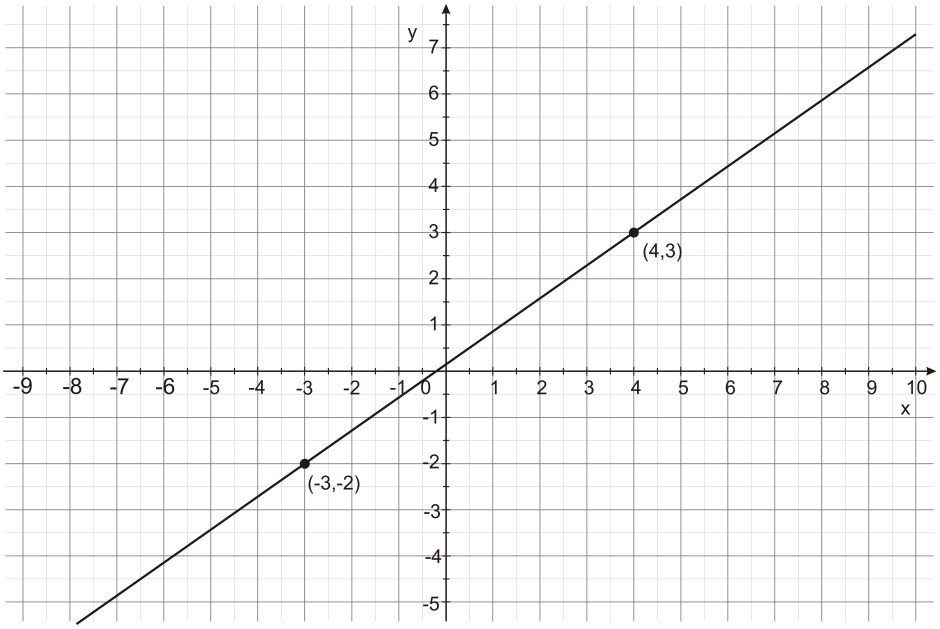
The slopes of perpendicular lines will be the *opposite reciprocal* of each other.

Another way to say this theorem is, if the slopes of two lines multiply to -1, then the two lines are perpendicular.

The opposite reciprocal can be found in two steps. First, find the reciprocal of the given slope. If the slope is a fraction, you can simply switch the numbers in the numerator and denominator. If the value is not a fraction, you can make it into one by putting a 1 in the numerator and the given value in the denominator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ and the reciprocal of 5 is $\frac{1}{5}$. The second step is to find the opposite of the given number. If the value is positive, make it negative. If the value is negative, make it positive. The opposite reciprocal of $\frac{2}{3}$ is $-\frac{3}{2}$ and the opposite reciprocal of 5 is $-\frac{1}{5}$.

Example 6

Which of the following could represent the slope of a line perpendicular to the one shown below?



A. $-\frac{7}{5}$
B. $-\frac{5}{7}$
C. $\frac{5}{7}$

D. $\frac{7}{5}$

Since you are looking for the slope of a perpendicular line, it will be the opposite reciprocal of the slope of the line in the diagram. First identify the slope of the line given, then find the opposite reciprocal, and finally select the answer with that value. You can use the slope formula to find the original slope. Pick two points on the line. For example, $(-3, -2)$ and $(4, 3)$.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(3 - (-2))}{(4 - (-3))}$$
$$\text{slope} = \frac{(3 + 2)}{(4 + 3)}$$
$$\text{slope} = \frac{5}{7}$$

The slope of the line in the diagram is $\frac{5}{7}$. Now find the opposite reciprocal of that value. First swap the numerator and denominator in the fraction, then find its opposite. The opposite reciprocal of $\frac{5}{7}$ is $-\frac{7}{5}$. The answer is A.

Graphing Strategies

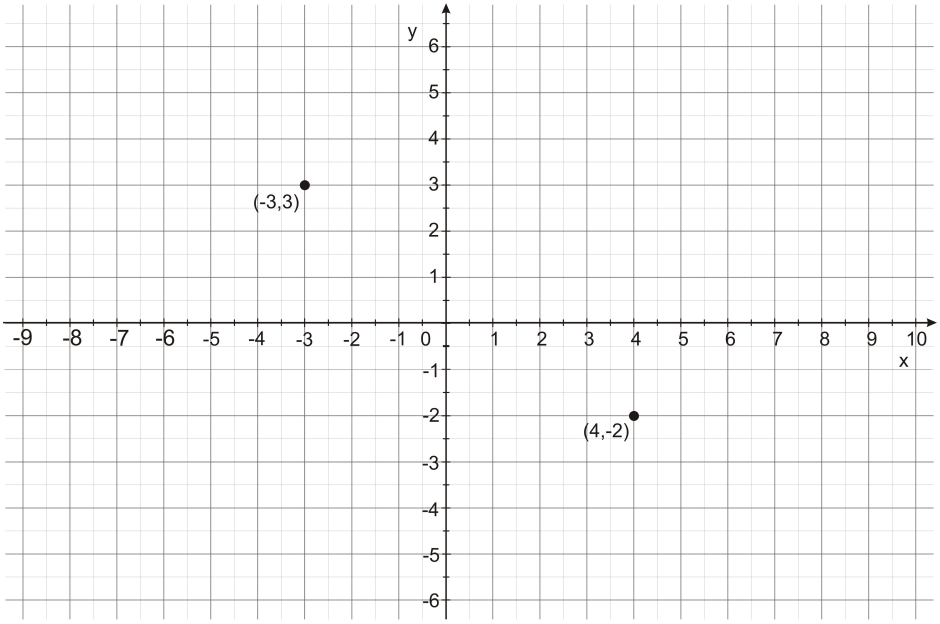
There are a number of ways to graph lines using slopes and points. This is an important skill to use throughout algebra and geometry. If you write an equation in algebra, it can help you to see the general slope of a line and understand its trend. This could be particularly helpful if you are making a financial analysis of a business plan, or are trying to figure out how long it will take you save enough money to buy something special. In geometry, knowing the behavior of different types of functions can be helpful to understand and make predictions about shapes, sizes, and trends.

There are two simple ways to create a linear graph. The first is to use two points that are given to you. Plot them on a coordinate grid, and draw a line segment connecting them. This segment can be expanded to represent the entire line that passes through those two points.

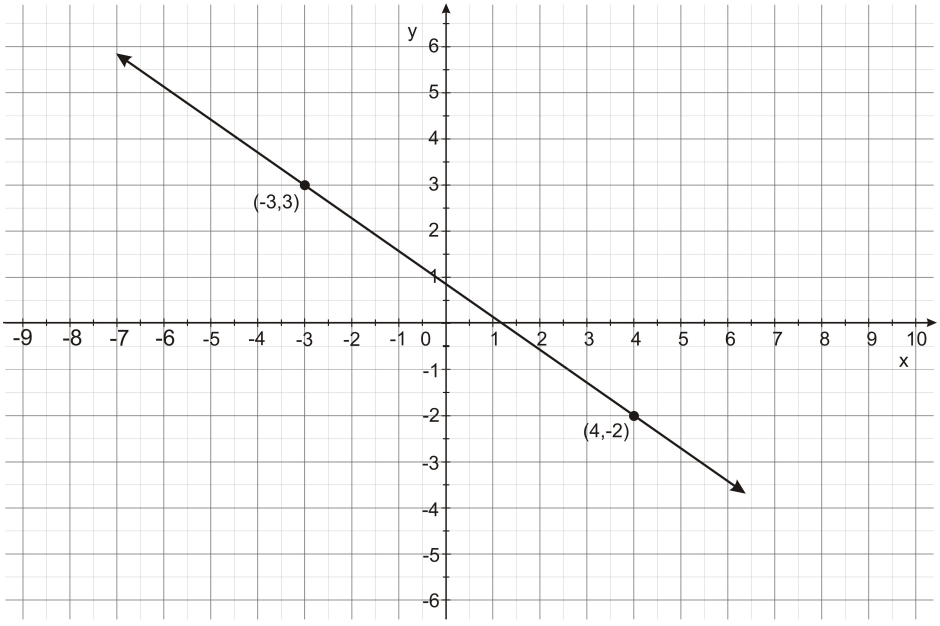
Example 7

Draw the line that passes through $(-3, 3)$ *and* $(4, -2)$.

Begin by plotting these points on a coordinate grid. Remember that the first number in the ordered pair represents the $x-$value and the second number represents the $y-$value.



Draw a segment connecting these two points and extend that segment in both directions, adding arrows to both ends. This shows the only line that passes through points $(-3, 3)$ and $(4, -2)$.

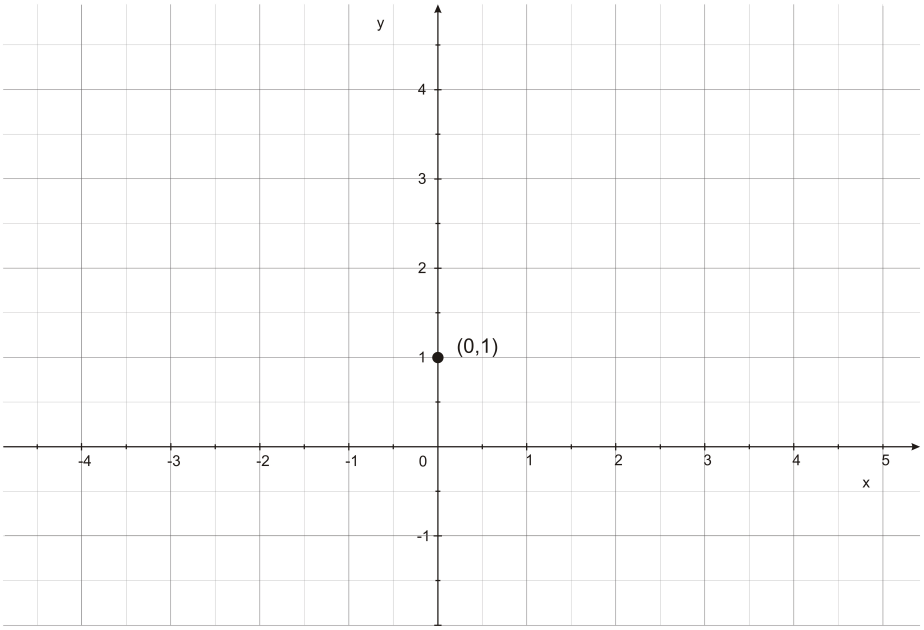


The other way to graph a line is using one point and the slope. Start by plotting the given point and using the slope to calculate another point. Then you can draw the segment and extend it as you did in the previous example.

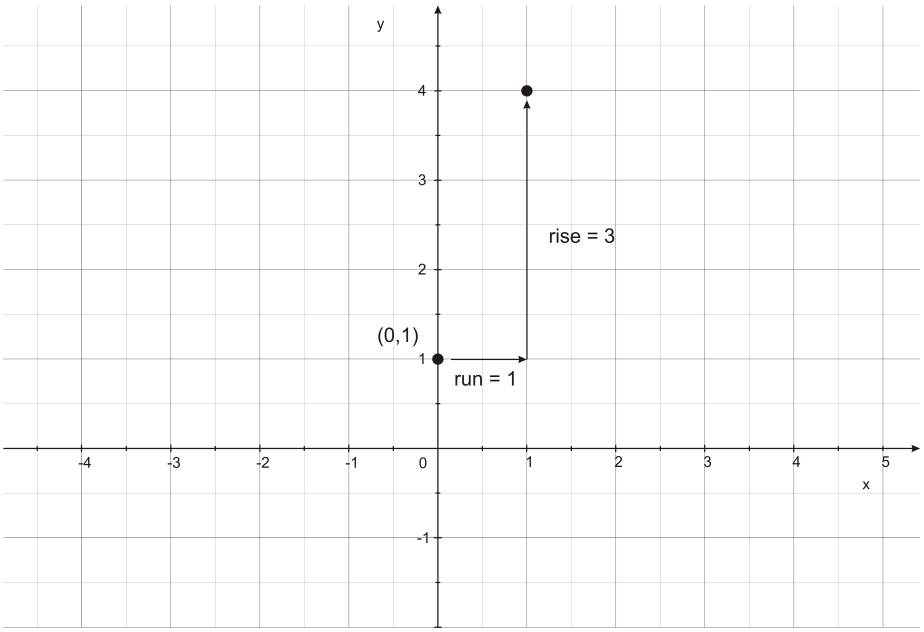
Example 8

Draw the line that passes through $(0, 1)$ *and has a slope of* 3.

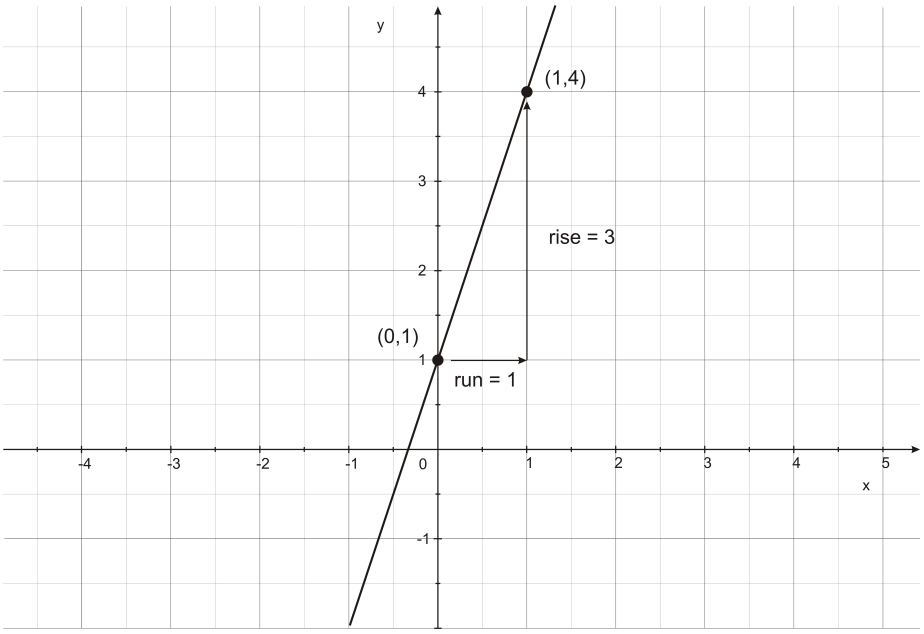
Begin by plotting the given point on a coordinate grid.



If the slope is 3, you can interpret that as $\frac{3}{1}$. The fractional expression makes it easier to identify the rise and the run. So, the rise is 3 and the run is 1. Find and plot a point that leaves the given coordinate and travels up three units and one unit to the right. This point will also be on the line.



Now you have plotted a second point on the line at $(1, 4)$. You can connect these two points, extend the segment, and add arrows to show the line that passes through $(0, 1)$ with a slope of 3.



Lesson Summary

In this lesson, we explored how to work with lines in the coordinate plane. Specifically, we have learned:

- How to identify slope in the coordinate plane.
- How to identify the relationship between slopes of parallel lines.
- How to identify the relationship between slopes of perpendicular lines.
- How to plot a line on a coordinate plane using different methods.

These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply concepts of slope, parallel and perpendicular lines, and graphing to mathematical situations.

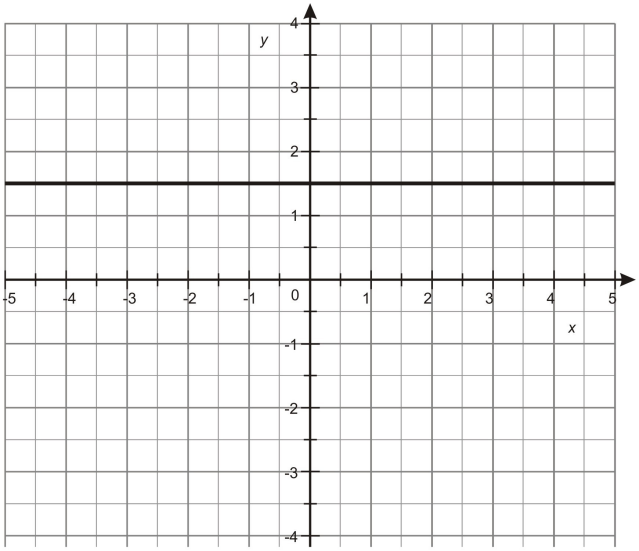
Points to Consider

Now that you have studied slope, graphing techniques, and other issues related to lines, you can learn about their algebraic properties. In the next lesson, you'll learn how to write different types of equations that represent lines in the coordinate plane.

Review Questions

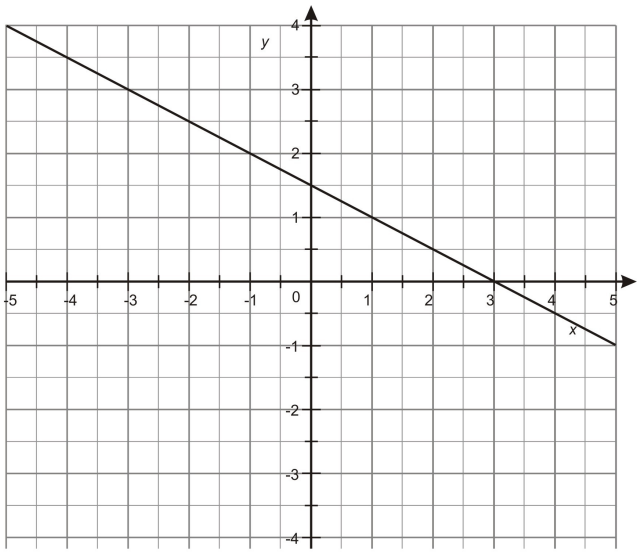
Solve each problem.

1. Which term best describes the slope of the line below?



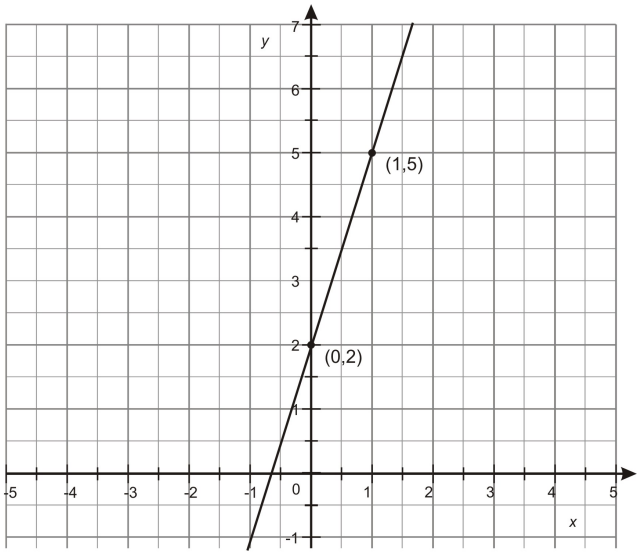
(a) positive
(b) negative
(c) zero
(d) undefined

2. Which term best describes the slope of the following line?

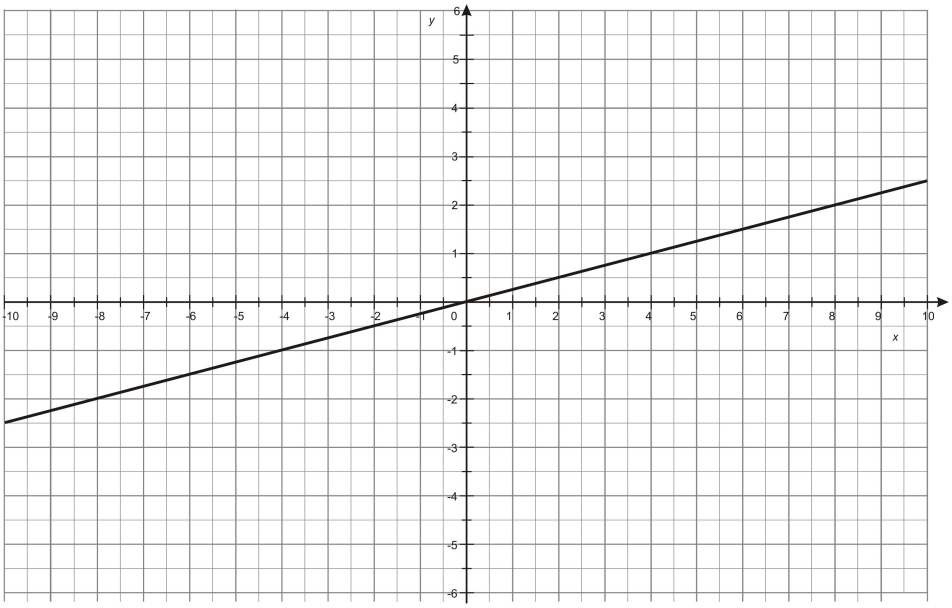


(a) positive
(b) negative
(c) zero
(d) undefined

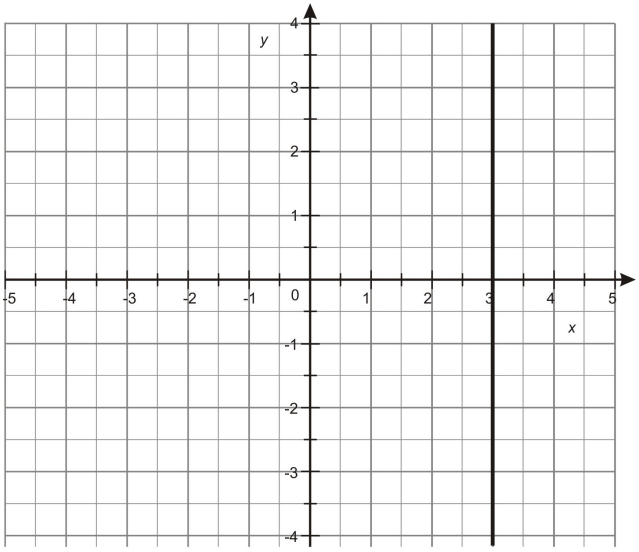
3. What is the slope of the following line?



4. What would the slope be of a line parallel to the one following?

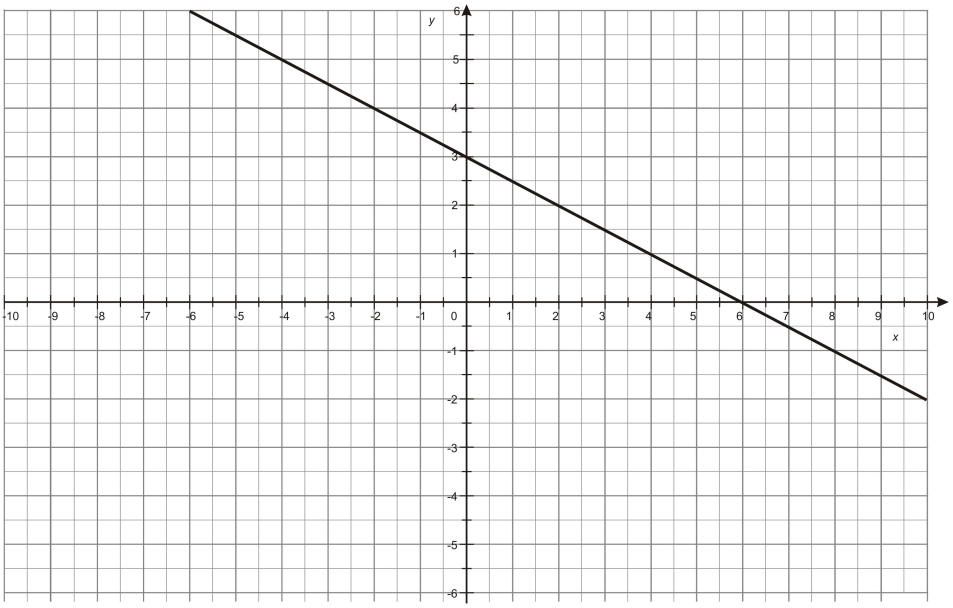


5. Which term best describes the slope of the following line?

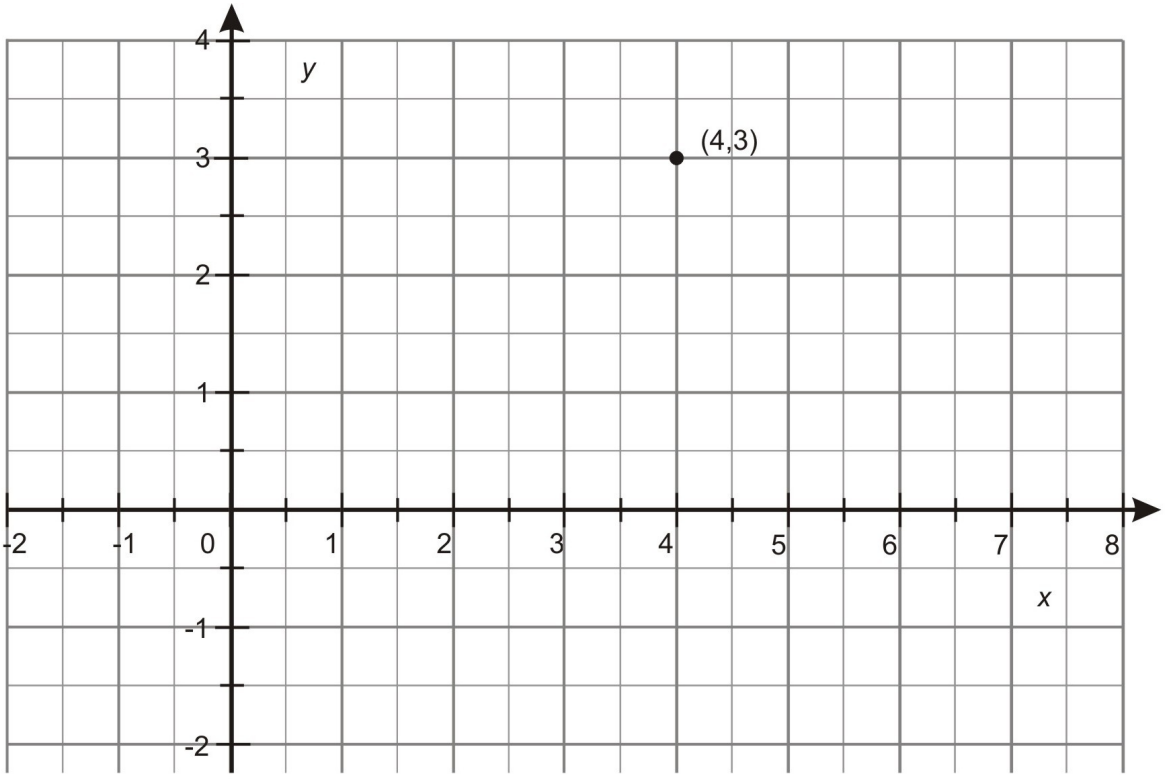


(a) positive
(b) negative
(c) zero
(d) undefined

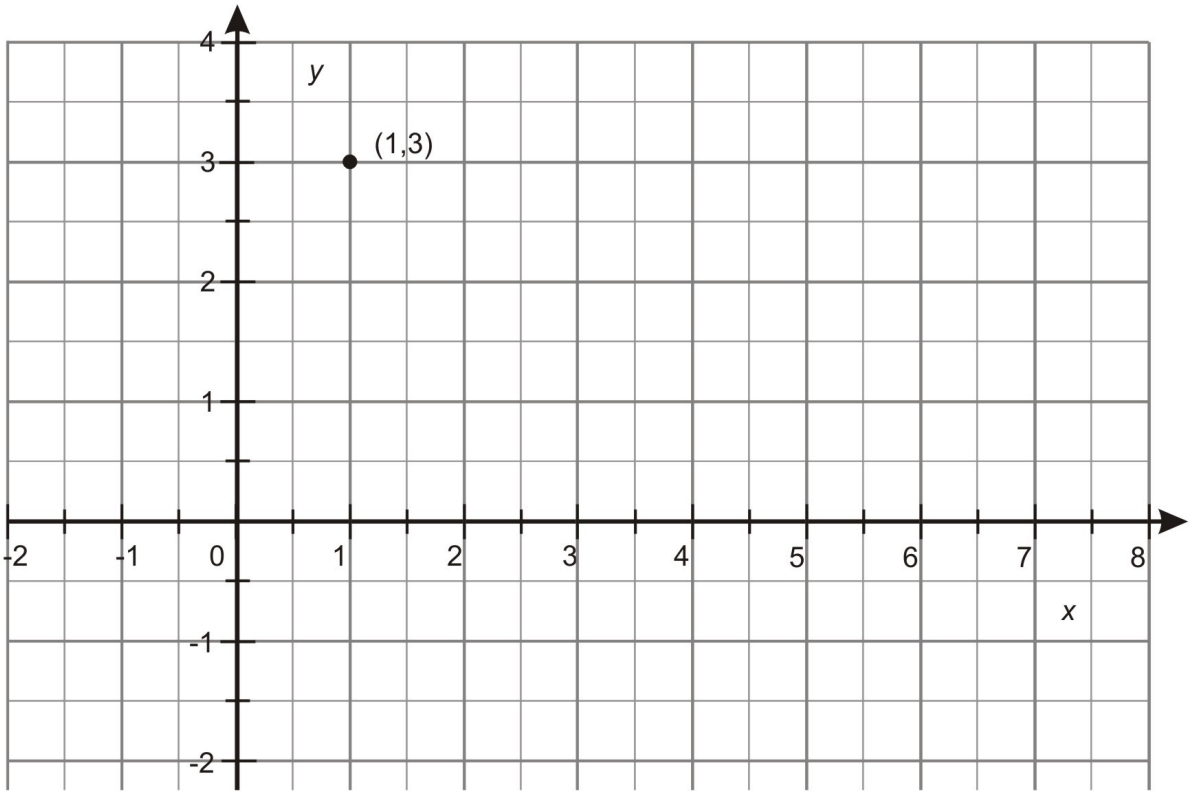
6. What is the slope of the following line?



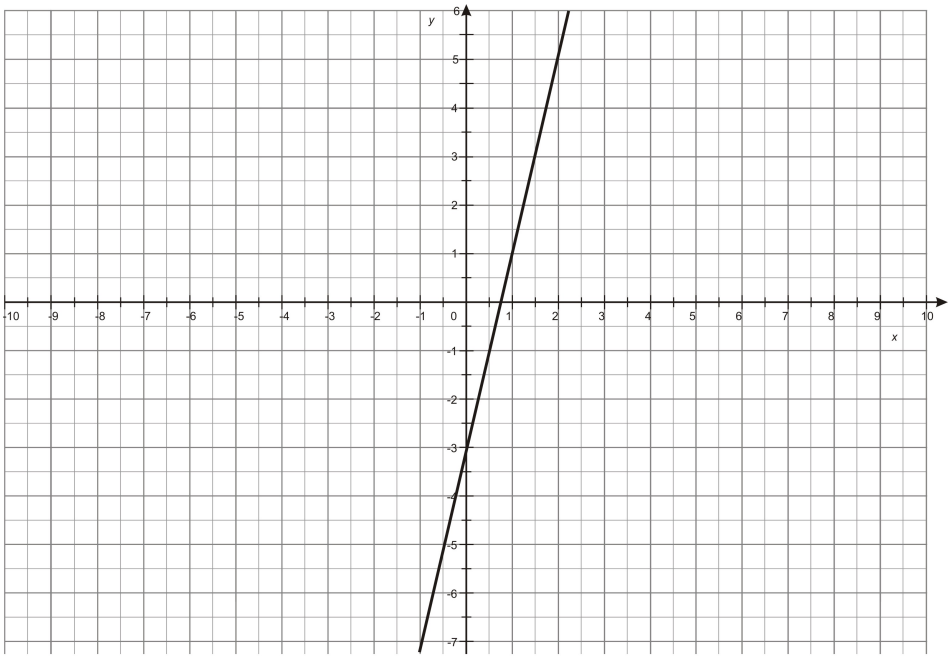
7. Plot a line through the point below with a slope of 0. What is the equation of that line?



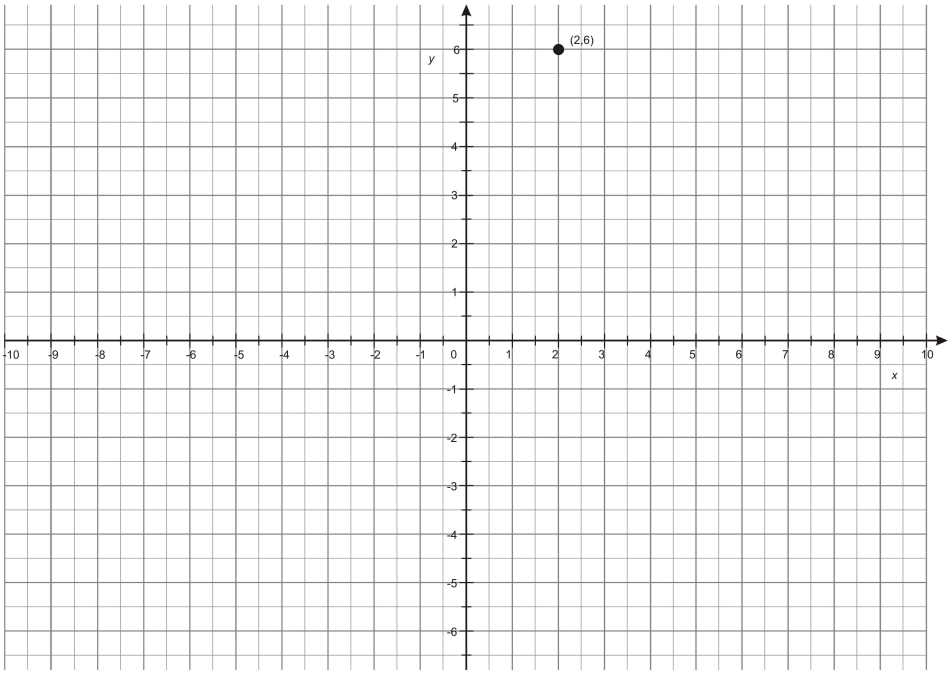
8. Plot a line through the point below with a slope of $\frac{1}{5}$.



9. What would the slope be of a line perpendicular to the line below?



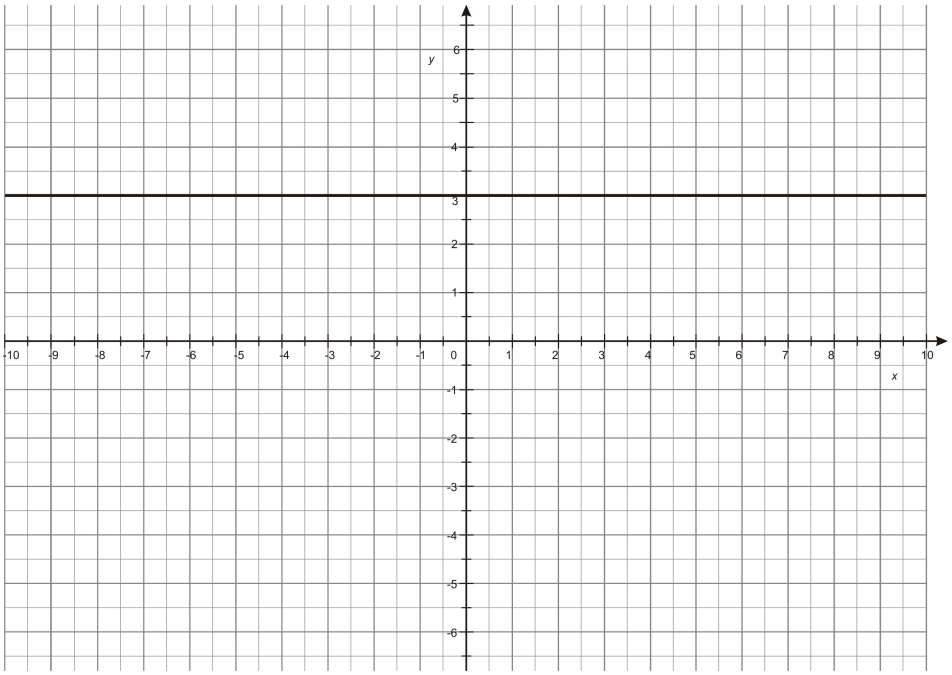
10. Plot a line that travels through the point below and has a slope of -2.



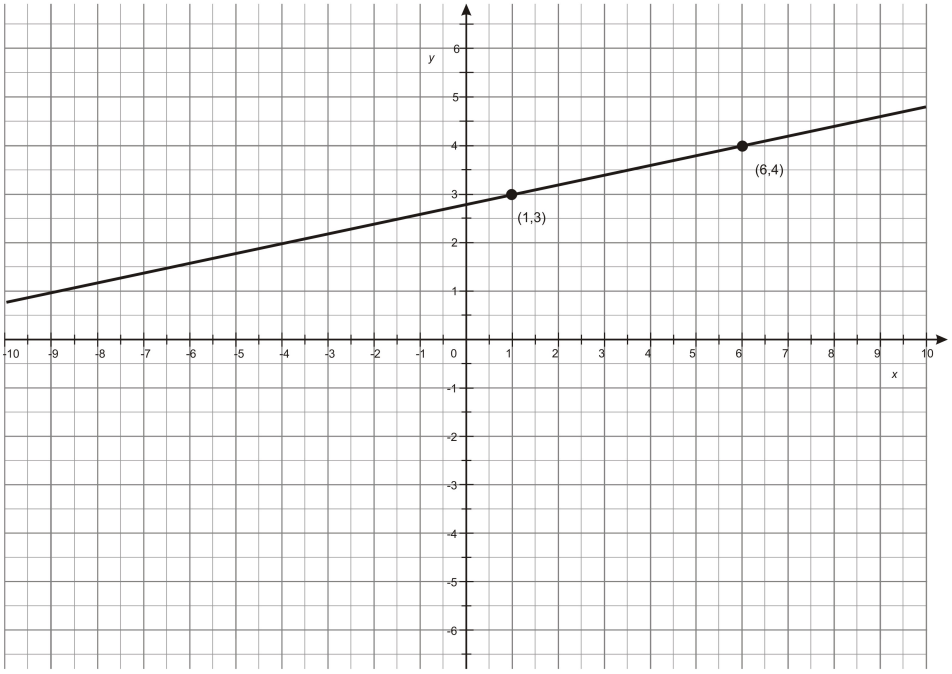
Review Answers

1. c
2. b
3. 3
4. $\frac{1}{4}$
5. d
6. $-\frac{1}{2}$

7.

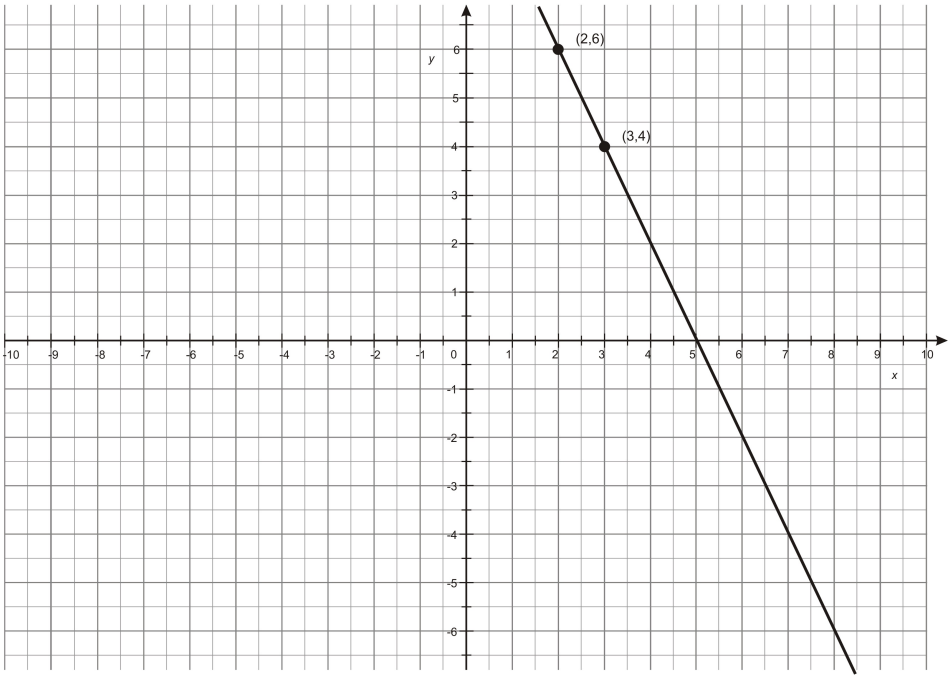


8.



9. $-\frac{1}{4}$

10.



3.5 Equations of Lines

Learning Objectives

- identify and write equations in slope-intercept form.
- identify equations of parallel lines.
- identify equations of perpendicular lines.
- identify and write equations in standard form.

Introduction

Every line that you can represent graphically on the coordinate plane can also be represented algebraically. That means that you can create an equation relating x and y that corresponds to any graph of a straight line. In this lesson, you'll learn how to create an equation from a graph or points given, identify equations of parallel and perpendicular lines, and practice using both slope-intercept and standard form.

Slope-Intercept Equations

The first type of linear equation to study is the most straightforward. It is called **slope-intercept form** and involves both the slope of the line and its $y-$ **intercept.** A $y-$intercept is the point at which the line crosses the vertical (y) axis. So, it will be the value of y when x is equal to 0. The generic formula for an equation in slope-intercept form is as follows.

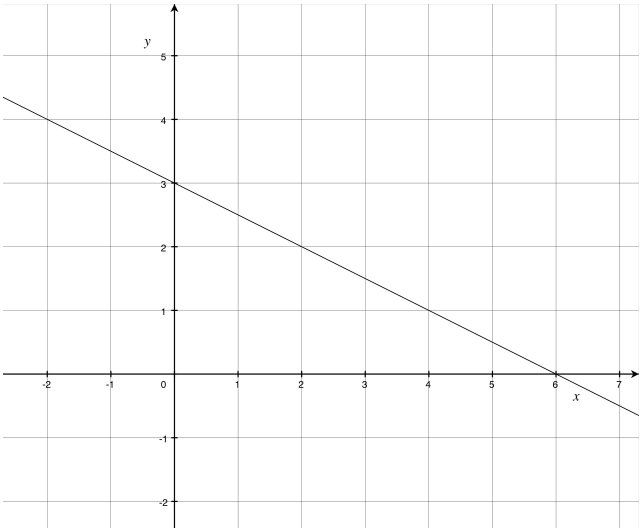
$$y = mx + b$$

In this equation, y and x remain as variables, m is the slope of the line, and b is the $y-$intercept of the line. So, if you know that a line has a slope of 4 and it crosses the $y-$axis at $(0, 8)$, its equation in slope-intercept form would be $y = 4x + 8$.

This form is especially useful for identifying the equation of a line given its graph. You already know how to deduce the slope by finding two points and using the slope formula. You can identify the $y-$intercept by sight by finding where the line crosses the $y-$axis on the graph. The value of b is the $y-$coordinate of this point.

Example 1

Write an equation in slope-intercept form that represents the following line.



First find the slope of the line. You already know how to do this using the slope formula. In this scenario, pick two points on the line to complete the formula. Use $(0, 3)$ and $(2, 2)$.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(2 - 3)}{(2 - 0)}$$
$$\text{slope} = -\frac{1}{2}$$

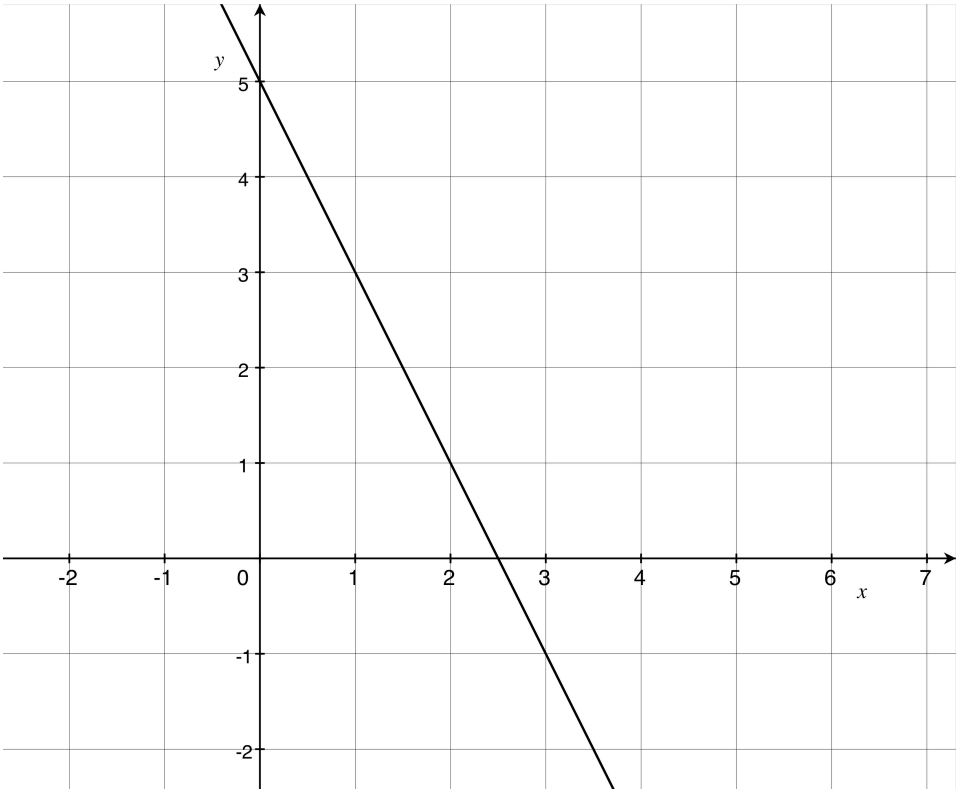
The slope of the line is $-\frac{1}{2}$. This value will replace m in the slope-intercept equation. Now you need to find the $y-$intercept. Identify on the graph where the line intersects the $y-$axis. It crosses the axes at $(0, 3)$, so the $y-$intercept is 3. This will replace b in the slope-intercept equation, so now you have all the information you need to write the full equation. The equation for the line shown in the graph is $y = -\frac{1}{2}x + 3$.

Equations of Parallel Lines

You studied parallel lines and their graphical relationships in the last lesson. In this lesson, you will learn how to easily identify equations of parallel lines. It's simple—look for equations that have the same slope. As long as the $y-$intercepts are not the same and the slopes are equal, the lines are parallel. (If the $y-$intercept and the slope are the same, then the two equations would be for the same line, and a line cannot be parallel to itself.)

Example 2

Millicent drew the line below.



Which of the following equations could represent a line parallel to the one Millicent drew?

A. $y = -\frac{1}{2}x - 6$
B. $y = \frac{1}{2}x + 9$
C. $y = -2x - 18$
D. $y = 2x + 1$

All you really need to do to solve this problem is identify the slope of the line in Millicent's graph. Identify two points on the graph, and find the slope using the slope formula. Use points $(0, 5)$ and $(1, 3)$.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(3 - 5)}{(1 - 0)}$$
$$\text{slope} = -\frac{2}{1}$$
$$\text{slope} = -2$$

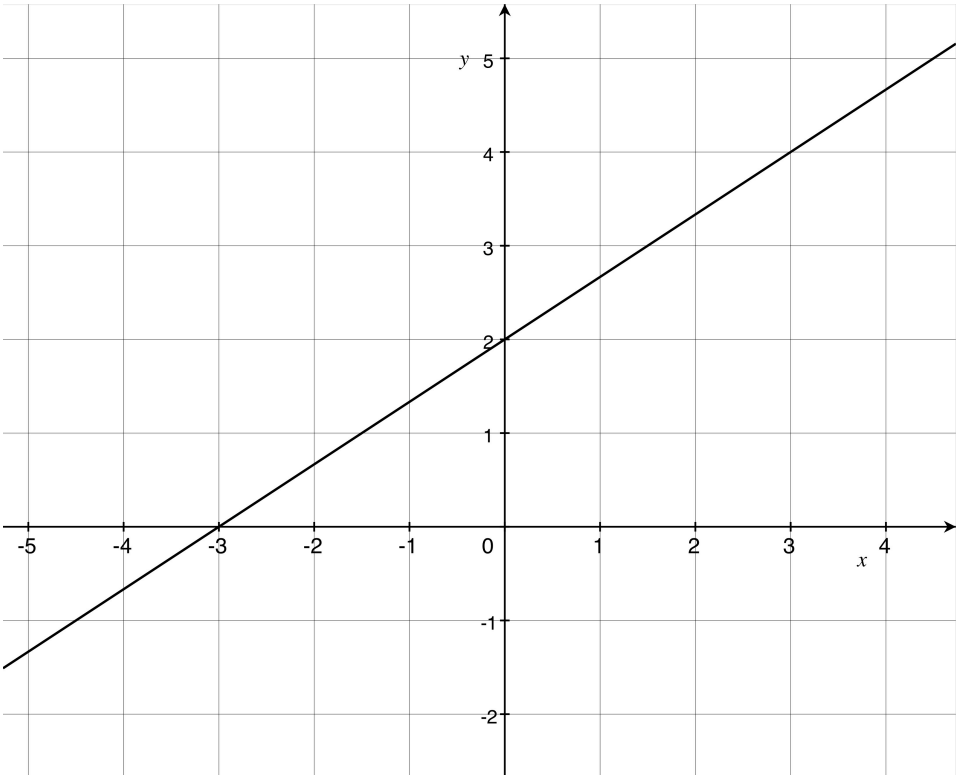
The slope of Millicent's line is -2. All you have to do is identify which equation among the four choices has a slope of -2. You can disregard all other information. The only equation that has a slope of -2 is choice C, so it is the correct answer.

Equations of Perpendicular Lines

You also studied perpendicular lines and their graphical relationships in the last lesson. Remember that the slopes of perpendicular lines are opposite reciprocals. In this lesson, you will learn how to easily identify equations of perpendicular lines. Look for equations that have the slopes that are opposite reciprocals of each other. In this case it doesn't matter what the $y-$intercept is; as long as the slopes are opposite reciprocals, the lines are perpendicular.

Example 3

Kieran drew the line in this graph.



Which of the following equations could represent a line perpendicular to the one Kieran drew?

A. $y = \frac{3}{2}x + 10$
B. $y = \frac{2}{3}x - 4$
C. $y = -\frac{2}{3}x - 1$

D. $y = -\frac{3}{2}x + 6$

All you really need to do to solve this problem is identify the slope of the line in Kieran's graph and find its opposite reciprocal. To begin, identify two points on the graph, and find the slope using the slope formula. Use points $(0, 2)$ and $(3, 4)$.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(4 - 2)}{(3 - 0)}$$
$$\text{slope} = \frac{2}{3}$$

The slope of Millicent's line is $\frac{2}{3}$. Now find the opposite reciprocal of this value. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, and the opposite of $\frac{3}{2}$ is $-\frac{3}{2}$. So, $-\frac{3}{2}$ is the opposite reciprocal of $\frac{2}{3}$. Now find the equation that has a slope of $-\frac{3}{2}$. The only equation that has a slope of $-\frac{3}{2}$ is choice D, so it is the correct answer.

Equations in Standard Form

There are other ways to write the equation of a line besides the slope intercept form. One alternative is **standard form.** Standard form is represented by the equation below.

$$Ax + By = C$$

In this equation, both A and B cannot be 0. Also, if possible, A and B should be integers.

Example 4

Convert the equation $y = -\frac{1}{3}x + \frac{5}{7}$ into standard form.

The goal is to remove the fractions and have x and y on the same side of the equals sign. To start, multiply the entire equation by 7 to eliminate the denominator of $\frac{5}{7}$.

$$7y = -\frac{7}{3}x + 5$$

Next multiply the equation by 3 to eliminate the denominator of $-\frac{7}{3}$.

$$21y = -7x + 15$$

Now add $7x$ to both sides of the equation to get x and y on the same side.

$$21y + 7x = -7x + 15 + 7x$$
$$7x + 21y = 15$$

We are done. The equation in standard form is $7x + 21y = 15$.

Lesson Summary

In this lesson, we explored how to understand equations of lines. Specifically, we have learned:

- How to identify and write equations in slope-intercept form.
- How to identify equations of parallel lines.
- How to identify equations of perpendicular lines.
- How to identify and write equations in standard form.

Always be on the lookout for ways to apply your knowledge of slope, parallel and perpendicular lines, and graphing on the coordinate plane to mathematical situations. Many problems in geometry can be solved by representing a geometric situation in the coordinate plane.

Points To Consider

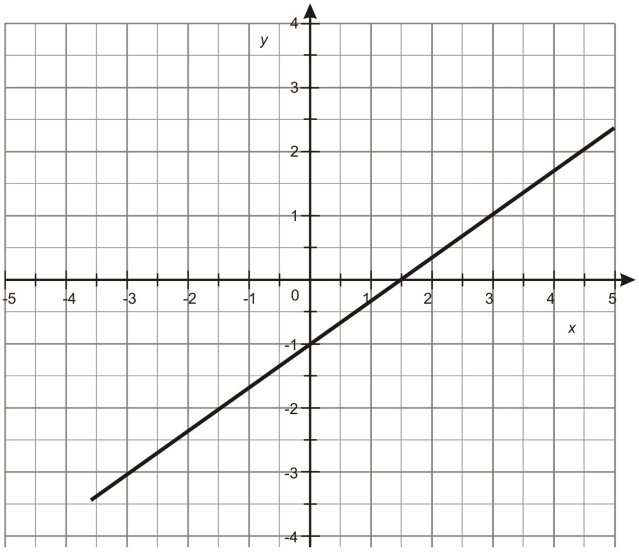
Now that you understand equations of lines, you are going to take a closer look at perpendicular lines and their properties.

Review Questions

Solve each problem.

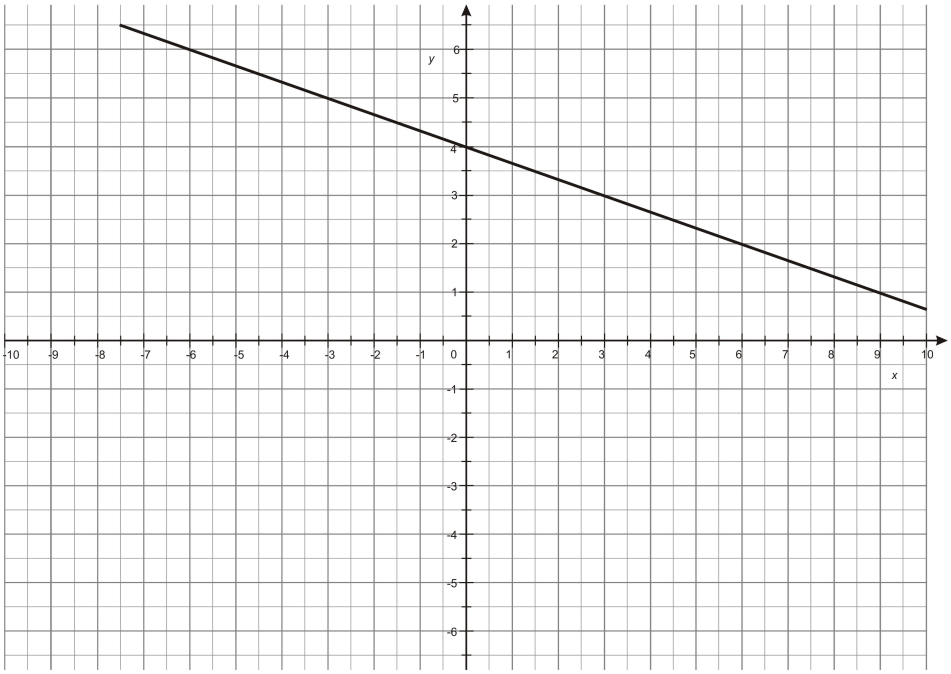
1. What equation could represent the line parallel to $y = -\frac{1}{4}x + 18$?
   (a) $y = -\frac{1}{4}x - \frac{1}{2}$
   (b) $y = -4x + \frac{2}{3}$
   (c) $y = \frac{1}{4}x - 7$
   (d) $y = 4x + 1$

2. What is the equation for the line shown below?

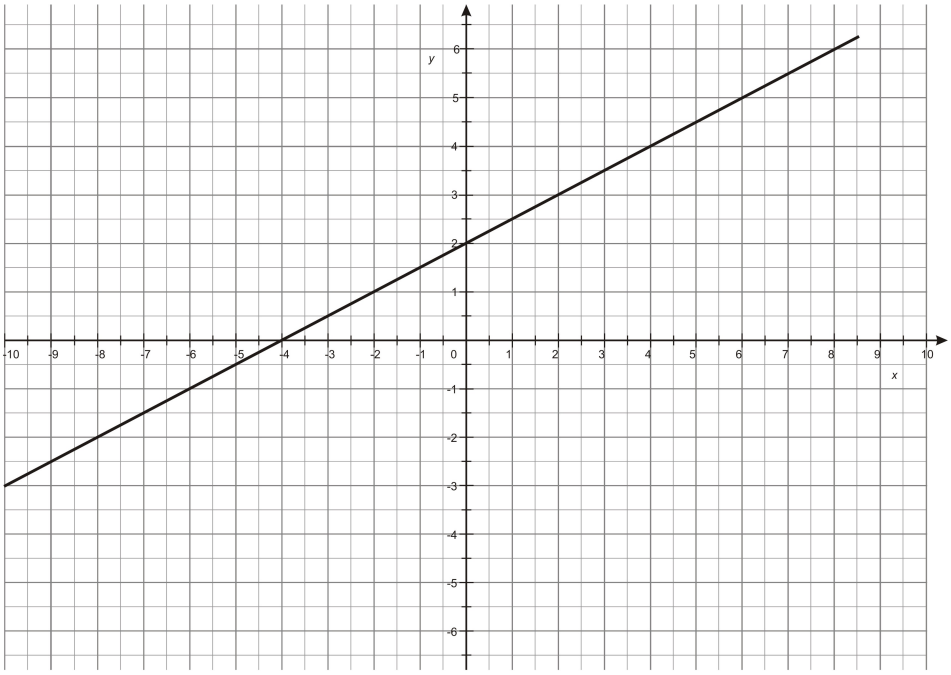


3. What equation could represent a line parallel to $y = \frac{3}{2}x - 1$?
   (a) $y = -\frac{3}{2}x + 2$
   (b) $y = -\frac{2}{3}x - 8$
   (c) $y = \frac{3}{2}x - 12$
   (d) $y = \frac{2}{3}x + 6$

4. What is the equation for the line shown below?



5. What equation could represent a line perpendicular to $y = -\frac{4}{7}x - 4$?
   (a) $y = -\frac{4}{7}x + 4$
   (b) $y = -\frac{7}{4}x - 8$
   (c) $y = \frac{7}{4}x$
   (d) $y = \frac{4}{7}x - 10$
6. What is the equation for the line shown below?



7. What equation could represent a line perpendicular to $y = \frac{5}{6}x - 2$?
   (a) $y = -\frac{5}{6}x + 5$
   (b) $y = -\frac{6}{5}x - 7$
   (c) $y = \frac{5}{6}x + 6$
   (d) $y = \frac{6}{5}x - 7$
8. Write the equation $y = \frac{1}{7}x + \frac{2}{3}$ in standard form.
9. Write the equation $y = \frac{6}{5}x + \frac{1}{4}$ in standard form.
10. Write the equation $y = -\frac{2}{3}x + \frac{1}{8}$ in standard form.

Review Answers

1. a
2. $y = \frac{2}{3}x - 1$
3. c
4. $y = -\frac{1}{3}x + 4$

5. c
6. $y = \frac{1}{2}x + 2$
7. b
8. $21y - 3x = 14$
9. $20y - 24x = 5$
10. $24y + 16x = 3$

3.6 Perpendicular Lines

Learning Objectives

- Identify congruent linear pairs of angles
- Identify the angles formed by perpendicular intersecting lines
- Identify complementary adjacent angles

Introduction

Where they intersect, perpendicular lines form right (90°) angles. This lesson explores the different properties of perpendicular lines and how to understand them in various geometrical contexts.

Congruent linear pairs

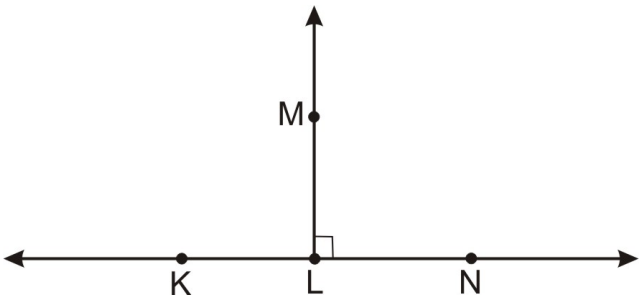
A Linear Pair of Angles is a pair of adjacent angles whose outer sides form a straight line. The Linear Pair Postulate states that the angles of a linear pair are supplementary, that is, their measures must sum to 180°. This makes sense because 180° is the measure of a straight angle. When two angles that form a linear pair are congruent, there is only one possible measure for each of them—90°. Remembering that they must sum to 180°, you can imagine how to find two equal angles. The easiest way to do this is to divide 180° by 2, the number of congruent angles.

$$180 \div 2 = 90$$

Congruent angles that form a linear pair must each measure 90°. You can use this information to fill in missing measures in diagrams and solve problems.

Example 1

What is the measure of $\angle KLM$ below?



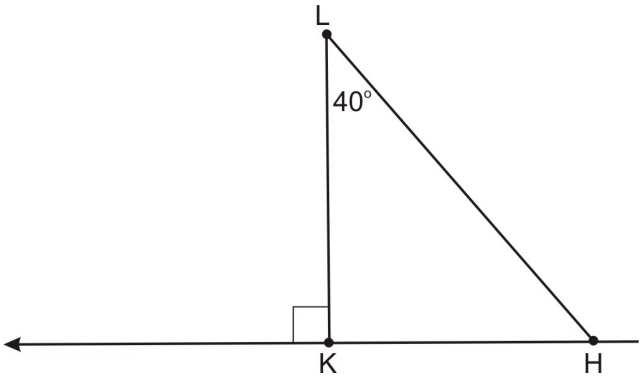
Since the two angles form a linear pair, they must sum to 180°. You can see that $\angle MLN$ is a right angle by the square marking in the angle, so that angle measures 90°. Use subtraction to find the missing angle.

$$m\angle KLM = 180 - 90$$
$$m\angle KLM = 90^\circ$$

Angle KLM will also equal 90° because they are congruent linear angles.

Example 2

What is the measure of $\angle LHK$ below?



For now you can assume that the interior angles in a triangle must sum to 180° *(this is a fact that you have used in the past and we will prove it soon!).*

Since you may assume that the interior angles in a triangle must sum to 180°, you can find $m\angle LHK$ if you know the measures of the other two angles. Use the exterior right angle to find the measure of the interior angle adjacent to it. The two angles together are a linear pair. Since you know that the outer angle measures 90°, find the value of the internal angle using subtraction.

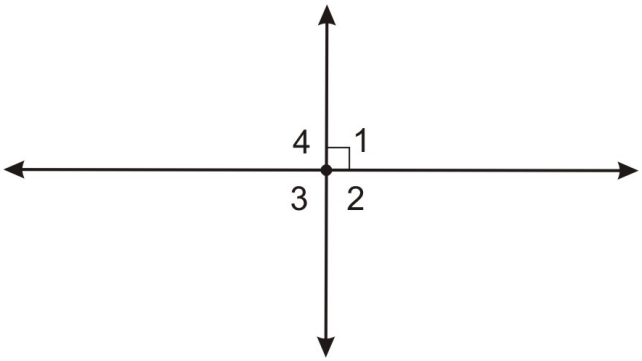
$$180 - 90 = 90$$

$m\angle LKH$ will also equal 90° because they are congruent linear angles. Now you know two of the internal angle measures in the triangle—40° and 90°. Use subtraction to find the measure of $\angle LHK$.

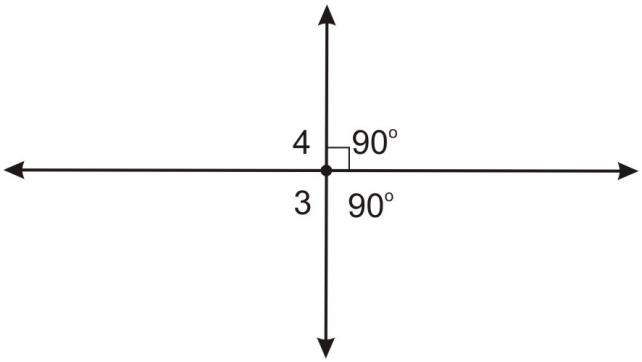
$$\begin{aligned} m\angle LHK + 40 + 90 &= 180 \\ m\angle LHK + 130 &= 180 \\ m\angle LHK + 130 - 130 &= 180 - 130 \\ m\angle LHK &= 50 \\ m\angle LHK &= 50^\circ \end{aligned}$$

Intersecting Perpendicular Lines

We can extend what we just said about linear pairs to all pairs of congruent supplementary angles: Congruent supplementary angles will always measure 90° each. When you have perpendicular lines, however, four different angles are formed.

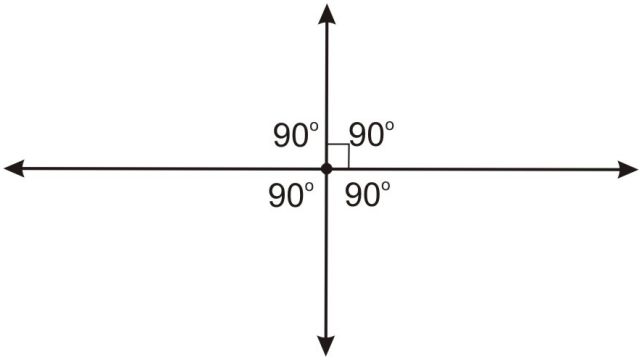


Think about what you just learned. If two angles are a linear pair, and one of them measures 90°, the other will also measure 90°. Fill in the measures of $\angle 1$ and $\angle 2$ in the diagram to show that they are both right angles.



Now think back to what you learned earlier in this chapter. Vertical angles are two angles

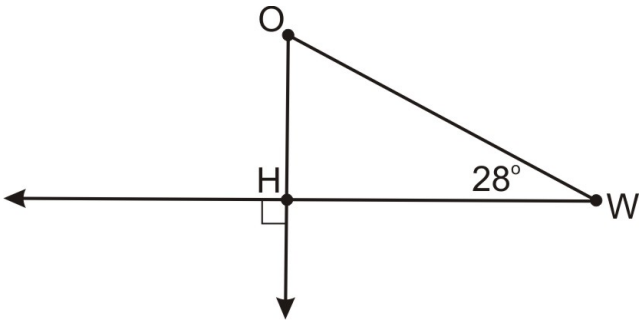
on opposite sides of intersecting lines. Applying the Vertical Angles Theorem, we know that vertical angles are also congruent. Using this logic you can prove that all four of the angles in this diagram are right angles.



Now that you know how to identify right angles formed by perpendicular lines, you can use this theorem for many different applications. Always be on the lookout for angles whose measures you know because of perpendicular lines.

Example 3

What is $m\angle O$ below?



Again you may assume that the interior angles in a triangle must sum to 180°

Since you know that the interior angles in a triangle must sum to 180°, you can find $m\angle O$ if you know the measures of the other two angles. Use the exterior right angle to find the measure of its interior angle. Since the intersecting lines form one right angle, all angles formed will measure 90°, and in particular, $m\angle WHO = 90°$. Now you know two of the internal angle measures in the triangle: 28° and 90°. Use subtraction to find $m\angle O$.

$$\begin{aligned} m\angle O + 28 + 90 &= 180 \\ m\angle O + 118 &= 180 \\ m\angle O + 118 - 118 &= 180 - 118 \\ m\angle O &= 62 \end{aligned}$$

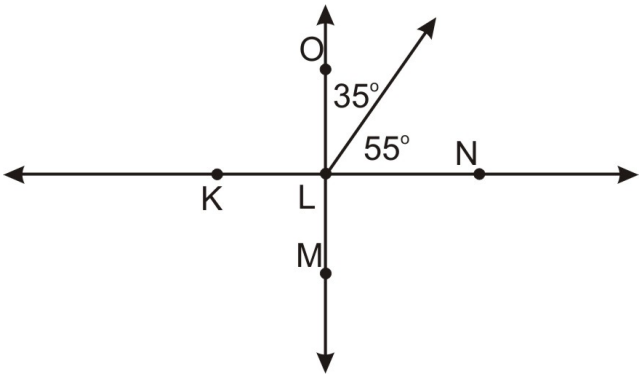
$\angle O$ measures 62°

Adjacent Complementary Angles

Remember that complementary angles are angles that sum to 90°. If complementary angles are adjacent, they form perpendicular rays. You can then apply everything you have learned about perpendicular lines to the situation to find missing angle values.

Example 4

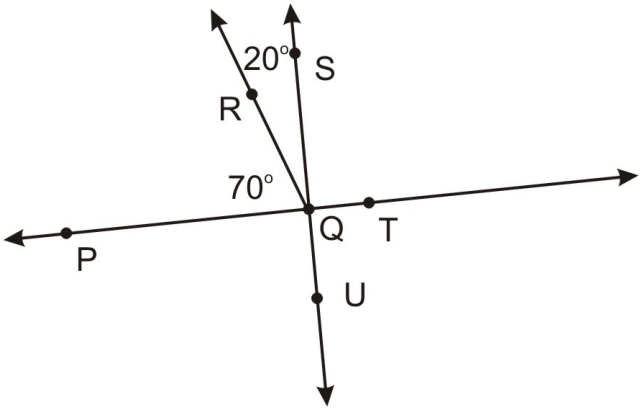
What is the measure of $\angle MLK$ in the diagram below?



$35° + 55° = 90°$, so the two angles in the upper right are complementary and sum to 90°. Since $m\angle OLN$ is 90°, its vertical angle will also measure 90°. Therefore $m\angle MLK = 90°$.

Example 5

What is the measure of $\angle TQU$ in the diagram below?



Not to scale

Because the diagram is not to scale, you cannot find the measure just by looking at this diagram. $\angle PQR$ and $\angle RQS$ measure 70° and 20° respectively. Add these two angles to find

their sum.

$$70 + 20 = 90$$

The two angles are complementary. Since $\angle TQU$ is vertical to the right angle, it must also be a right angle. So $m\angle TQU = 90°$.

Lesson Summary

In this lesson, we explored perpendicular lines. Specifically, we have learned:

- The properties of congruent angles that form a linear pair.
- How to identify the angles formed by perpendicular intersecting lines.
- How to identify complementary adjacent angles.

These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply concepts perpendicular lines to mathematical situations.

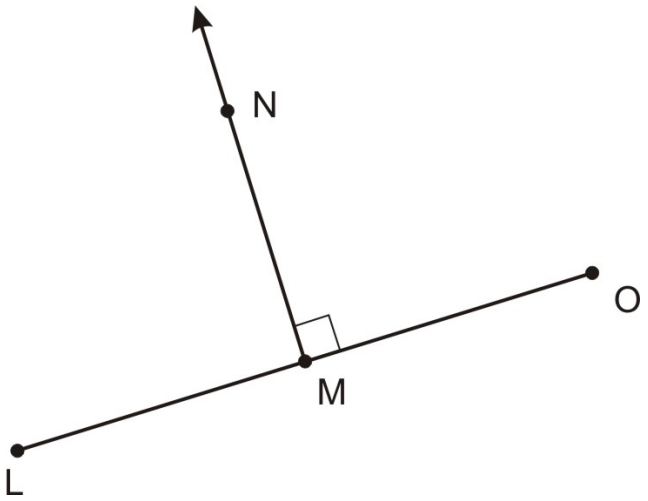
Points to Consider

Now that you understand perpendicular lines, you are going to take a closer look at perpendicular transversals.

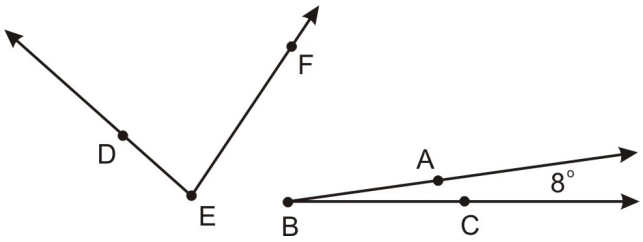
Review Questions

Solve each problem.

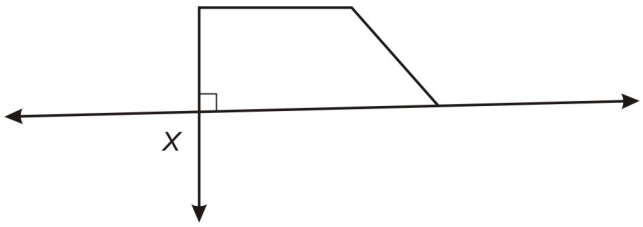
1. What is the measure of $\angle LMN$ in the diagram below?



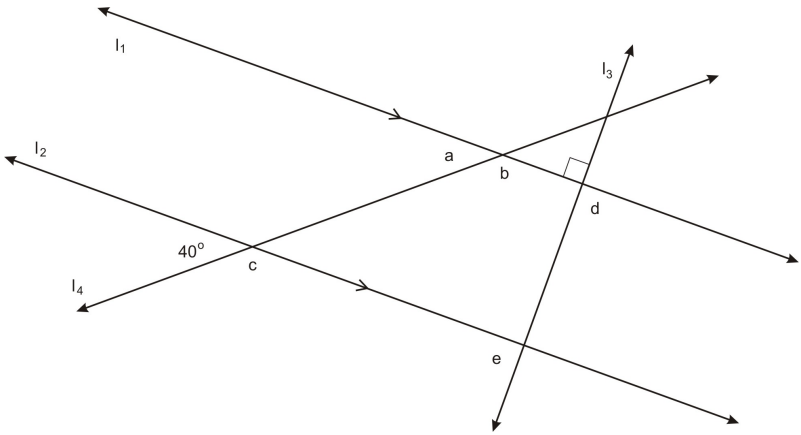
2. $\angle B$ and $\angle E$ below are complementary. What is the measure of $\angle E$?



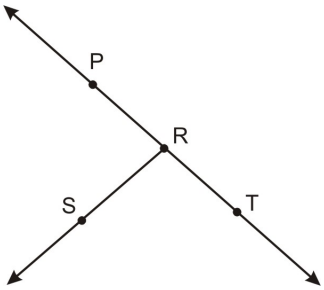
3. If x represents the measure of the angle, what is x in the diagram below?



4-8. Given $l_1 || l_2$ and $l_3 \perp l_1$, find the measure of each angle in the diagram below.



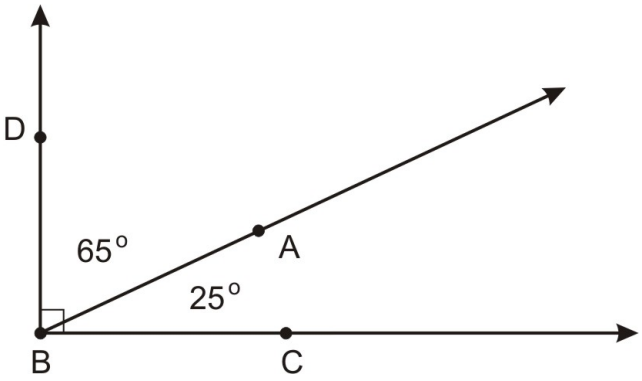
4. $a =$
5. $b =$
6. $c =$
7. $d =$
8. $e =$
9. Sketch and label a pair of adjacent complementary angles.
10. If two angles are complementary and congruent, what is the measure of each angle? Write one or two sentences to convince a reader that your answer is correct.
11. Given $\angle PRS \cong \angle SRT$ in the diagram below, find the measure of each angle. (You may assume $\overleftrightarrow{PT}$ is a line.)



12. Explain in why you know your answer to 11 is true. Write one or two sentences that will convince a reader your answer is correct.

Review Answers

1. $90°$
2. $82°$
3. $90°$
4. $a = 40°$
5. $b = 140°$
6. $c = 140°$
7. $d = 90°$
8. $e = 90°$
9. Answers will vary. One possible solution is below. Note to emphasize that students should use the right angle symbol in their sketch.



10. Each angle will measure $45°$. If two angles are complementary, then the sum of their measures is $90°$. Since the angles are congruent, they must have the same measure, $90° \div 2 = 45°$
11. $m\angle PRS = m\angle SRT = 90°$
12. This is similar to number 10. The two angles form a linear pair so they are supplementary. $m\angle PRS + m\angle SRT = 180°$. Since we know $\angle PRS \cong \angle SRT$, we can conclude that each angle measures $180° \div 2 = 90°$

3.7 Perpendicular Transversals

Learning Objectives

- Identify the implications of perpendicular transversals on parallel lines.
- Identify the converse theorems involving perpendicular transversals and parallel lines.
- Understand and use the distance between parallel lines.

Introduction

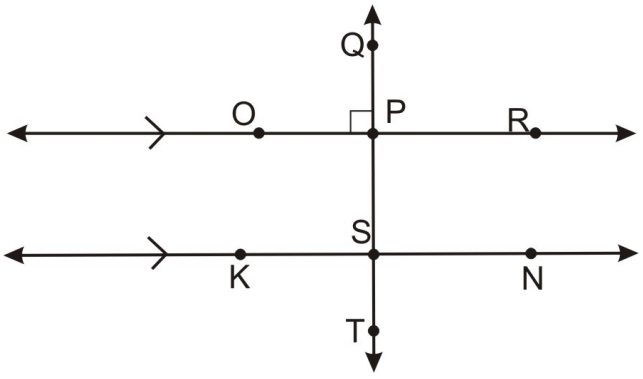
In the last lesson, you learned about perpendicular intersections. You know that when two lines are perpendicular they form four right (90°) angles. This lesson combines your knowledge of perpendicular lines with your knowledge of parallel lines and transversals.

Perpendicular Transversals and Parallel lines

When two lines are cut by a transversal, a number of special angles are formed. In previous lessons, you learned to identify corresponding angles, alternate interior angles, alternate exterior angles, and consecutive interior angles. You learned that the lines crossed by the transversal are parallel if and only if corresponding angles are congruent. Likewise alternate interior and alternate exterior angles are congruent and interior angles on the same side of the transversal are supplementary if the lines crossed by the transversal are parallel. When the transversal is perpendicular, something interesting happens with these angles. Observe the angle measures in the example below.

Example 1

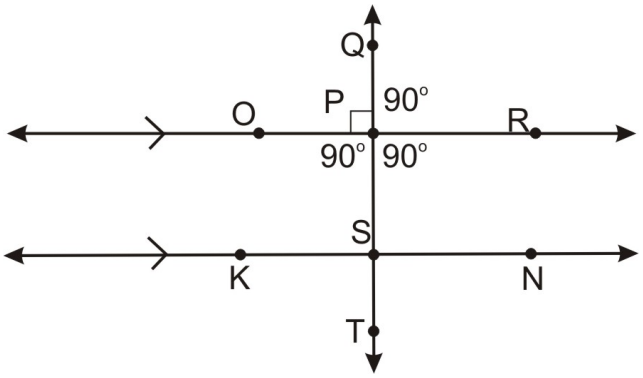
Lines $\overleftrightarrow{OR}$ *and* $\overleftrightarrow{KN}$ *are parallel and* $\overleftrightarrow{QT} \perp \overleftrightarrow{OR}$.



What is $m\angle TSN$?

Since you know that line $\overleftrightarrow{QT}$ is perpendicular to line $\overleftrightarrow{OR}$, you can fill in the four right angles

at that intersection.



The angle that corresponds to $\angle TSN$ is a right angle. This is true because you know lines $\overleftrightarrow{OR}$ and $\overleftrightarrow{KN}$ are parallel. Thus, the corresponding angles must be congruent. So $\angle TSN$ is a right angle. It measures 90°.

Notice in this example that if $\angle QPO$ is a right angle, then all of the angles formed by the intersection of lines $\overleftrightarrow{OR}$ and $\overleftrightarrow{QT}$ are right angles. Lines $\overleftrightarrow{KN}$ and $\overleftrightarrow{QT}$ are perpendicular as well. This is a result of the Corresponding Angles Postulate.

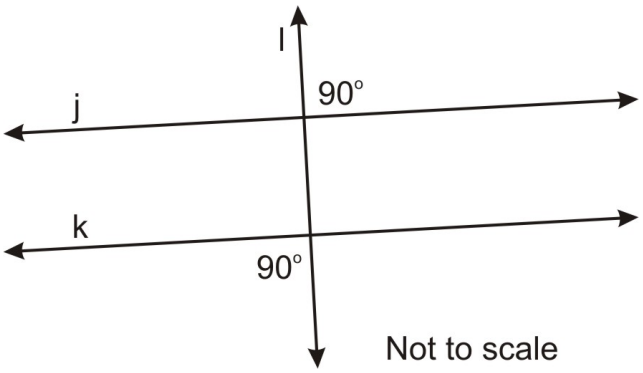
As in previous problems involving parallel lines crossed by a transversal, all pairs of angles remain either congruent or supplementary. When dealing with perpendicular lines, however, all of the angles are right angles.

Converse Theorem with Perpendicular Transversals

When examining the scenario of a perpendicular transversal with parallel lines, a converse theorem can be applied. The converse statement says that if a transversal forms right angles on two different coplanar lines, those two lines are parallel. Think back to the converse theorems you studied earlier in this chapter. They stated that if corresponding angles were congruent, consecutive interior angles were supplementary, or other specific relationship, then the two lines were parallel. Use this converse theorem to understand different graphic situations.

Example 2

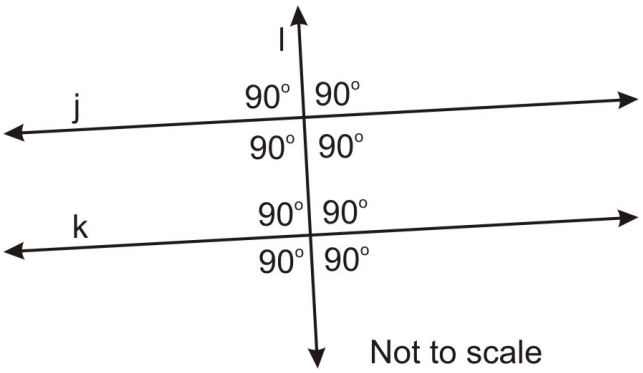
Line l below is a transversal, cutting through lines k and j.



Note: Figure NOT to scale

Are lines j and k parallel?

First, notice that the diagram is labeled "not to scale." Do not make your decision based on how this diagram looks. Remember that if one angle at an intersection measures 90°, all four angles measure 90°. Fill in the angle measures you can identify with this information.



Since the corresponding angles are all 90°, these two lines are parallel. The transversal is perpendicular to both lines j and k, so they must be parallel.

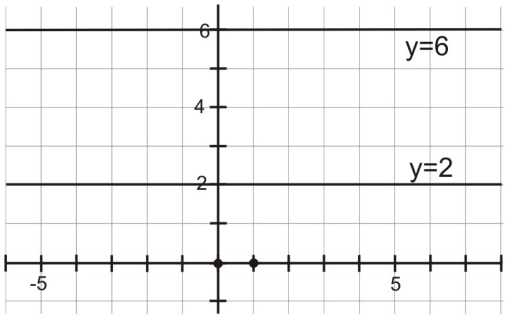
Distance Between Parallel Lines

When we talk about the distance between two points, what we are really talking about is the shortest or most direct distance between those two points. On paper, you can use a ruler or a taut string to find the distance between two points. When you measure the distance between a line and a point, the most direct path from a point to a line is always measured along the perpendicular from the point to the line.

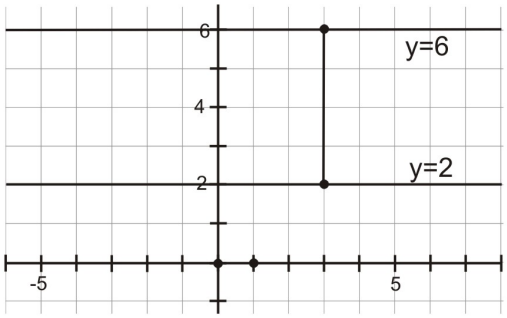
Similarly, sometimes you might be asked to find the distance between two parallel lines. When you need to find this value, you need to find the length of a perpendicular segment that connects the two lines. Remember that if a line is perpendicular to one parallel line, it is perpendicular to both of them. Let's look at an example on a coordinate grid.

Example 3

What is the distance between the two lines shown on the grid below?



In this image, the distance is found by drawing a perpendicular segment on the graph and calculating its length.

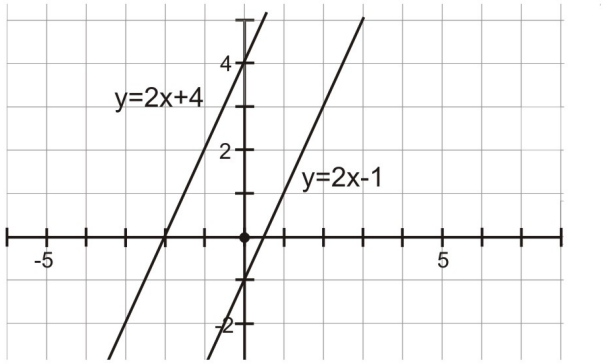


Because this line segment rises 4 units and has no run, it is 4 units long. The distance between the two lines on the graph is 4 units.

Problems involving distances between lines will not always be this straightforward. You may have to use other skills and tools to solve the problem. The following example shows how you can apply tools from algebra to find the distance between two "slanted" parallel lines.

Example 4

What is the distance between the lines shown on the graph?

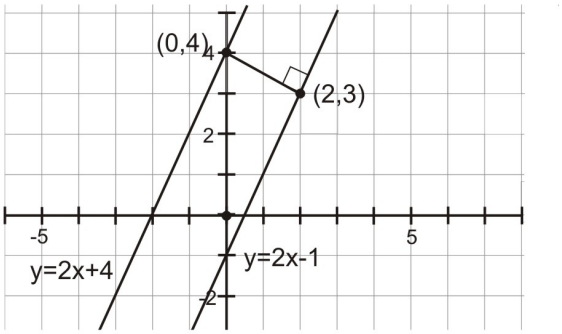


To start, you may think that you can just find the distance on the $y-$axis between the two lines. That distance is 5 units. However, this is not correct, as the shortest distance between two lines will be a perpendicular segment between them. The $y-$axis is perpendicular to neither line.

You'll need to draw a perpendicular segment and calculate its length. To begin, identify the slope of the lines in the diagram. You can then identify the slope of a perpendicular because the slopes will be opposite reciprocals. To find the slope of a line, use the formula to calculate. Pick two points on one of the lines—here we will use $(0, 4)$ and $(2, 8)$ from the line $y = 2x + 4$.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$
$$\text{slope} = \frac{(8 - 4)}{(2 - 0)}$$
$$\text{slope} = \frac{4}{2}$$
$$\text{slope} = 2$$

The slope of the parallel lines in the diagram is 2. The slope of the perpendicular segment will be the opposite reciprocal of 2. The reciprocal of 2 is $\frac{1}{2}$ and the opposite of $\frac{1}{2}$ is $-\frac{1}{2}$. The slope of the perpendicular line is $-\frac{1}{2}$. Pick a point on the line and use the slope to draw a perpendicular segment. Remember that the line will go down 1 unit for every 2 units you move the right.



Notice that there are points where the perpendicular segment intersects both parallel lines. It intersects the top line at $(0, 4)$ and the bottom line at $(2, 3)$. You need to find the length of this segment, and you know two points. You can use the distance formula which you learned in algebra and which we briefly reviewed in Chapter 1. Substitute the $x-$ and $y-$coordinates from these points into the formula, and you'll have the distance between the two parallel lines.

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d &= \sqrt{(2 - 0)^2 + (3 - 4)^2} \\
d &= \sqrt{(2)^2 + (-1)^2} \\
d &= \sqrt{4 + 1} \\
d &= \sqrt{5}
\end{aligned}
$$

The distance between the two lines is $\sqrt{5}$, or about 2.24 units.

Lesson Summary

In this lesson, we explored perpendicular transversals. Specifically, we have learned:

- How to identify the implications of perpendicular transversals on parallel lines.
- How to identify the converse theorems involving perpendicular transversals and parallel lines.
- To understand and use the distance between parallel lines.

These will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply the concepts of perpendicular transversals to new mathematical situations.

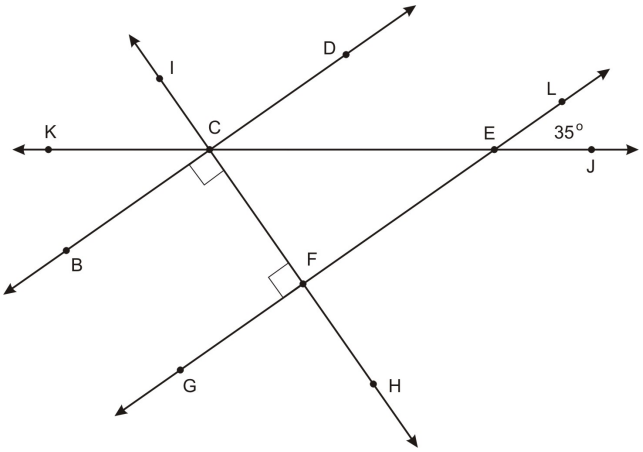
Points to Consider

Finding the distance from a point to a line and the distance between two lines are two good examples of ways to apply skills you learned in algebra, such as finding slopes and using the distance formula to geometry problems. Even when geometric problems are given without a coordinate system, you can often define a convenient coordinate system to help you solve the problem.

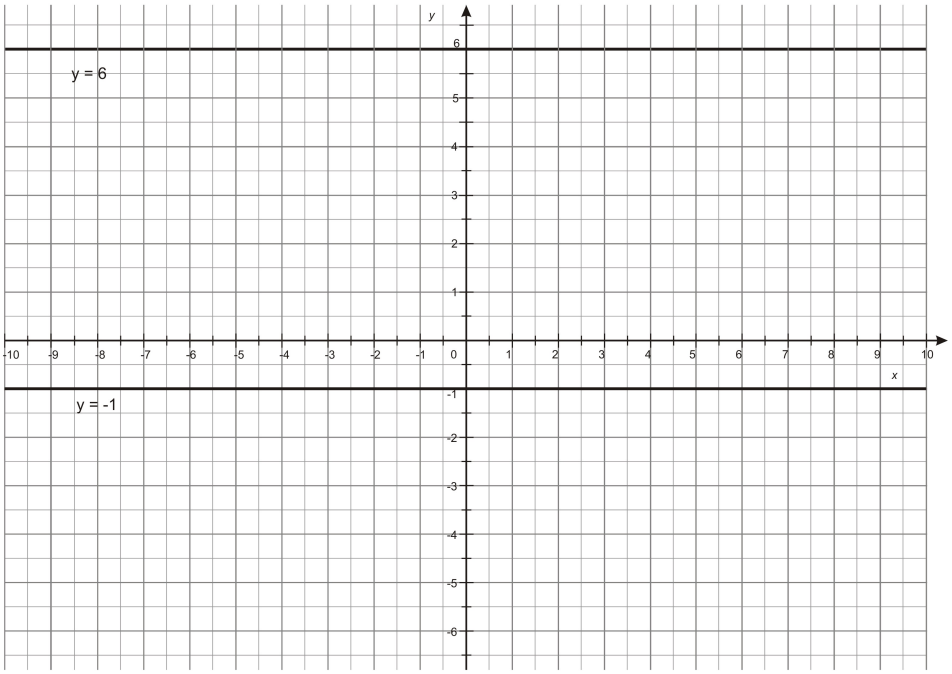
Review Questions

Solve each problem.

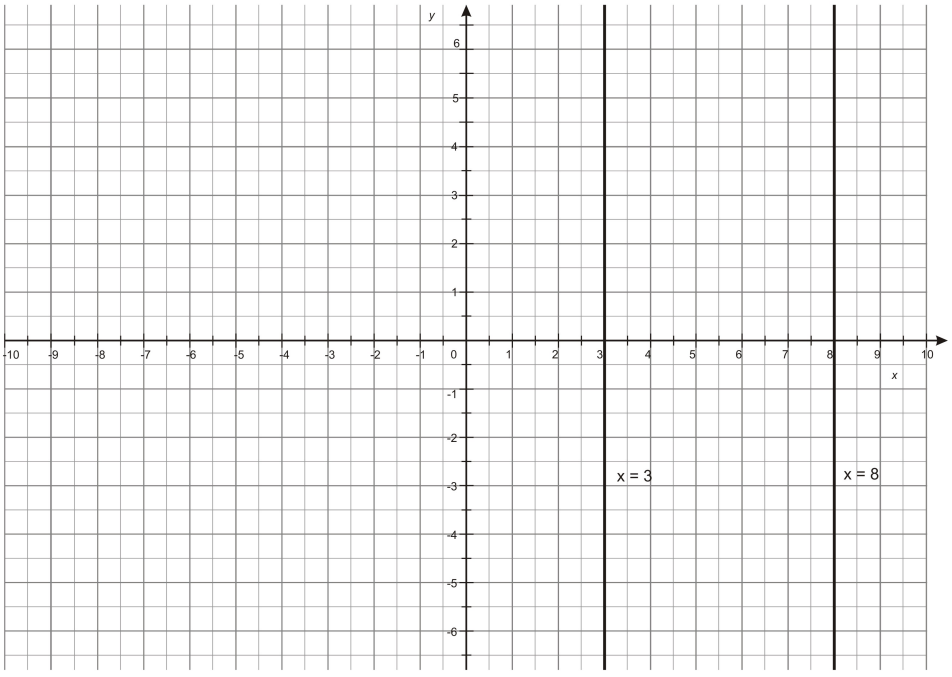
Use the diagram below for questions 1 and 2. $\angle BCF$ and $\angle CFG$ are right angles.



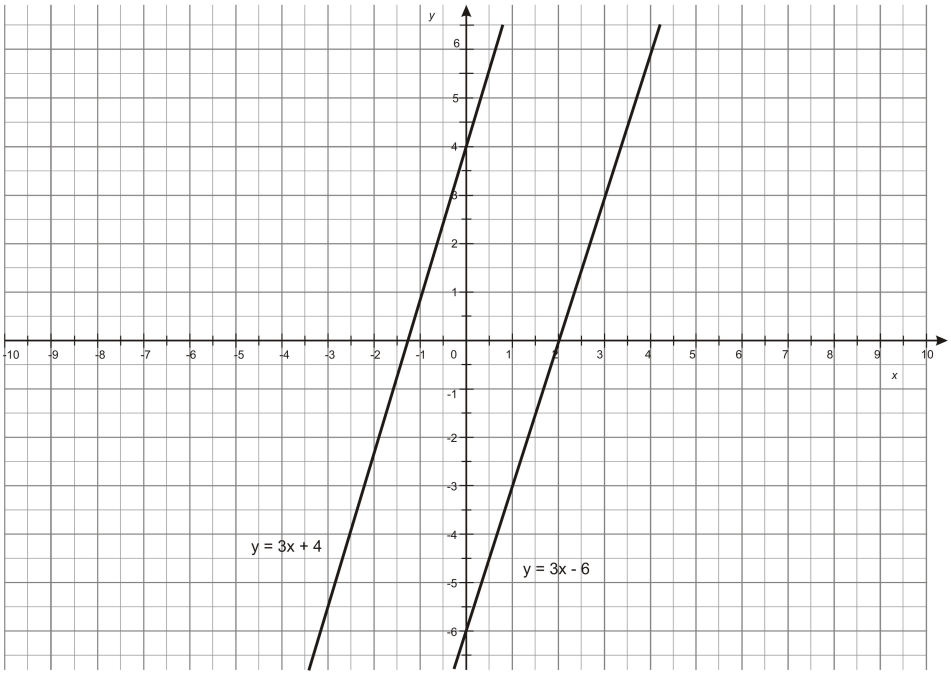
1. What is the relationship between $\overleftrightarrow{BC}$ and $\overleftrightarrow{GF}$? How do you know?
2. What is $m\angle HFE$? How do you know?
3. What is $m\angle BCK$? How do you know?
4. What is $m\angle KCI$? How do you know?
5. What is $m\angle BCE$? How do you know?
6. What is the distance between the two lines on the grid below?



7. What is the distance between the two lines on the grid below?



8–10: Here we will "walk through" the process of finding the distance between two lines, $y = 3x + 4$ and $y = 3x - 6$.

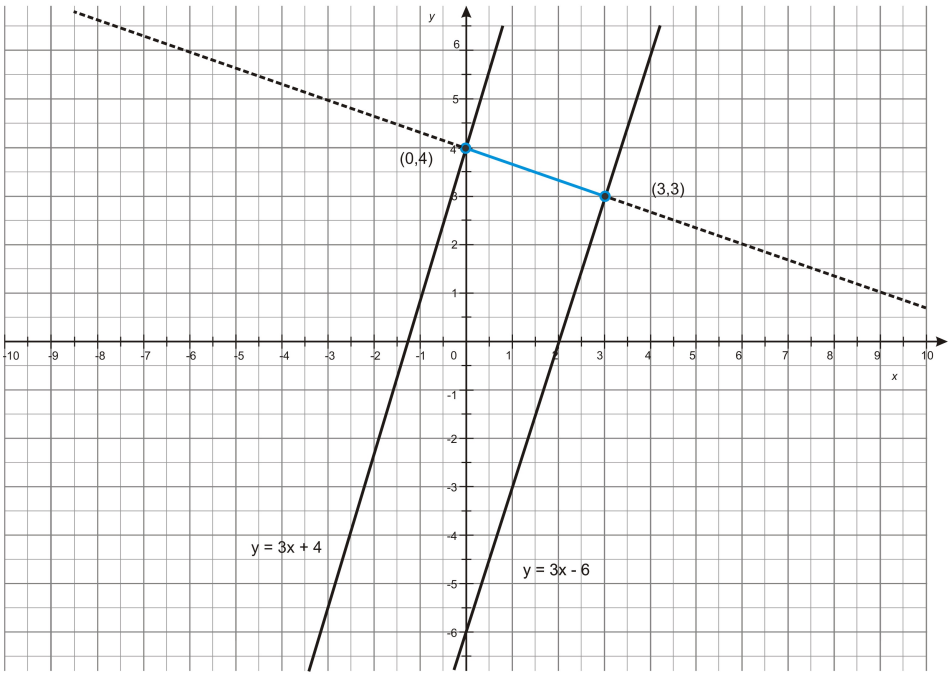


8. What is the slope of these lines? How do you know?
9. What is the slope of a line perpendicular to these lines? How do you know?
10. Draw the line perpendicular to $y = 3x + 4$ that passes through the $y-$intercept, $(0, 4)$.
11. Where does that line intersect $y = 3x - 6$?
12. Find the distance between $(0, 4)$ and the point you found in 11. This is the distance between the two lines.
13. Now solve one on your own: Find the distance between $y = -\frac{1}{2}x + 8$ and $y = -\frac{1}{2}x + 3$.

Review Answers

1. $\overleftrightarrow{BC} || \overleftrightarrow{GF}$.The converse theorem with perpendicular transversals
2. 90°. This angle is a vertical angle with a right angle
3. 35°. It is an alternate exterior angle with $\angle LEJ$
4. 55°. $\angle KCI$ is the complement of $\angle LEJ$
5. 145°. Use the angle addition postulate: $m\angle BCF = 90°$ and $m\angle FCE = 55°$
6. a
7. 4 units
8. The slope of each line is 3. You can find it by looking at the coefficient of x, or by finding two points on each line and computing the slope
9. The slope of the perpendicular line is $-\frac{1}{3}$
10. See dashed line in the image below. The equation of the line is $y = -\frac{1}{3}x + 4$

11. The second intersection is at $(3, 3)$
12. The distance is $\sqrt{10} \approx 3.16$.



13. The distance is $\sqrt{20} \approx 4.47$ units

3.8 Non-Euclidean Geometry

Learning Objectives

- Understand non-Euclidean geometry concepts.
- Find taxicab distances.
- Identify and understand taxicab circles.
- Identify and understand taxicab midpoints.

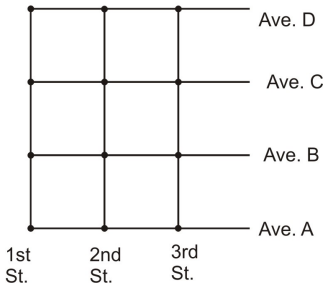
Introduction

What if we changed the rules of a popular game? For example, what if batters in baseball got five strikes instead of three? How would the game be different? How would it be the same? Up to this point, you have been studying what is called Euclidean geometry. Based on the work of the Greek mathematician Euclid, this type of geometry is based on the assumption that given a line and a point not on the line, there is only one line through that point parallel to the given line (this was one of our postulates). What if we changed

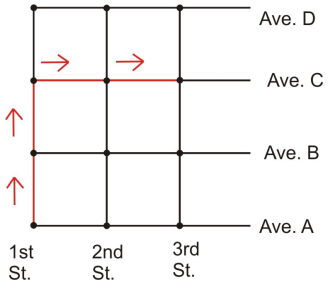
that rule? Or what if we changed another rule (such as the ruler postulate)? What would happen? **Non-Euclidean** geometry is the term used for all other types of geometric study that are based on different rules than the rules Euclid used. It is a large body of work, involving many different types of theories and ideas. One of the most common introductions to non-Euclidean geometry is called **taxicab geometry.** That will be the principal focus of this lesson. There are many other types of non-Euclidean geometry, such as spherical and hyperbolic geometry that are useful in different contexts.

Basic Concepts

In previous lessons, you have learned to find distances on a plane, and that the shortest distance between two points is always along a straight line connecting the two points. This is true when dealing with theoretical situations, but not necessarily when approaching real-life scenarios. Examine the map below.



Imagine that you wanted to find the distance you would cover if you walked from the corner of 1^{st} and A to the corner of 3^{rd} and C. Using the kind of geometry you have studied until now, you would draw a straight line and calculate its distance.



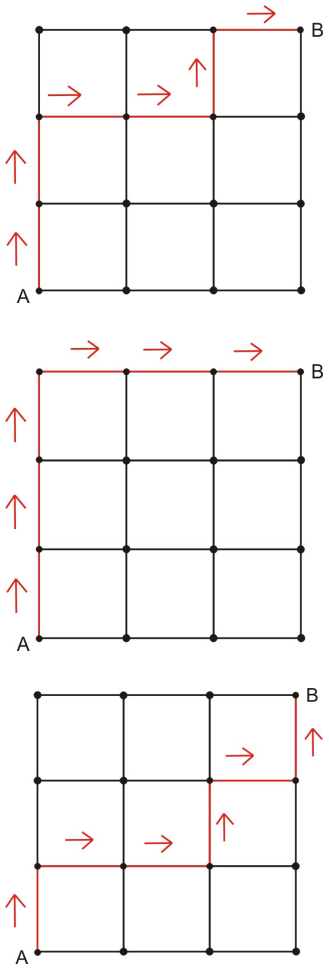
But to walk the route shown, you would have to walk through buildings! As that isn't possible, you will have to walk on streets, working your way over to the other corner. This route will be longer, but since walking through a building is not an option, it is the only choice.

Our everyday world is not a perfect plane like the $xy-$coordinate grid, so we have developed language to describe the difference between an ideal world (like the $xy-$plane) and our real

world. For example, the direct line between two points is often referred to by the phrase "as the crow flies," talking about if you could fly from one point to another regardless of whatever obstacles lay in the path. When referring to the real-world application of walking down different streets, mathematicians refer to taxicab geometry. In other words, taxicab geometry represents the path that a taxi driver would have to take to get from one point to another. This language will help you understand when you should use the theoretical geometry that you have been practicing and when to use taxicab geometry.

Taxicab Distance

Now that you understand the basic concepts that separate taxicab geometry from Euclidean geometry, you can apply them to many different types of problems. It may seem daunting to find the correct path when there are many options on a map, but it is interesting to see how their distances relate. Examine the diagram below.



Each of the drawings above show different paths between points A and B. Take a moment

to calculate the length (in units) of each path.

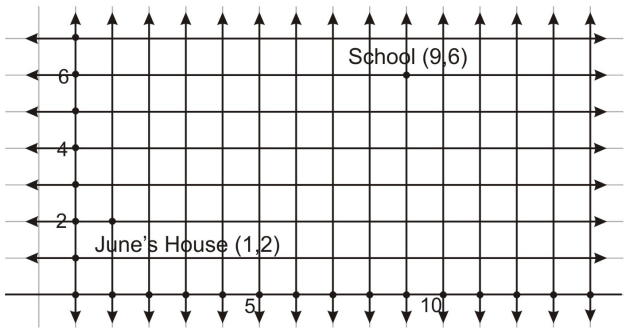
Path 1: 6 units
Path 2: 6 units
Path 3: 6 units

Each of those distances is equal, even though the paths are different. The point is that the shortest distance between A and B is 3 units to the right and 3 units up. Since $3 + 3 = 6$, this is consistent with the findings above. What you can learn from this is that it doesn't matter the order in which you move up or over. As long as you do not backtrack, the length will always be the same.

Note that the taxicab geometry system looks familiar—in fact it is the same as the $xy-$coordinate grid system, with the added rule that you can only travel up and down or right and left.

Example 1

In the grid below, each vertical and horizontal line represents a street on a map. The streets are evenly spaced.



Scale: 1 unit = 100ft

June rides her bike from home to school each day along the roads in her town. How far, in feet, does June ride her bike to get to school?

This is a taxicab geometry question, as June only rides her bike on the streets. Count how many units to the right June travels—8 units. Now count how may units up June travels—4 units. Add these two values.

$$8 + 4 = 12$$

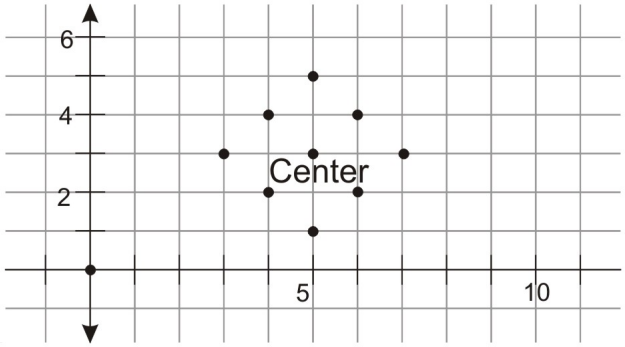
June travels 12 units to get to school. Because the scale shows that 1 unit is equal to 100 feet, you can calculate the distance in feet.

$$12 \text{ units} \times \frac{100 \text{ feet}}{1 \text{ unit}} = 1,200 \text{ feet}$$

June rides her bike 1,200 feet to school.

Taxicab Circles

From your previous work in geometry you should already know the definition of a circle—a circle is the set of points equidistant from a center point. Taxicab "circles" look a little different. Imagine selecting a point on a grid and finding every point that was 2 units away from it using taxicab geometry. The result is as follows.

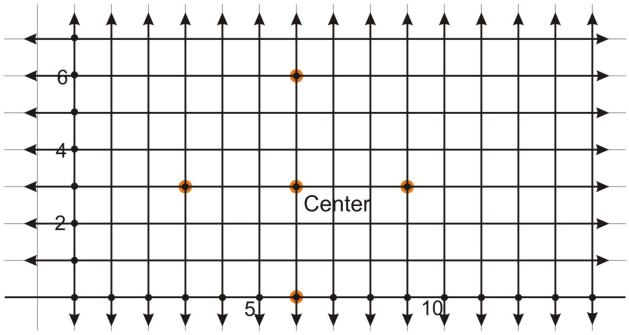


Use logic to work through problems involving taxicab circles. If you work carefully and slowly, you should be able to find the desired answer.

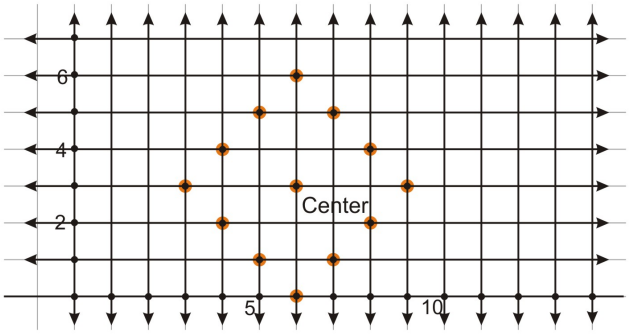
Example 2

A passenger in a taxi wants to see how many distinct points she could visit if a cab travels exactly three blocks from where she is standing without turning around. Count the points and draw the taxicab circle with a radius of 3 units.

Start with a coordinate grid with a point in the middle. Count 3 units straight in each direction and mark the points that result.



Now fill in the other points that involve a combination of moving up, down, and over.



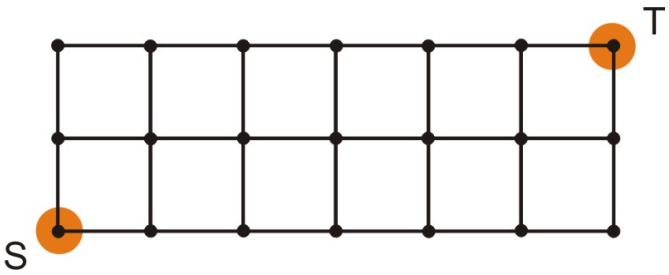
Count the points to find that there are 12 distinct points on the circle.

Taxicab Midpoints

Much like finding taxicab distances, you can also identify taxicab midpoints. However, unlike traditional midpoints, there may be more than one midpoint between two points in taxicab geometry. To find a taxicab midpoint, trace a path between the given paths along the roads, axes, or lines. Then, divide the distance by 2 and count that many units along your path. This results in identifying a taxicab midpoint. As you will see, there may be more than one midpoint between any two points.

Example 3

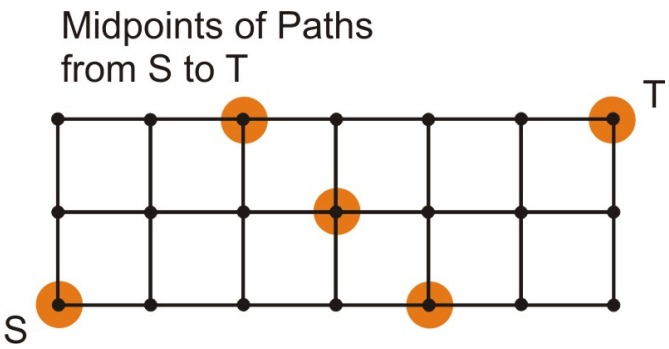
Find the taxicab midpoints between S and T in the diagram below.



Start by finding the taxicab distance ST. You will have to travel 6 units to the right and 2 units up. Add these two values to find the distance.

$$6 + 2 = 8$$

The taxicab distance, ST is 8 units. Use the diagram and identify how many points are 4 units away from S and T. These will be the taxicab midpoints.



There are three taxicab midpoints in this scenario, shown in the diagram above.

Now we have seen two major differences between taxicab geometry and Euclidean geometry. Based on a new definition of the "distance between two points" in taxicab geometry, the look of a circle has changed, and one of the fundamental postulates about the midpoint has changed.

These short examples illustrate how one small change in the rules results in different rules for many parts of the taxicab geometry system.

Other non-Euclidean geometries apply to other situations, such as navigating on the globe, or finding the shape of surfaces on bubbles. All of these different systems of geometry follow postulates and definitions, but by changing a few key rules (either the postulates or definitions) the entire system changes. For example, in Taxicab geometry we see that by changing the definition of "the distance between two points" we also changed the meaning of midpoint.

Lesson Summary

In this lesson, we explored one example of non-Euclidean geometry. Specifically, we have learned:

- Where non-Euclidean geometry concepts come from.
- How to find taxicab distances.
- How to identify and understand taxicab circles.
- How to identify and understand taxicab midpoints.

These will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply concepts of non-Euclidean geometry to mathematical situations.

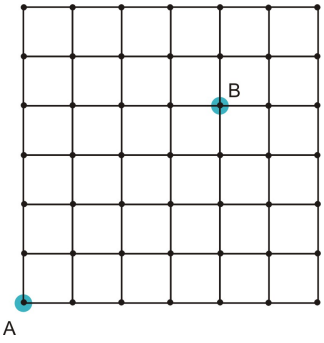
Points To Consider

Now that you understand lines and angles, you are going to learn about triangles and their special relationships.

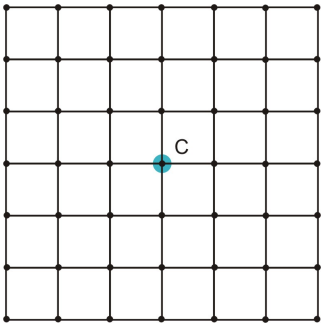
Review Questions

Solve each problem.

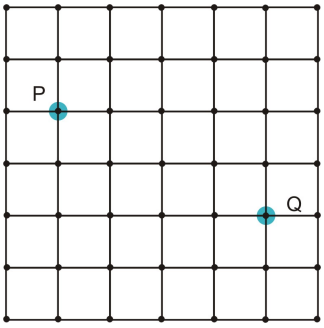
1. What is the taxicab distance between points A and B in the diagram below?



2. Draw a taxicab circle on the diagram below with a radius of 3 units.



3. What is the taxicab distance between P and Q in the diagram below?

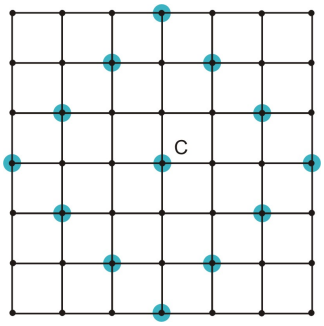


4. Draw one of the taxicab midpoints between P and Q on the diagram above.
5. How many taxicab midpoints will there be between P and Q? How do you know you have found them all?

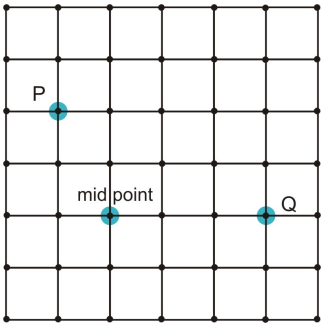
6. What is the taxicab distance between points $(3, 5)$ and $(22, 9)$ in a coordinate grid?
7. Is there a taxicab midpoint between $(3, 5)$ and $(22, 9)$? Why?
8. What are the coordinates of one of the taxicab midpoints between $(2, 5)$ and (10,1)?
9. How many taxicab midpoints will there be between points $(3, 10)$ and $(6, 7)$ on a coordinate grid? What are their coordinates?
10. If you know that two points have a taxicab distance of 12 between them, do you have enough information to tell how many taxicab midpoints there will be between those two points? Why or why not?
11. What are some similarities and differences between taxicab geometry and Euclidean geometry?

Review Answers

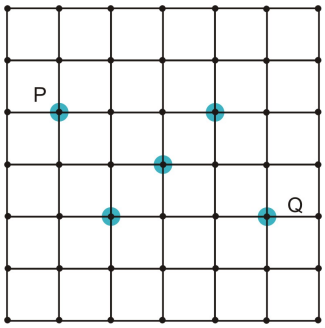
1. 8 units
2. See below:



3. 6 units
4. One possible answer:



5. Any of the following points are correct. There are a total of three midpoints, and by systematic checking there are no more.



6. 23 units
7. No, since the distance between the two points is odd, there are no midpoints as they happen in the "middle of a block"—or they are not at a point in the coordinate grid with integer coordinates.
8. Any of the following coordinate pairs is correct: $(4, 1), (5, 2), (6, 3), (7, 4)$, and $(8, 5)$
9. There are four midpoints between $(3, 10)$ and $(6, 7)$. The midpoints are at $(3, 7), (4, 8), (5, 9)$ and $(6, 10)$
10. No, there may be one midpoint if the two points have the same $x-$ or $y-$ coordinate, or as many as 6
11. Answers will vary, but some major ideas: In taxicab geometry all distances are integers, while in Euclidean geometry distances can be rational and real values. In Euclidean geometry there is only one midpoint of a segment, but in taxicab geometry there may be multiple midpoints for a segment. Both types of geometry use "lines" between points but in the case of taxicab geometry, lines must be vertical or horizontal (along the grid). One other interesting difference is that taxicab circles appear to be squares in Euclidean geometry

Image Sources

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Chapter 4

Congruent Triangles

4.1 Triangle Sums

Learning Objectives

- Identify interior and exterior angles in a triangle.
- Understand and apply the Triangle Sum Theorem.
- Utilize the complementary relationship of acute angles in a right triangle.
- Identify the relationship of the exterior angles in a triangle.

Introduction

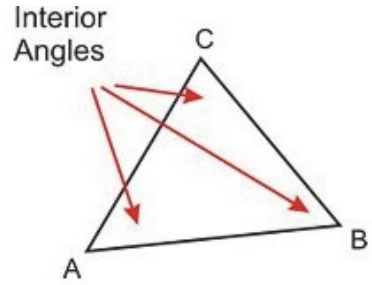
In the first chapter of this course, you developed an understanding of basic geometric principles. The rest of this course explores specific ideas, techniques, and rules that will help you be a successful problem solver. If you ever want to review the basic problem solving in geometry return to Chapter 1. This chapter explores triangles in more depth. In this lesson, you'll explore some of their basic components.

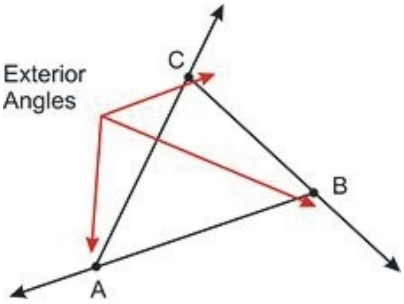
Interior and Exterior Angles

Any closed structure has an inside and an outside. In geometry we use the words **interior** and **exterior** for the inside and outside of a figure. An *interior* designer is someone who furnishes or arranges objects *inside* a house or office. An *external* skeleton (or exo-skeleton) is on the outside of the body. So the prefix "ex" means outside and exterior refers to the outside of a figure.

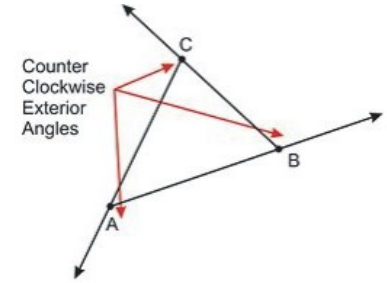
The terms interior and exterior help when you need to identify the different angles in tri-

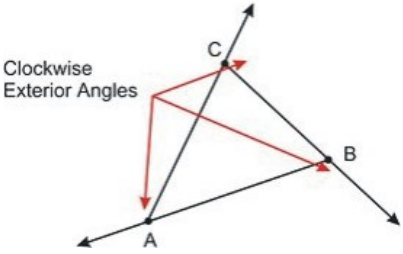
angles. The three angles inside the triangles are called **interior angles.** On the outside, **exterior angles** are the angles formed by extending the sides of the triangle. The exterior angle is the angle formed by one side of the triangle and the extension of the other.





Note: In triangles and other polygons there are TWO sets of exterior angles, one "going" clockwise, and the other "going" counterclockwise. The following diagram should help.



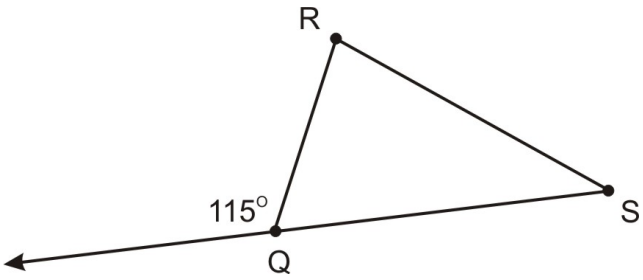


But, if you look at one vertex of the triangle, you will see that the interior angle and an exterior angle form a linear pair. Based on the Linear Pair Postulate, we can conclude that interior and exterior angles at the same vertex will always be supplementary. This tells us

that the two exterior angles at the same vertex are congruent.

Example 1

What is $m\angle RQS$ in the triangle below?



The question asks for $m\angle RQS$. The exterior angle at vertex $\angle RQS$ measures 115°. Since interior and exterior angles sum to 180°, you can set up an equation.

$$\begin{aligned} \text{interior angle} + \text{exterior angle} &= 180^\circ \\ m\angle RQS + 115 &= 180 \\ m\angle RQS + 115 - 115 &= 180 - 115 \\ m\angle RQS &= 65 \end{aligned}$$

Thus, $m\angle RQS = 65^\circ$.

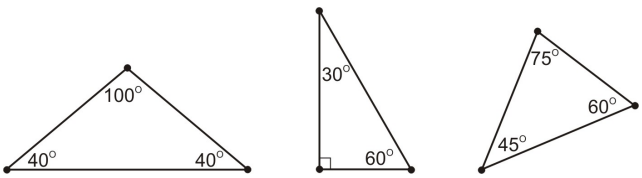
Triangle Sum Theorem

Probably the single most valuable piece of information regarding triangles is the **Triangle Sum Theorem.**

Triangle Sum Theorem

The sum of the measures of the interior angles in a triangle is 180°

Regardless of whether the triangle is right, obtuse, acute, scalene, isosceles, or equilateral, the interior angles will always add up to 180°. Examine each of the triangles shown below.



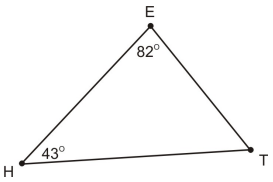
Notice that each of the triangles has an angle that sums to 180°.

$$100^\circ + 40^\circ + 40^\circ = 180^\circ$$
$$90^\circ + 30^\circ + 60^\circ = 180^\circ$$
$$45^\circ + 75^\circ + 60^\circ = 180^\circ$$

You can also use the triangle sum theorem to find a missing angle in a triangle. Set the sum of the angles equal to 180° and solve for the missing value.

Example 2

What is $m\angle T$ in the triangle below?



Set up an equation where the three angle measures sum to 180°. Then, solve for $m\angle T$.

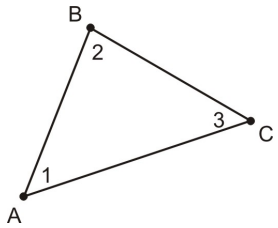
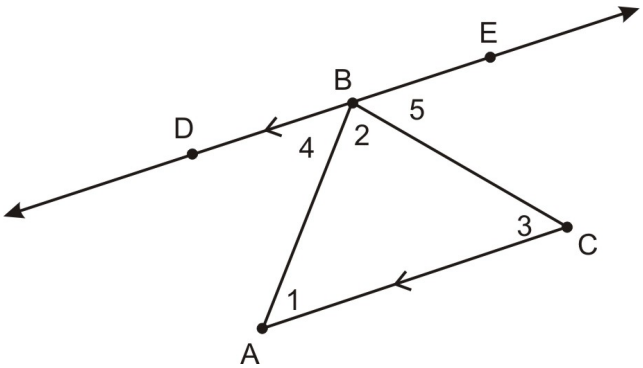
$$82 + 43 + m\angle T = 180$$
$$125 + m\angle T = 180$$
$$125 - 125 + m\angle T = 180 - 125$$
$$m\angle T = 55$$

$$m\angle T = 55^\circ$$

Now that you have seen an example of the triangle sum theorem at work, you may wonder, why it is true. The answer is actually surprising: The measures of the angles in a triangle add to 180° because of the Parallel line Postulate. Here is a proof of the triangle sum theorem.

- Given: $\triangle ABC$ as in the diagram below,
- Prove: that the measures of the three angles add to 180°, or in symbols, that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$.

Table 4.1:

Statement	Reason
1. Given $\triangle ABC$ in the diagram 	1. Given
2. Through point B, draw the line parallel to $\overline{AC}$. We will call it $\overleftrightarrow{BD}$.	2. Parallel Postulate
	
3. $\angle 4 \cong \angle 1$ ($m\angle 4 = m\angle 1$)	3. Alternate interior Angles Theorem
4. $\angle 5 \cong \angle 3$ ($m\angle 5 = m\angle 3$)	4. Alternate interior Angles Theorem
5. $m\angle 4 + m\angle 2 = m\angle DBC$	5. Angle Addition postulate
6. $m\angle DBC + m\angle 5 = 180°$	6. Linear Pair Postulate
7. $m\angle 4 + m\angle 2 + m\angle 5 = 180°$	7. Substitution (also known as "transitive property of equality")
8. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$	8. Substitution (Combining steps 3, 4, and 7).

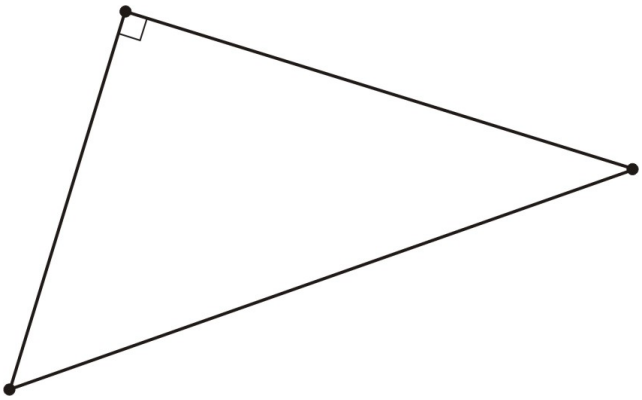
And that proves that the sum of the measures of the angles in ANY triangle is 180°.

Acute Angles in a Right Triangle

Expanding on the triangle sum theorem, you can find more specific relationships. Think about the implications of the triangle sum theorem on right triangles. In any right triangle, by definition, one of the angles is a right angle—it will always measure 90°. This means that the sum of the other two angles will always be 90°, resulting in a total sum of 180°.

Therefore the two acute angles in a right triangle will always be complementary and as one of the angles gets larger, the other will get smaller so that their sum is 90°.

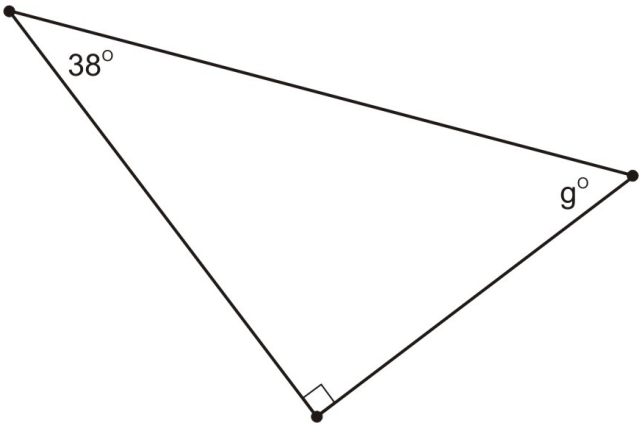
Recall that a right angle is shown in diagrams by using a small square marking in the angle, as shown below.



So, when you know that a triangle is right, and you have the measure of one acute angle, you can easily find the other.

Example 3

What is the measure of the missing angle in the triangle below?



Since the triangle above is a right triangle, the two acute angles must be complementary. Their sum will be 90°. We will represent the missing angle with the variable g and write an

equation.

$$38^\circ + g = 90^\circ$$

Now we can use inverse operations to isolate the variable, and then we will have the measure of the missing angle.

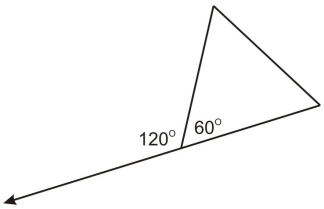
$$\begin{aligned} 38 + g &= 90 \\ 38 + g - 38 &= 90 - 38^\circ \\ g &= 52 \end{aligned}$$

The measure of the missing angle is 52°.

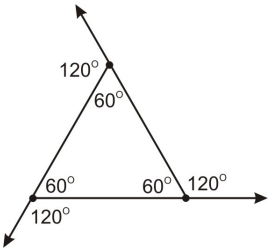
Exterior Angles in a Triangle

One of the most important lessons you have learned thus far was the triangle sum theorem, stating that the sum of the measure of the interior angles in any triangle will be equal to 180°. You know, however, that there are two types of angles formed by triangles: interior and exterior. It may be that there is a similar theorem that identifies the sum of the exterior angles in a triangle.

Recall that the exterior and interior angles around a single vertex sum to 180°, as shown below.



Imagine an equilateral triangle and the exterior angles it forms. Since each interior angle measures 60°, each exterior angle will measure 120°.



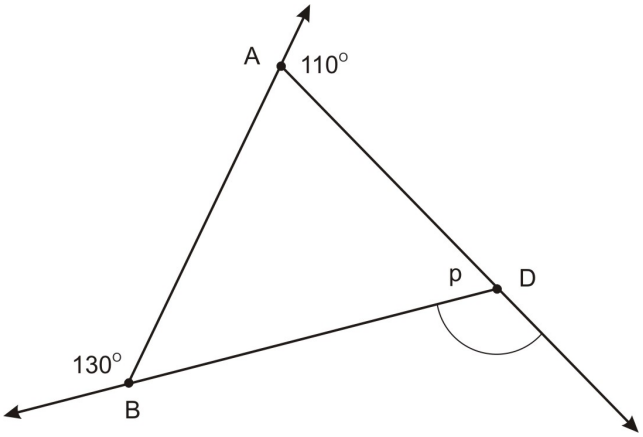
What is the sum of these three angles? Add them to find out.

$$120^\circ + 120^\circ + 120^\circ = 360^\circ$$

The sum of these three angles is 360°. In fact, the sum of the exterior angles in any triangle will always be equal to 360°. You can use this information just as you did the triangle sum theorem to find missing angles and measurements.

Example 4

What is the value of p in the triangle below?



You can set up an equation relating the three exterior angles to 360°. Remember that p does not represent an exterior angle, so do not use that variable. Solve for the value of the exterior angle. Let's call the measure of the exterior angle e.

$$\begin{aligned} 130^\circ + 110^\circ + e &= 360^\circ \\ 240^\circ + e &= 360^\circ \\ 240^\circ + e - 240^\circ &= 360^\circ - 240^\circ \\ e &= 120^\circ \end{aligned}$$

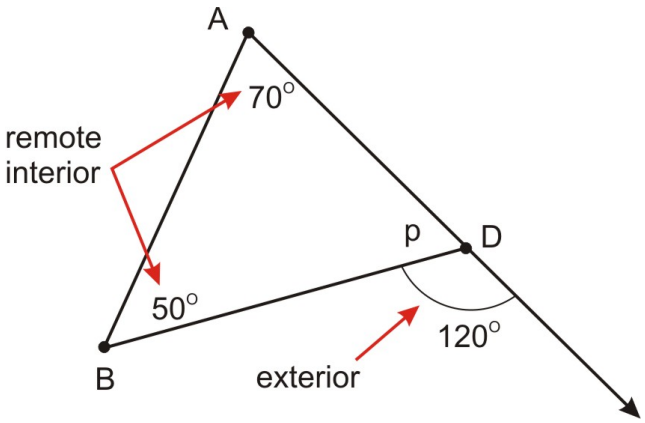
The missing exterior angle measures 120°. You can use this information to find the value of p, because the interior and exterior angles form a linear pair and therefore they must sum to 180°.

$$\begin{aligned} 120^\circ + p &= 180^\circ \\ 120^\circ + p - 120^\circ &= 180^\circ - 120^\circ \\ p &= 60^\circ \end{aligned}$$

Exterior Angles in a Triangle Theorem

In a triangle, the measure of an exterior angle is equal to the sum of the remote interior angles.

We won't prove this theorem with a two-column proof (that will be an exercise), but we will use the example above to illustrate it. Look at the diagram from the previous example for a moment. If we look at the exterior angle at D, then the interior angles at A and B are called "remote interior angles."



Notice that the exterior angle at point D measured 120°. At the same time, the interior angle at point A measured 70° and the interior angle at B measured 50°. The sum of interior angles $m\angle A + m\angle B = 70° + 50° = 120°$. Notice the measures of the remote interior angles sum to the measure of the exterior angle at D. This relationship is always true, and it is a result of the linear pair postulate and the triangle sum theorem. Your job will be to show how this works.

Lesson Summary

In this lesson, we explored triangle sums. Specifically, we have learned:

- How to identify interior and exterior angles in a triangle.
- How to understand and apply the Triangle Sum Theorem
- How to utilize the complementary relationship of acute angles in a right triangle.
- How to identify the relationship of the exterior angles in a triangle.

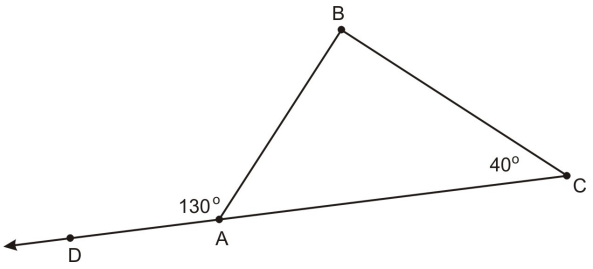
These skills will help you understand triangles and their unique qualities. Always look for triangles in diagrams, maps, and other mathematical representations.

Points to Consider

Now that you understand the internal qualities of triangles, it is time to explore the basic concepts of triangle congruence.

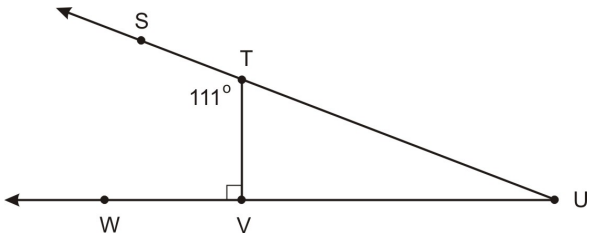
Review Questions

Questions 1 and 2 use the following diagram:

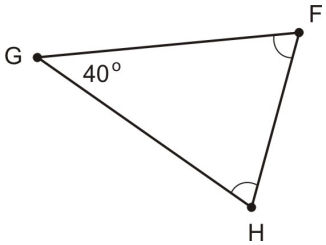


1. Find $m\angle BAC$ in the triangle above.
2. What is $m\angle ABC$ in the triangle above?

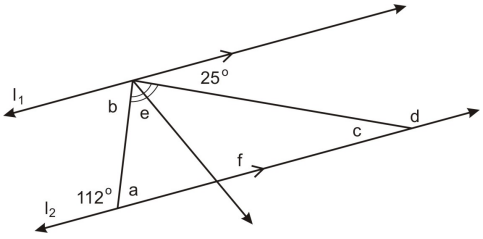
Questions 3-6 use the following diagram:



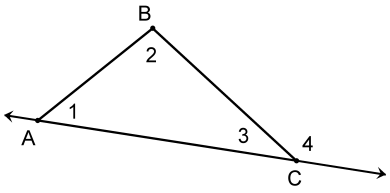
3. What is $m\angle VTU$?
4. What is $m\angle TVU$?
5. What is $m\angle TUV$?
6. What is the relationship between $\angle VTU$ and $\angle TUV$? Write one or two sentences to explain how you know this is the relationship.
7. Find $m\angle F$ in the diagram below:



Use the diagram below for questions 8-13. (Note $l_1 || l_2$)



8. $a =$ _____. Why?
9. $b =$ _____. Why?
10. $c =$ _____. Why?
11. $d =$ _____. Why?
12. $e =$ _____. Why?
13. $f =$ _____. Why?
14. Prove the Remote Exterior Angle Theorem: The measure of an exterior angle in a triangle equals the sum of the measures of the remote interior angles. To get started, you may use the following: Given triangle ABC as in the diagram below, prove $m\angle 1 + m\angle 2 = m\angle 4$.



Review Answers

1. 50°
2. 90°
3. 69°
4. 90°
5. 21°
6. $\angle VTU$ and $\angle TUV$ are complementary. Since the measures of the three angles of the triangle must add up to 180°, we can use the fact that $\angle TVU$ is a right angle to conclude that $m\angle VTU + m\angle TUV = 90°$
7. 70°
8. $a = 68°$. a and 112° add up to 180°
9. $b = 68°$. b is an alternate interior angle with a
10. $c = 25°$. c is an alternate interior angle with the labeled 25°
11. $d = 155°$. d is a linear pair with c

12. $e = 43.5°$. Use the triangle sum theorem with $a + e + e + c = 180°$ and solve for e
13. $f = 111.5°$. Use the triangle sum theorem with $f + c + e = 180°$
14. We will prove this using a two-column proof.

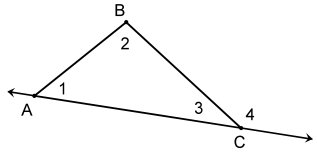


Table 4.2:

Statement	Reason
1. $\triangle ABC$	1. Given
2. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$	2. Triangle Sum Theorem
3. $m\angle 3 + m\angle 4 = 180°$	3. Linear Pair Postulate
4. $m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$	4. Substitution
5. $m\angle 1 + m\angle 2 = m\angle 4$	5. Subtraction property of equality (subtracted $m\angle 3$ on both sides)

4.2 Congruent Figures

Learning Objectives

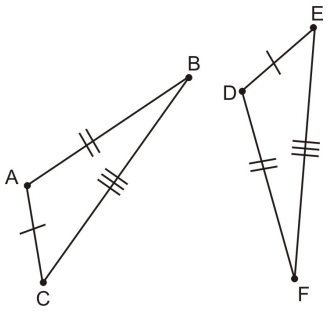
- Define congruence in triangles.
- Create accurate congruence statements.
- Understand that if two angles of a triangle are congruent to two angles of another triangle, the remaining angles will also be congruent.
- Explore properties of triangle congruence.

Introduction

Triangles are important in geometry because every other polygon can be turned into triangles by cutting them up (formally we call this adding **auxiliary lines**). Think of a square: If you add an auxiliary line such as a diagonal, then it is two right triangles. If we understand triangles well, then we can take what we know about triangles and apply that knowledge to all other polygons. In this chapter you will learn about congruent triangles, and in subsequent chapters you will use what you know about triangles to prove things about all kinds of shapes and figures.

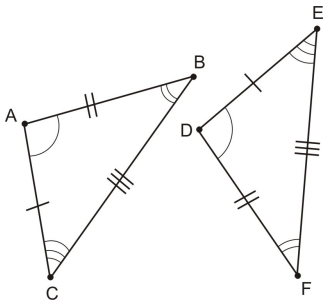
Defining Congruence in Triangles

Two figures are **congruent** if they have exactly the same size and shape. Another way of saying this is that the two figures can be perfectly aligned when one is placed on top of the other—but you may need to rotate or flip the figures over to make them line up. When that alignment is done, the angles that are matched are called **corresponding angles,** and the sides that are matched are called **corresponding sides.**



In the diagram above, sides $\overline{AC}$ and $\overline{DE}$ have the same length, as shown by the tic marks. If two sides have the same number of tic marks, it means that they have the same length. Since $\overline{AC}$ and $\overline{DE}$ each have one tic mark, they have the same length. Once we have established that $\overline{AC} \cong \overline{DE}$, we need to examine the other sides of the triangles. $\overline{BA}$ and $\overline{DF}$ each have two tic marks, showing that they are also congruent. Finally, as you can see, $\overline{BC} \cong \overline{EF}$ because they each have three tic marks. Each of these pairs corresponds because they are congruent to each other. Notice that the three sides of each triangle do not need to be congruent to each other, as long as they are congruent to their corresponding side on the other triangle.

When two triangles are congruent, the three pairs of corresponding angles are also congruent. Notice the tic marks in the triangles below.

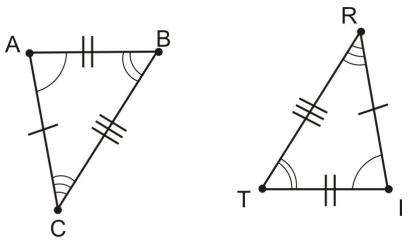


We use arcs inside the angle to show congruence in angles just as tic marks show congruence in sides. From the markings in the angles we can see $\angle A \cong \angle D, \angle B \cong \angle F$, and $\angle C \cong \angle E$.

By definition, if two triangles are congruent, then you know that all pairs of corresponding sides are congruent and all pairs of corresponding angles are congruent. This is sometimes called **CPCTC**: Corresponding parts of congruent triangles are congruent.

Example 1

Are the two triangles below congruent?



The question asks whether the two triangles in the diagram are congruent. To identify whether or not the triangles are congruent, each pair of corresponding sides and angles must be congruent.

Begin by examining the sides. $\overline{AC}$ and $\overline{RI}$ both have one tic mark, so they are congruent. $\overline{AB}$ and $\overline{TI}$ both have two tic marks, so they are congruent as well. $\overline{BC}$ and $\overline{RT}$ have three tic marks each, so each pair of sides is congruent.

Next you must check each angle. $\angle I$ and $\angle A$ both have one arc, so they are congruent. $\angle T \cong \angle B$ because they each have two arcs. Finally, $\angle R \cong \angle C$ because they have three arcs.

We can check that each angle in the first triangle matches with its corresponding angle in the second triangle by examining the sides. $\angle B$ corresponds with $\angle T$ because they are formed by the sides with two and three tic marks. Since all pairs of corresponding sides and angles are congruent in these two triangles, we conclude that the two triangles are congruent.

Creating Congruence Statements

We have already been using the congruence sign $\cong$ when talking about congruent sides and congruent angles.

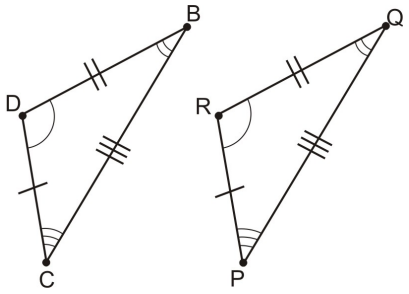
For example, if you wanted to say that $\overline{BC}$ was congruent to $\overline{CD}$, you could write the following statement.

$$\overline{BC} \cong \overline{CD}$$

In Chapter 1 you learned that the line above BC with no arrows means that BC is a segment (and not a line or a ray). If you were to read this statement out loud, you could say "Segment BC is congruent to segment CD."

When dealing with congruence statements involving angles or triangles, you can use other symbols. Whereas the symbol $\overline{BC}$ means "segment BC," the symbol $\angle B$ means "angle B." Similarly, the symbol $\triangle ABC$ means "triangle ABC."

When you are creating a congruence statement of two triangles, the order of the letters is very important. **Corresponding parts must be written in order**. That is, the angle at first letter of the first triangle corresponds with the angle at the first letter of the second triangle, the angles at the second letter correspond, and so on.



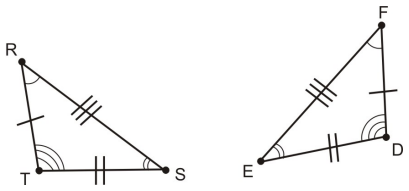
In the diagram above, if you were to name each triangle individually, they could be $\triangle BCD$ and $\triangle PQR$. Those names seem the most appropriate because the letters are in alphabetical order. However, if you are writing a congruence statement, you could NOT say that $\triangle BCD \cong \triangle PQR$. If you look at $\angle B$, it *does not* correspond to $\angle P$. $\angle B$ corresponds to $\angle Q$ instead (indicated by the two arcs in the angles). $\angle C$ corresponds to $\angle P$, and $\angle D$ corresponds to $\angle R$. Remember, you must compose the congruent statement so that the vertices are lined up for congruence. The statement below is correct.

$$\triangle BCD \cong \triangle QPR$$

This form may look strange at first, but this is how you must create congruence statements in any situation. Using this standard form allows your work to be easily understood by others, a crucial element of mathematics.

Example 2

Compose a congruence statement for the two triangles below.



To write an accurate congruence statement, you must be able to identify the corresponding pairs in the triangles above. Notice that $\angle R$ and $\angle F$ each have one arc mark. Similarly, $\angle S$ and $\angle E$ each have two arcs, and $\angle T$ and $\angle D$ have three arcs. Additionally, $RS = FE$ (or $\overline{RS} \cong \overline{FE}$), $ST = ED$, and $RT = FD$.

So, the two triangles are congruent, and to make the most accurate statement, this should be expressed by matching corresponding vertices. You can spell the first triangle in alphabetical

order and then align the second triangle to that standard.

$$\triangle RST \cong \triangle FED$$

Notice in example 2 that you don't need to write the angles in alphabetical order, as long **as corresponding parts match up.** If you're feeling adventurous, you could also express this statement as shown below.

$$\triangle DEF \cong \triangle TSR$$

Both of these congruence statements are accurate because corresponding sides and angles are aligned within the statement.

The Third Angle Theorem

Previously, you studied the triangle sum theorem, which states that the sum of the measures of the interior angles in a triangle will always be equal to 180°. This information is useful when showing congruence. As you practiced, if you know the measures of two angles within a triangle, there is only one possible measurement of the third angle. Thus, if you can prove two corresponding angle pairs congruent, the third pair is also guaranteed to be congruent.

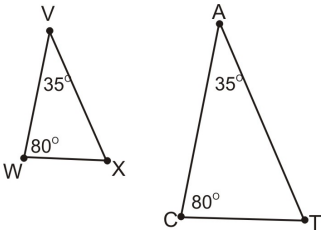
Third Angle Theorem

If two angles in one triangle are congruent to two angles in another triangle, then the third pair of angles are also congruent.

This may seem like an odd statement, but use the exercise below to understand it more fully.

Example 3

Identify whether or not the missing angles in the triangles below are congruent.



To identify whether or not the third angles are congruent, you must first find their measures. Start with the triangle on the left. Since you know two of the angles in the triangle, you can use the triangle sum theorem to find the missing angle. In $\triangle WVX$ we know

$$\begin{aligned} m\angle W + m\angle V + m\angle X &= 180^\circ \\ 80^\circ + 35^\circ + m\angle X &= 180^\circ \\ 115^\circ + m\angle X &= 180^\circ \\ m\angle X &= 65^\circ \end{aligned}$$

The missing angle of the triangle on the left measures 65°. Repeat this process for the triangle on the right.

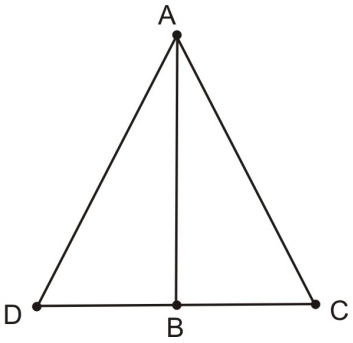
$$\begin{aligned} m\angle C + m\angle A + m\angle T &= 180^\circ \\ 80^\circ + 35^\circ + m\angle T &= 180^\circ \\ 115^\circ + m\angle T &= 180^\circ \\ m\angle T &= 65^\circ \end{aligned}$$

So, $\angle X \cong \angle T$. Remember that you could also identify this without using the triangle sum theorem. If two pairs of angles in two triangles are congruent, then the remaining pair of angles also must be congruent.

Congruence Properties

In earlier mathematics courses, you have learned concepts like the reflexive or commutative properties. These concepts help you solve many types of mathematics problems. There are a few properties relating to congruence that will help you solve geometry problems as well.

The **reflexive property of congruence** states that any shape is congruent to itself. This may seem obvious, but in a geometric proof, you need to identify every possibility to help you solve a problem. If two triangles share a line segment, you can prove congruence by the reflexive property.



In the diagram above, you can say that the shared side of the triangles is congruent because of the reflexive property. Or in other words, $\overline{AB} \cong \overline{AB}$.

The **symmetric property of congruence** states that congruence works frontwards and backwards, or in symbols, if $\angle ABC \cong \angle DEF$ then $\angle DEF \cong \angle ABC$.

The **transitive property of congruence** states that if two shapes are congruent to a third, they are also congruent to each other. In other words, if $\triangle ABC \cong \triangle JLM$, and $\triangle JLM \cong \triangle WYZ$, then $\triangle ABC \cong \triangle WYZ$. This property is very important in identifying congruence between different shapes.

Example 4

Which property can be used to prove the statement below?

If $\triangle MNO \cong \triangle PQR$ and $\triangle PQR \cong \triangle XYZ$, then $\triangle MNO \cong \triangle XYZ$.

A. reflexive property of congruence

B. identity property of congruence

C. transitive property of congruence

D. symmetric property of congruence

The transitive property is the one that allows you to transfer congruence to different shapes. As this states that two triangles are congruent to a third, they must be congruent to each other by the transitive property. The correct answer is C.

Lesson Summary

In this lesson, we explored congruent figures. Specifically, we have learned:

- How to define congruence in triangles.
- How to create accurate congruence statements.
- To understand that if two angles of a triangle are congruent to two angles of another triangle, the remaining angles will also be congruent.
- How to employ properties of triangle congruence.

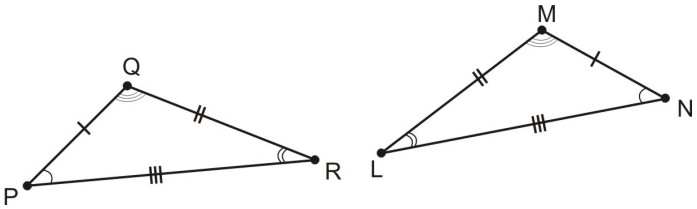
These skills will help you understand issues of congruence involving triangles. Always look for triangles in diagrams, maps, and other mathematical representations.

Points to Consider

Now that you understand the issues inherent in triangle congruence, you will create your first congruence proof.

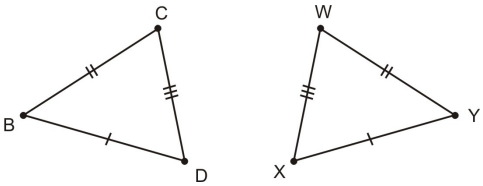
Review Questions

Use the diagram below for problem 1.



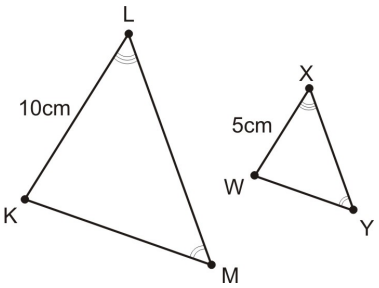
1. Write a congruence statement for the two triangles above.

Exercises 2-3 use the following diagram.



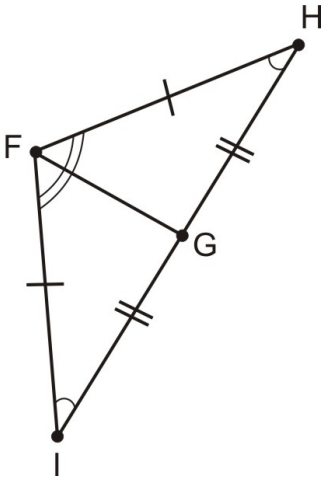
2. Suppose the two triangles above are congruent. Write a congruence statement for these two triangles.
3. Explain how we know that if the two triangles are congruent, then $\angle B \cong \angle Y$.

Use the diagram below for exercises 4-5.



4. Explain how we know $\angle K \cong \angle W$.
5. Are these two triangles congruent? Explain why (note, "looks" are not enough of a reason!).
6. If you want to know the measure of all three angles in a triangle, how many angles do you need to measure with your protractor? Why?

Use the following diagram for exercises 7-10.



7. What is the relationship between $\angle FGH$ and $\angle FGI$? How do you know?
8. What is $m\angle FGH$? How do you know?
9. What property tells us $\overline{FG} \cong \overline{FG}$?
10. Write a congruence statement for these triangles.

Review Answers

1. $\triangle PQR \cong \triangle NML$
2. $\triangle BCD \cong \triangle YWX$ (Note the order of the letters is important!)
3. If the two triangles are congruent, then $\angle B$ corresponds with $\angle Y$ and therefore they are congruent to each other by the definition of congruence.
4. The third angle theorem states that if two pairs of angles are congruent in two triangles, then the third pair of angles must also be congruent
5. No. $\overline{KL}$ corresponds with $\overline{WX}$ but they are not the same length
6. You only need to measure two angles. The triangle sum theorem will help you find the measure of the third angle
7. $\angle FGH$ and $\angle FGI$ are supplementary since they are a linear pair
8. $m\angle FGH = 90^\circ$
9. The reflexive property of congruence
10. $\triangle IGF \cong \triangle HGF$

4.3 Triangle Congruence using SSS

Learning Objectives

- Use the distance formula to analyze triangles on a coordinate grid.
- Understand and apply the SSS postulate of triangle congruence.

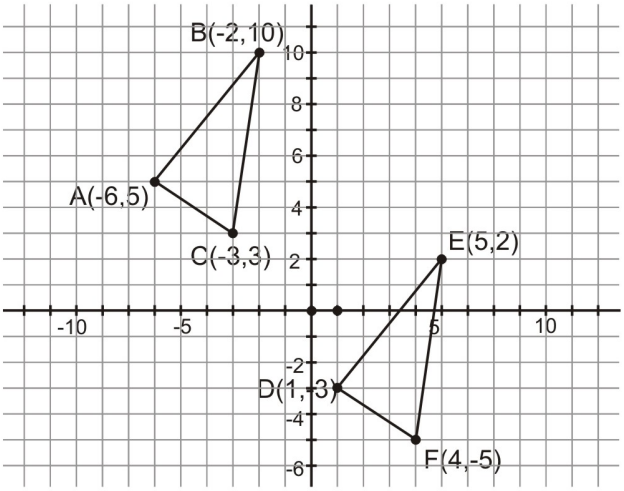
Introduction

In the last section you learned that if two triangles are congruent then the three pairs of corresponding sides are congruent and the three pairs of corresponding angles are congruent. In symbols, $\triangle CAT \cong \triangle DOG$ means $\angle C \cong \angle D, \angle A \cong \angle O, \angle T \cong \angle G, \overline{GA} \cong \overline{DO}, \overline{AT} \cong \overline{OG}$, and $\overline{CT} \cong \overline{DG}$.

Wow, that's a lot of information—in fact, one triangle congruence statement contains *six* different congruence statements! In this section we show that proving two triangles are congruent does not necessarily require showing all six congruence statements are true. Lucky for us, there are shortcuts for showing two triangles are congruent—this section and the next explore some of these shortcuts.

Triangles on a Coordinate Grid

To begin looking at rules of triangle congruence, we can use a coordinate grid. The following grid shows two triangles.



The first step in finding out if these triangles are congruent is to identify the lengths of the sides. In algebra, you learned the distance formula, shown below.

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

You can use this formula to find the distances on the grid.

Example 1

Find the distances of all the line segments on the coordinate grid above using the distance formula.

Begin with $\triangle ABC$. First write the coordinates.

A is $(-6, 5)$

B is $(-2, 10)$

C is $(-3, 3)$

Now use the coordinates to find the lengths of each segment in the triangle.

$$\begin{aligned} AB &= \sqrt{(-2 - (-6))^2 + (10 - 5)^2} \\ &= \sqrt{(-2 + 6)^2 + (10 - 5)^2} \\ &= \sqrt{(4)^2 + (5)^2} \\ &= \sqrt{16 + 25} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-3 - (-2))^2 + (3 - 10)^2} \\ &= \sqrt{(-3 + 2)^2 + (3 - 10)^2} \\ &= \sqrt{(-1)^2 + (-7)^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-3 - (-6))^2 + (3 - 5)^2} \\ &= \sqrt{(-3 + 6)^2 + (3 - 5)^2} \\ &= \sqrt{(3)^2 + (-2)^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} \end{aligned}$$

So, the lengths are as follows.

$AB = \sqrt{41}$, $BC = \sqrt{50}$, and $AC = \sqrt{13}$

Next, find the lengths in triangle DEF. First write the coordinates.

D is $(1, -3)$

E is $(5, 2)$

F is $(4, -5)$

Now use the coordinates to find the lengths of each segment in the triangle.

$$\begin{aligned} DE &= \sqrt{(5-1)^2 + (2-(-3))^2} \\ &= \sqrt{(5-1)^2 + (2+3)^2} \\ &= \sqrt{(4)^2 + (5)^2} \\ &= \sqrt{16+25} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} EF &= \sqrt{(4-5)^2 + (-5-2)^2} \\ &= \sqrt{(-1)^2 + (-7)^2} \\ &= \sqrt{1+49} \\ &= \sqrt{50} \end{aligned}$$

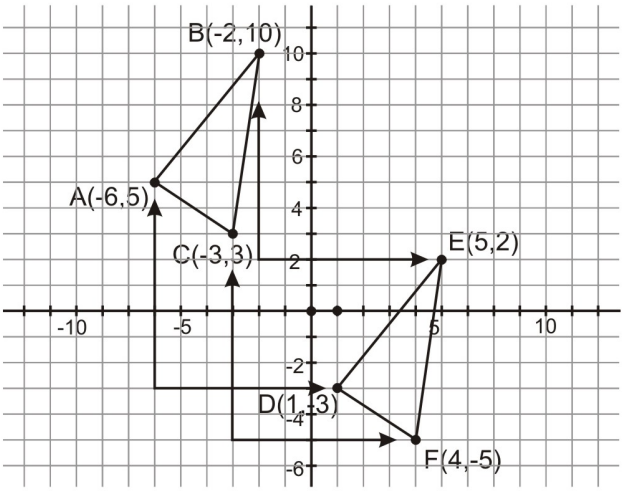
$$\begin{aligned} DF &= \sqrt{(4-1)^2 + (-5-(-3))^2} \\ &= \sqrt{(4-1)^2 + (-5+3)^2} \\ &= \sqrt{(3)^2 + (-2)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \end{aligned}$$

So, the lengths are as follows:

$DE = \sqrt{41}$, $EF = \sqrt{50}$, and $DE = \sqrt{13}$

Using the distance formula, we showed that the corresponding sides of the two triangles have the same length. We don't have the tools to find the measures of the angles in these triangles, so we can show congruence in a different way.

Imagine you could pick up $\triangle ABC$ without changing its shape and move the whole triangle 7 units right and 8 units down. If you did this, then points A and D would be on top of each other, B and E would also be on top of each other, and C and F would also coincide.



To analyze the relationship between the points, the distance formula is not necessary. Simply look at how far (and in what direction) the vertices may have moved.

Points A and D: A is $(-6, 5)$ and D is $(1, -1)$. D is 7 units to the right and 8 units below A.

Points B and E: B is $(-2, 10)$ and E is $(5, 2)$. E is 7 units to the right and 8 units below B.

Points C and F: C is $(-3, 3)$ and F is $(4, -5)$. F is 7 units to the right and 8 units below C.

Since the same relationship exists between the vertices, you could move the entire triangle ABC 7 units to the right and 8 units down. It would exactly cover triangle DEF. These triangles are therefore congruent.

$$\triangle ABC \cong \triangle DEF$$

SSS Postulate of Triangle Congruence

The extended example above illustrates that when three sides of one triangle are equal in length to three sides of another, then the triangles are congruent. We did not need to measure the angles—the lengths of the corresponding sides being the same "forced" the corresponding angles to be congruent. This leads us to one of the triangle congruence postulates:

Side-Side-Side (SSS) Triangle Congruence Postulate: If three sides in one triangle are congruent to the three corresponding sides in another triangle, then the triangles are

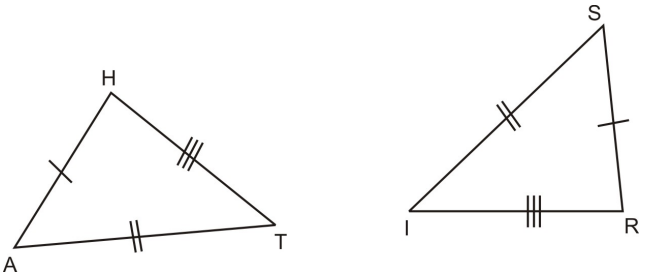
congruent to each other.

This is a *postulate* so we accept it as true without proof.

You can perform a quick experiment to test this postulate. Cut two pieces of spaghetti (or a straw, or some segment-like thing) exactly the same length. Then cut another set of pieces that are the same length as each other (but not necessarily the same length as the first set). Finally, cut one more pair of pieces of spaghetti that are identical to each other. Separate the pieces into two piles. Each pile should have three pieces of different lengths. Build a triangle with one set and leave it on your desk. Using the other pieces, attempt to make a triangle with a different shape or size by matching the ends. Notice that no matter what you do, you will always end up with congruent triangles (though they might be "flipped over" or rotated). This demonstrates that if you can identify three pairs of congruent sides in two triangles, the two triangles are fully congruent.

Example 2

Write a triangle congruence statement based on the diagram below:



We can see from the tick marks that there are three pairs of corresponding congruent sides: $\overline{HA} \cong \overline{RS}$, $\overline{AT} \cong \overline{SI}$, and $\overline{TH} \cong \overline{IR}$. Matching up the corresponding sides, we can write the congruence statement $\triangle HAT \cong \triangle RSI$.

Don't forget that ORDER MATTERS when writing triangle congruence statements. Here, we lined up the sides with one tic mark, then the sides with two tic marks, and finally the sides with three tic marks.

Lesson Summary

In this lesson, we explored triangle congruence using only the sides. Specifically, we have learned:

- How to use the distance formula to analyze triangles on a coordinate grid
- How to understand and apply the SSS postulate of triangle congruence.

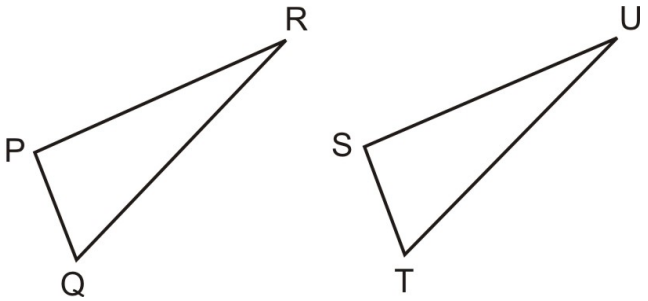
These skills will help you understand issues of congruence involving triangles, and later you will apply this knowledge to all types of shapes.

Points to Consider

Now that you have been exposed to the SSS Postulate, there are other triangle congruence postulates to explore. The next chapter deals with congruence using a mixture of sides and angles.

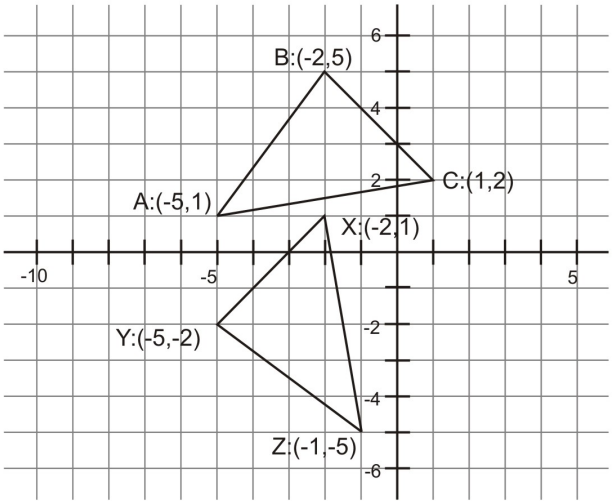
Review Questions

1. If you know that $\triangle PQR \cong \triangle STU$ in the diagram below, what are six congruence statements that you also know about the parts of these triangles?



2. Redraw these triangles using geometric markings to show all congruent parts.

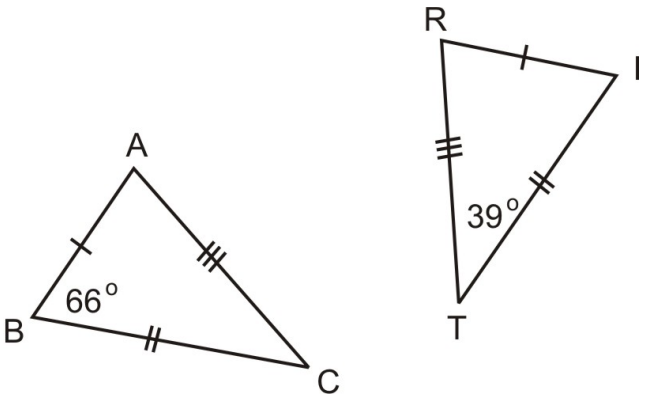
Use the diagram below for exercises 3-7 .



3. Find the length of each side in $\triangle ABC$.
   (a) $AB =$
   (b) $BC =$
   (c) $AC =$

4. Find the length of each side in $\triangle XYZ$.
   (a) $XY =$
   (b) $YZ =$
   (c) $XZ =$
5. Write a congruence statement relating these two triangles.
6. Write another equivalent congruence statement for these two triangles.
7. What postulate guarantees these triangles are congruent?

Exercises 8-10 use the following diagram:



8. Write a congruence statement for the two triangles in this diagram. What postulate did you use?
9. Find $m\angle C$. Explain how you know your answer.
10. Find $m\angle R$. Explain how you know your answer.

Review Answers

1. $\overline{PQ} \cong \overline{ST}, \overline{QR} \cong \overline{TU}, \overline{PR} \cong \overline{SU}, \angle P \cong \angle S, \angle Q \cong \angle T$, and $\angle R \cong \angle U$
2. One possible answer:
3. (a) $AB = 5$ units
   (b) $BC = 3\sqrt{2}$ units
   (c) $AC = \sqrt{37}$ units
4. (a) $XY = 3\sqrt{2}$ units
   (b) $YZ = 5$ units
   (c) $XZ = \sqrt{37}$ units
5. $\triangle ABC \cong \triangle ZYX$ (Note, other answers are possible, but the relative order of the letters does matter.)
6. $\triangle BAC \cong \triangle YZX$
7. SSS

8. $\triangle ABC \cong \triangle RIT$, the side-side-side triangle congruence postulate
9. $m\angle c = 39°$. We know this because it corresponds with $\angle T$, so $\angle C \cong \angle T$
10. $m\angle R = 75°$. Used the triangle sum theorem together with my answer for 9.

4.4 Triangle Congruence Using ASA and AAS

Learning Objectives

- Understand and apply the ASA Congruence Postulate.
- Understand and apply the AAS Congruence Theorem.
- Understand and practice two-column proofs.
- Understand and practice flow proofs.

Introduction

The SSS Congruence Postulate is one of the ways in which you can prove two triangles are congruent without measuring six angles and six sides. The next two lessons explore other ways in which you can prove triangles congruent using a combination of sides and angles. It is helpful to know all of the different ways you can prove congruence between two triangles, or rule it out if necessary.

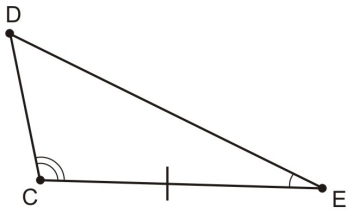
ASA Congruence

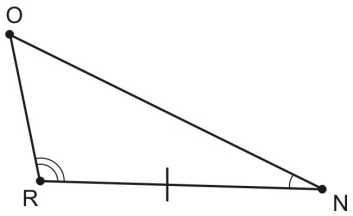
One of the other ways you can prove congruence between two triangles is the **ASA Congruence Postulate.** The "S" represents "side," as it did in the SSS Theorem. "A" stands for "angle" and the order of the letters in the name of the postulate is crucial in this circumstance. To use the ASA postulate to show that two triangles are congruent, you must identify two angles and the side in between them. If the corresponding sides and angles are congruent, the entire triangles are congruent. In formal language, the ASA postulate is this:

Angle-Side-Angle (ASA) Congruence Postulate: If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent.

To test out this postulate, you can use a ruler and a protractor to make two congruent triangles. Start by drawing a segment that will be one side of your first triangle and pick two angles whose sum is less than 180°. Draw one angle on one side of the segment, and draw the second angle on the other side. Now, repeat the process on another piece of paper, using the same side length and angle measures. What you'll find is that there is only one possible triangle you could create—the two triangles will be congruent.

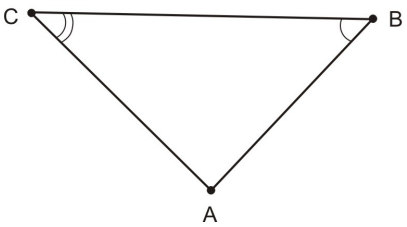
Notice also that by picking two of the angles of the triangle, you have determined the measure of the third by the Triangle Sum Theorem. So, in reality, you have defined the whole triangle; you have identified all of the angles in the triangle, and by picking the length of one side, you defined the scale. So, no matter what, if you have two angles, and the side in between them, you have described the whole triangle.

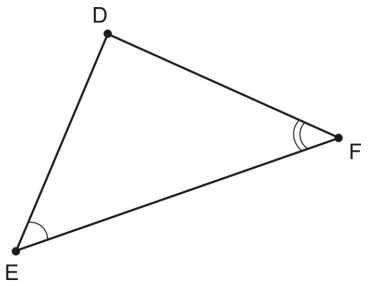




Example 1

What information would you need to prove that these two triangles are congruent using the ASA postulate?





A. the measures of the missing angles

B. the measures of sides $\overline{AB}$ and $\overline{BC}$

C. the measures of sides $\overline{BC}$ and $\overline{EF}$

D. the measures of sides $\overline{AC}$ and $\overline{DF}$

If you are to use the ASA postulate to prove congruence, you need to have two pairs of congruent angles and the *included* side, the side in between the pairs of congruent angles. The side in between the two marked angles in $\triangle ABC$ is side $\overline{BC}$. The side in between the two marked angles in $\triangle DEF$ is side $\overline{EF}$. You would need the measures of sides $\overline{BC}$ and $\overline{EF}$ to prove congruence. The correct answer is C.

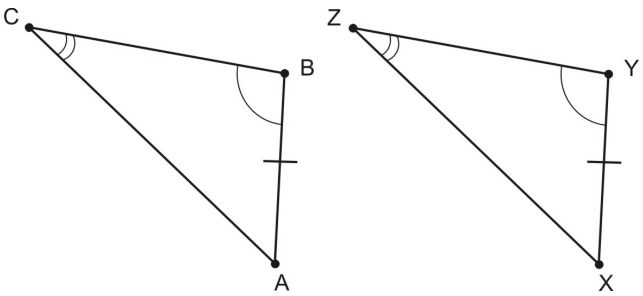
AAS Congruence

Another way you can prove congruence between two triangles is using two angles and the non-included side.

Angle-Angle-Side (AAS) Congruence Theorem: If two angles and a non-included side in one triangle are congruent to two corresponding angles and a non-included side in another triangle, then the triangles are congruent.

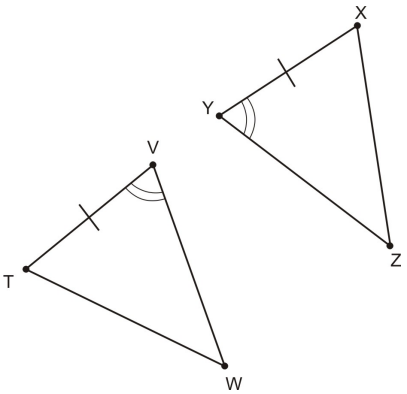
This is a *theorem* because it can be proven. First, we will do an example to see why this theorem is true, then we will prove it formally. Like the ASA postulate, the AAS theorem uses two angles and a side to prove triangle congruence. However, the order of the letters (and the angles and sides they stand for) is different.

The AAS theorem is equivalent to the ASA postulate because when you know the measure of two angles in a triangle, you also know the measure of the third angle. The pair of congruent sides in the triangles will determine the size of the two triangles.



Example 2

What information would you need to prove that these two triangles were congruent using the AAS theorem?



A. the measures of sides $\overline{TW}$ and $\overline{XZ}$

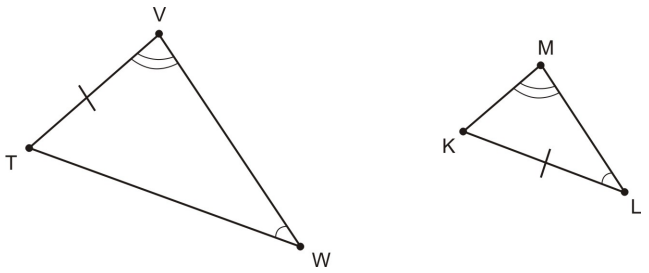
B. the measures of sides $\overline{VW}$ and $\overline{YZ}$

C. the measures of $\angle VTW$ and $\angle YXZ$

D. the measures of angles $\angle TWV$ and $\angle XZY$

If you are to use the AAS theorem to prove congruence, you need to know that pairs of two angles are congruent and the pair of sides adjacent to one of the given angles are congruent. You already have one side and its adjacent angle, but you still need another angle. It needs to be the angle not touching the known side, rather than adjacent to it. Therefore, you need to find the measures of $\angle TWV$ and $\angle XZY$ to prove congruence. The correct answer is D.

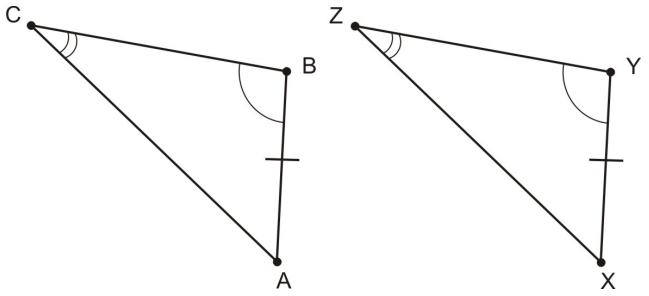
When you use AAS (or any triangle congruence postulate) to show that two triangles are congruent, *you need to make sure that the corresponding pairs of angles and sides actually align.* For instance, look at the diagram below:



Even though two pairs of angles and one pair of sides are congruent in the triangles, these triangles are NOT congruent. Why? Notice that the marked side in $\triangle TVW$ is $\overline{TV}$, which is between the unmarked angle and the angle with two arcs. However in $\triangle KML$, the marked side is between the unmarked angle and the angle with one arc. As the corresponding parts do not match up, you cannot use AAS to say these triangles are congruent.

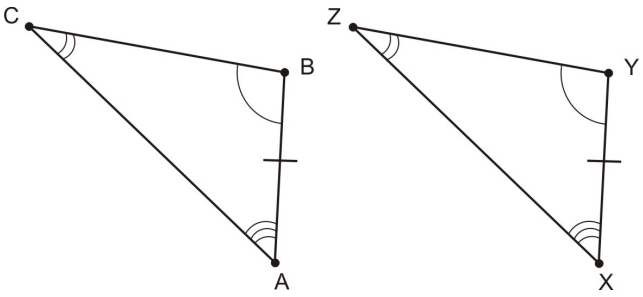
AAS and ASA

The AAS triangle congruence theorem is logically the exact same as the ASA triangle congruence postulate. Look at the following diagrams to see why.



Since $\angle C \cong \angle Z$ and $\angle B \cong \angle Y$, we can conclude from the **third angle theorem** that $\angle A \cong \angle X$. This is because the sum of the measures of the three angles in each triangle is 180° and if we know the measures of two of the angles, then the measure of the third angle

is already determined. Thus, marking $\angle A \cong \angle X$, the diagram becomes this:



Now we can see that $\angle A \cong \angle X$ (A), $\overline{AB} \cong \overline{XY}$(S), and $\angle B \cong \angle Y$ (A), which shows that $\triangle ABC \cong \triangle XYZ$ by ASA.

Proving Triangles Congruent

In geometry we use proofs to show something is true. You have seen a few proofs already—they are a special form of argument in which you have to justify every step of the argument with a reason. Valid reasons are definitions, postulates, or results from other proofs.

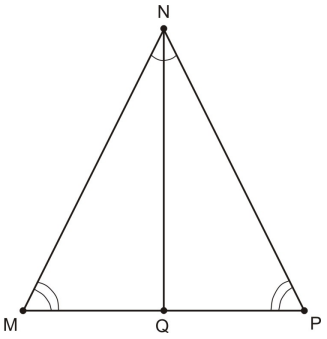
One way to organize your thoughts when writing a proof is to use a **two-column proof.** This is probably the most common kind of proof in geometry, and it has a specific format. In the left column you write statements that lead to what you want to prove. In the right hand column, you write a reason for each step you take. Most proofs begin with the "given" information, and the conclusion is the statement you are trying to prove. Here's an example:

Example 3

Create a two-column proof for the statement below.

Given: $\overline{NQ}$ is the bisector of $\angle MNP$, and $\angle NMQ \cong \angle NPQ$

Prove: $\triangle MNQ \cong \triangle PNQ$



Remember that each step in a proof must be clearly explained. You should formulate a

strategy before you begin the proof. Since you are trying to prove the two triangles congruent, you should look for congruence between the sides and angles. You know that if you can prove SSS, ASA, or AAS, you can prove congruence. Since the given information provides two pairs of congruent angles, you will most likely be able to show the triangles are congruent using the ASA postulate or the AAS theorem. Notice that both triangles share one side. We know that side is congruent to itself ($\overline{NQ} \cong \overline{NQ}$), and now you have pairs of two congruent angles and non-included sides. You can use the AAS congruence theorem to prove the triangles are congruent.

Table 4.3:

Statement	Reason
1.$\angle NQ$ is the bisector of $\angle MNP$	1. Given
2.$\angle MNQ \cong \angle PNQ$	2. Definition of an angle bisector (a bisector divides an angle into two congruent angles)
3.$\angle NMQ \cong \angle NPQ$	3. Given
4.$\overline{NQ} \cong \overline{NQ}$	4. Reflexive Property
5.$\triangle MNQ \cong \triangle PNQ$	5. AAS Congruence Theorem (if two pairs of angles and the corresponding non-included sides are congruent, then the triangles are congruent)

Notice how the markings in the triangles help in the proof. Whenever you do proofs, use arcs in the angles and tic marks to show congruent angles and sides.

Flow Proofs

Though two-column proofs are the most traditional style (in geometry textbooks, at least!), there are many different ways of solving problems in geometry. We already wrote a paragraph proof in an earlier lesson that simply described, step by step, the rationale behind an assertion (when we showed why AAS is logically equivalent to ASA). The two-column style is easy to read and organizes ideas clearly. Some students, however, prefer **flow proofs.** Flow proofs show the relationships between ideas more explicitly by using a chart that shows how one idea will lead to the next. Like two-column proofs, it is helpful to always remember the end goal so you can identify what it is you need to prove. Sometimes it is easier to work backwards!

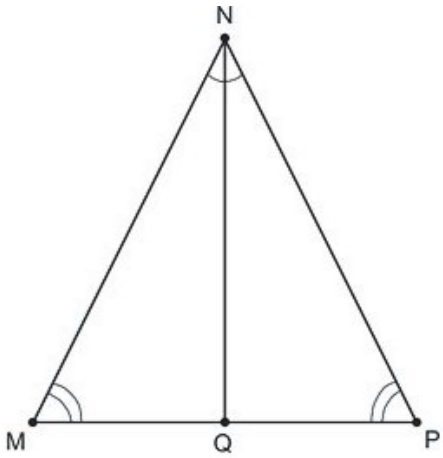
The next example repeats the same proof as the one above, but displayed in a flow style, rather than two columns.

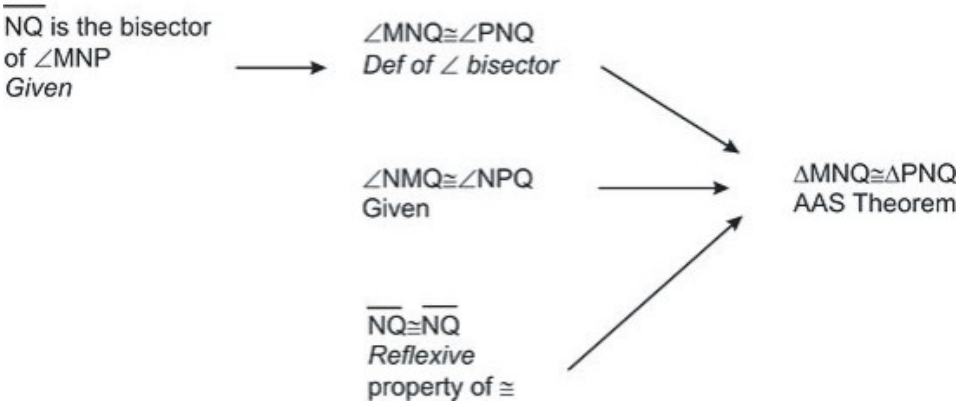
Example 4

Create a flow proof for the statement below.

Given: $\angle NQ$ is the bisector of $\angle MNP$ and $\angle NMQ \cong \angle NPQ$

Prove: $\triangle MNQ \cong \triangle PNQ$





As you can see from these two proofs of the theorem, there are many different ways of expressing the same information. It is important that you become familiar with proving things using all of these styles because you may find that different types of proofs are better suited for different theorems.

Lesson Summary

In this lesson, we explored triangle congruence. Specifically, we have learned to:

- Understand and apply the ASA Congruence Postulate.
- Understand and apply the AAS Congruence Postulate.
- Understand and practice Two-Column Proofs.

- Understand and practice Flow Proofs.

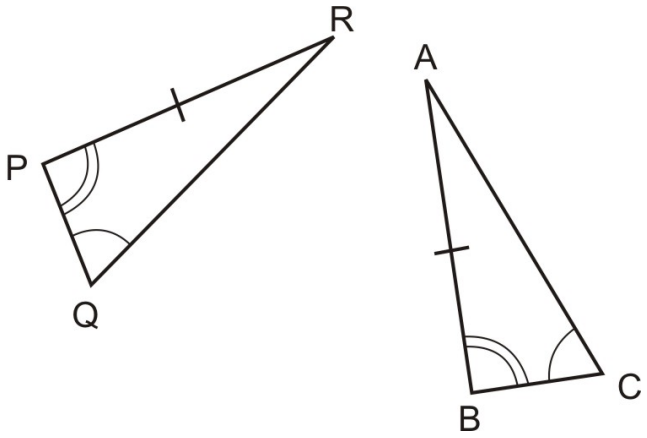
These skills will help you understand issues of congruence involving triangles. Always look for triangles in diagrams, maps, and other mathematical representations.

Points to Consider

Now that you have been exposed to the SAS and AAS postulates, there are even more triangle congruence postulates to explore. The next lesson deals with SAS and HL proofs.

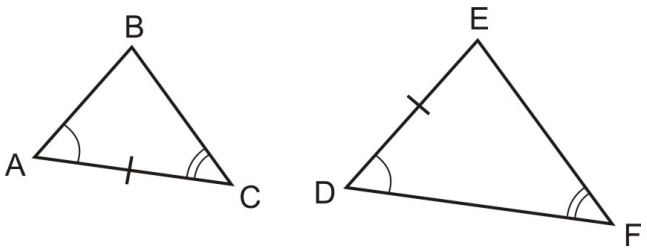
Review Questions

Use the following diagram for exercises 1-3.



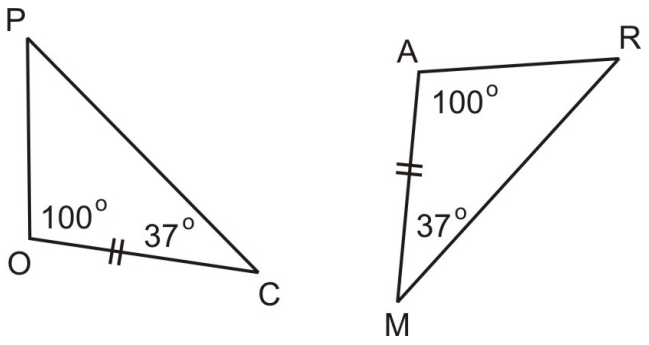
1. Complete the following congruence statement, if possible $\triangle PQR \cong$ _____.
2. What postulate allows you to make the congruence statement in 1, or, if it is not possible to make a congruence statement explain why.
3. Given the marked congruent parts, what *other* congruence statements do you now know based on your answers to 1 and 2?

Use the following diagram for exercises 4-6 .

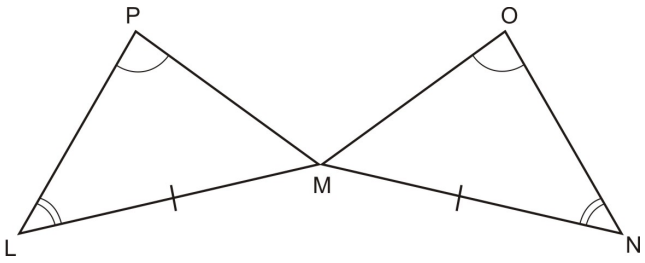


4. Complete the following congruence statement, if possible $\triangle ABC \cong$ _____.
5. What postulate allows you to make the congruence statement in 4, or, if it is not possible to make a congruence statement explain why.
6. Given the marked congruent parts in the triangles above, what *other* congruence statements do you now know based on your answers to 4 and 5?

Use the following diagram for exercises 7-9.



7. Complete the following congruence statement, if possible $\triangle POC \cong$ _____.
8. What postulate allows you to make the congruence statement in 7, or, if it is not possible to make a congruence statement explain why.
9. Given the marked congruent parts in the triangles above, what *other* congruence statements do you now know based on your answers to 7 and 8?
10. Complete the steps of this two-column proof:



Given $\angle L \cong \angle N, \angle P \cong \angle O$, and $\overline{LM} \cong \overline{MN}$
Prove: $\angle PML \cong \angle OMN$

Note: You cannot assume that P, M, and N are collinear or that L, M, and O are collinear.

Table 4.4:

Statement	Reason
1. $\angle L \cong \angle N$	1. Given
2. $\angle P \cong \angle O$	2. _____
3. _____	3. Given

Table 4.4: (continued)

Statement	Reason
4. $\triangle LMP \cong$ _____	4. _____ triangle congruence postulate
5. $\angle PML \cong \angle OMN$	5. _____

11. Bonus question: Why do we have to use three letters to name $\angle PML$ and $\angle OMN$, while we can use only one letter to name $\angle L$ or $\angle N$?

Review Answers

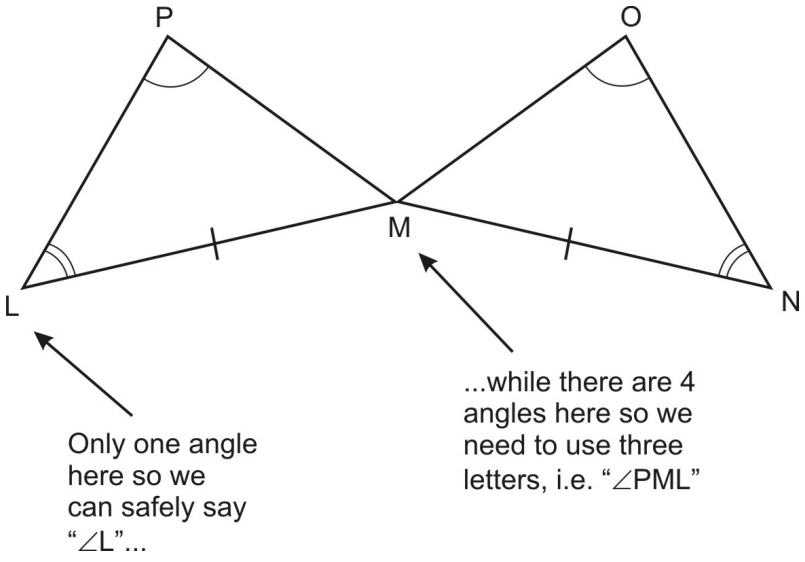
1. $\triangle PQR \cong \triangle BCA$
2. AAS triangle congruence postulate
3. $\overline{PQ} \cong \overline{BC}, \overline{QR} \cong \overline{CA}$, and $\angle R \cong \angle A$
4. No congruence statement is possible
5. We can't use either AAS or ASA because the corresponding parts do not match up
6. $\angle E \cong \angle B$. This is still true by the **third angle theorem,** even if the triangles are not congruent.
7. $\triangle POC \cong \triangle RAM \triangle PQR \cong \triangle BCA$
8. ASA triangle congruence postulate
9. $\angle P \cong \angle A, \overline{PO} \cong \overline{RA}$, and $\overline{PC} \cong \overline{RM}$

Table 4.5:

Statement	Reason
1. $\angle L \cong \angle N$	1. Given
2. $\angle P \cong \angle O$	2. Given
3. $\overline{LM} \cong \overline{MN}$	3. Given
4. $\triangle LMP \cong \triangle NMO$	4. AAS Triangle Congruence Postulate
5. $\angle PML \cong \angle OMN$	5. Definition of congruent triangles (if two triangles are $\cong$ then all corresponding parts are also $\cong$).

10.
11. We can use one letter to name an angle when there is no ambiguity. So at point L in

the diagram for 10 there is only one possible angle. At point M there are four angles, so we use the "full name" of the angles to be specific!



4.5 Proof Using SAS and HL

Learning Objectives

- Understand and apply the SAS Congruence Postulate.
- Identify the distinct characteristics and properties of right triangles.
- Understand and apply the HL Congruence Theorem.
- Understand that SSA does not necessarily prove triangles are congruent.

Introduction

You have already seen three different ways to prove that two triangles are congruent (without measuring six angles and six sides). Since triangle congruence plays such an important role in geometry, it is important to know all of the different theorems and postulates that can prove congruence, and it is important to know which combinations of sides and angles do *not* prove congruence.

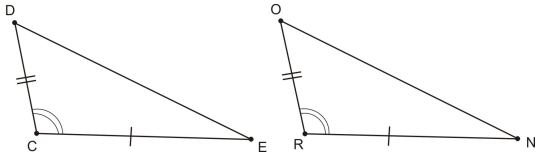
SAS Congruence

By now, you are very familiar with postulates and theorems using the letters S and A to represent triangle sides and angles. One more way to show two triangles are congruent is by the **SAS Congruence Postulate.**

SAS Triangle Congruence Postulate: If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.

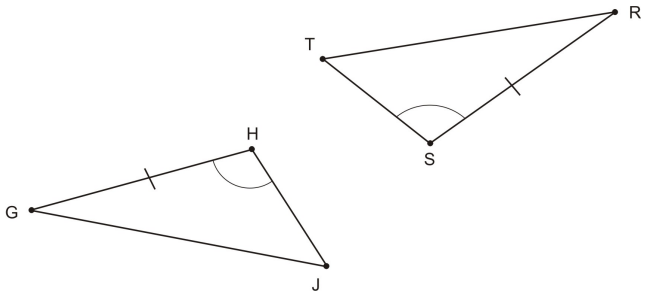
Like ASA and AAS congruence, the order of the letters is very significant. You must have the angles between the two sides for the SAS postulate to be valid.

Once again you can test this postulate using physical models (such as pieces of uncooked spaghetti) for the sides of a triangle. You'll find that if you make two pairs of congruent sides, and lay them out with the same included angle then the third side will be determined.



Example 1

What information would you need to prove that these two triangles were congruent using the SAS postulate?



A. the measures of $\angle HJG$ and $\angle STR$

B. the measures of $\angle HGJ$ and $\angle SRT$

C. the measures of $\overline{HJ}$ and $\overline{ST}$

D. the measures of sides $\overline{GJ}$ and $\overline{RT}$

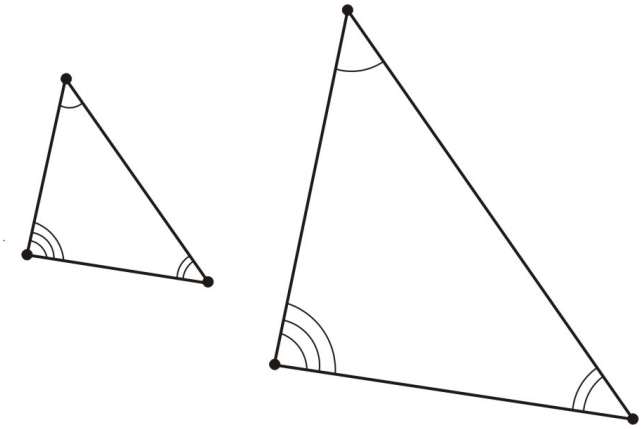
If you are to use the SAS postulate to establish congruence, you need to have the measures of two sides and the angle in between them for both triangles. So far, you have one side and one angle. So, you must use the other side adjacent to the same angle. In $\triangle GHJ$, that side is $\overline{HJ}$. In triangle $\triangle RST$, the corresponding side is $\overline{ST}$. So, the correct answer is C.

AAA and SSA relationships

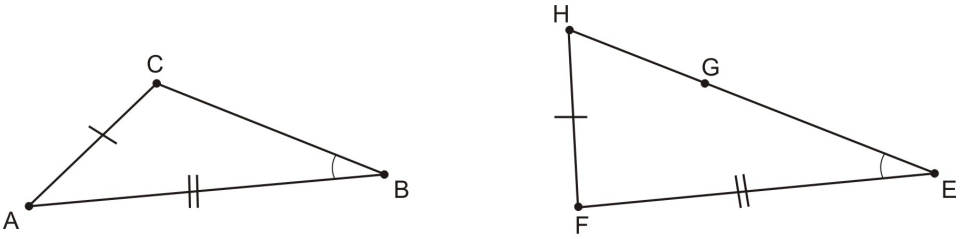
You have learned so many different ways to prove congruence between two triangles, it may be tempting to think that if you have any pairs of congruent three elements (combining sides

or angles), you can prove triangle congruence.

However, you may have already guessed that AAA congruence does not work. Even if all of the angles are equal between two triangles, the triangles may be of different scales. So, AAA can only prove **similarity,** not congruence.



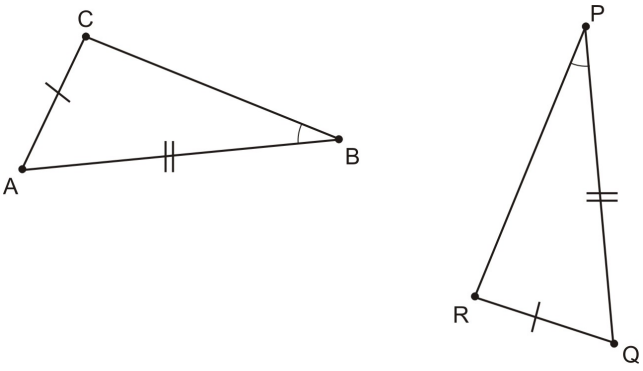
SSA relationships do not *necessarily* prove congruence either. Get your spaghetti and protractors back on your desk to try the following experiment. Choose two pieces of spaghetti at given length. Select a measure for an angle that is *not* between the two sides. If you keep that angle constant, you may be able to make two different triangles. As the angle in between the two given sides grows, so does the side opposite it. In other words, if you have two sides and an angle that is not between them, you cannot prove congruence.



In the figure, $\triangle ABC$ is NOT congruent to $\triangle FEH$ even though they have two pairs of congruent sides and a pair of congruent angles. $\overline{FG} \cong \overline{FH} \cong \overline{AC}$ and you can see that there are two possible triangles that can be made using this combination SSA.

Example 2

Can you prove that the two triangles below are congruent?



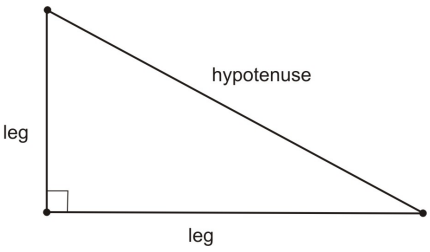
Note: Figure is not to scale.

The two triangles above look congruent, but are labeled, so you cannot assume that how they look means that they are congruent. There are two sides labeled congruent, as well as one angle. Since the angle is *not* between the two sides, however, this is a case of SSA. You cannot prove that these two triangles are congruent. Also, it is important to note that although two of the angles appear to be right angles, they are not marked that way, so you cannot assume that they are right angles.

Right Triangles

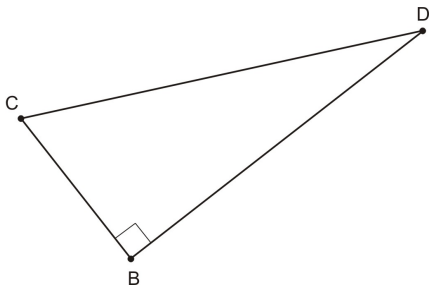
So far, the congruence postulates we have examined work on any triangle you can imagine. As you know, there are a number of types of triangles. **Acute triangles** have all angles measuring less than 90°. **Obtuse triangles** have one angle measuring between 90° and 180°. **Equilateral triangles** have congruent sides, and all angles measure 60°. **Right triangles** have one angle measuring exactly 90°.

In right triangles, the sides have special names. The two sides adjacent to the right angle are called **legs** and the side opposite the right angle is called the **hypotenuse.**



Example 3

Which side of right triangle BCD is the hypotenuse?



Looking at $\triangle BCD$, you can identify $\angle CBD$ as a right angle (remember the little square tells us the angle is a right angle). By definition, the hypotenuse of a right triangle is opposite the right angle. So, side $\overline{CD}$ is the hypotenuse.

HL Congruence

There is one special case when SSA does prove that two triangles are congruent-When the triangles you are comparing are right triangles. In any two right triangles you know that they have at least one pair of congruent angles, the right angles.

Though you will learn more about it later, there is a special property of right triangles referred to as the **Pythagorean theorem.** It isn't important for you to be able to fully understand and apply this theorem in this context, but it is helpful to know what it is. The Pythagorean Theorem states that for any right triangle with legs that measure a and b and hypotenuse measuring c units, the following equation is true.

$$a^2 + b^2 = c^2$$

In other words, if you know the lengths of two sides of a right triangle, then the length of the third side can be determined using the equation. This is similar in theory to how the Triangle Sum Theorem relates angles. You know that if you have two angles, you can find the third.

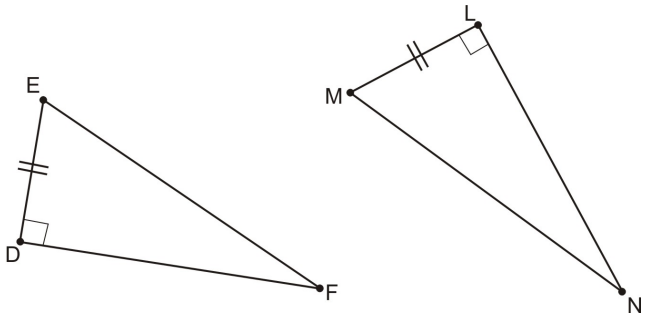
Because of the Pythagorean Theorem, if you know the length of the hypotenuse and a leg of a right triangle, you can calculate the length of the missing leg. Therefore, if the hypotenuse and leg of one right triangle are congruent to the corresponding parts of another right triangle, you could prove the triangles congruent by the SSS congruence postulate. So, the last in our list of theorems and postulates proving congruence is called the **HL Congruence Theorem.** The "H" and "L" stand for hypotenuse and leg.

HL Congruence Theorem: If the hypotenuse and leg in one right triangle are congruent to the hypotenuse and leg in another right triangle, then the two triangles are congruent.

The proof of this theorem is omitted because we have not yet proven the Pythagorean Theorem.

Example 4

What information would you need to prove that these two triangles were congruent using the HL theorem?



A. the measures of sides $\overline{EF}$ and $\overline{MN}$

B. the measures of sides $\overline{DF}$ and $\overline{LN}$

C. the measures of angles $\angle DEF$ and $\angle LMN$

D. the measures of angles $\angle DFE$ and $\angle LNM$

Since these are right triangles, you only need one leg and the hypotenuse to prove congruence. Legs $\overline{DE}$ and $\overline{LM}$ are congruent, so you need to find the lengths of the hypotenuses. The hypotenuse of $\triangle DEF$ is $\overline{EF}$. The hypotenuse of $\triangle LMN$ is $\overline{MN}$. So, you need to find the measures of sides $\overline{EF}$ and $\overline{MN}$. The correct answer is A.

Points to Consider

The HL congruence theorem shows that sometimes SSA is sufficient to prove that two triangles are congruent. You have also seen that sometimes it is not. In trigonometry you will study this in more depth. For now, you might try playing with objects or you may try using geometric software to explore under which conditions SSA does provide enough information to infer that two triangles are congruent.

Lesson Summary

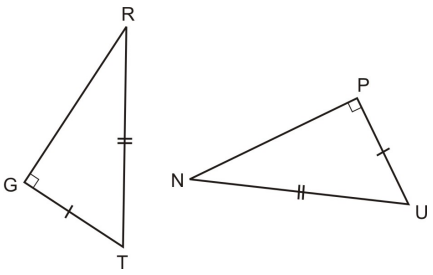
In this lesson, we explored triangle sums. Specifically, we have learned:

- How to understand and apply the SAS Congruence Postulate.
- How to identify the distinct characteristics and properties of right triangles.
- How to understand and apply the HL Congruence Theorem.
- That SSA does not necessarily prove triangles are congruent.

These skills will help you understand issues of congruence involving triangles. Always look for triangles in diagrams, maps, and other mathematical representations.

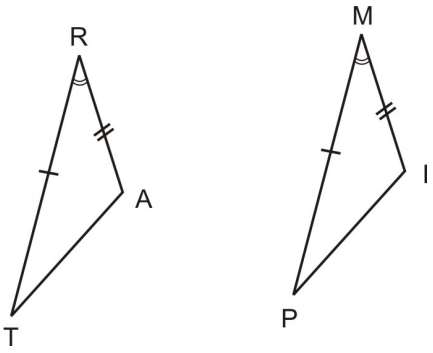
Review Questions

Use the following diagram for exercises 1-3.



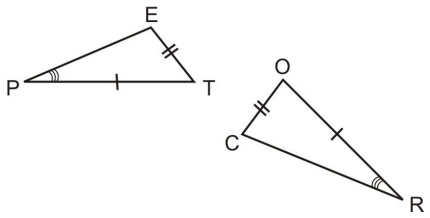
1. Complete the following congruence statement, if possible $\triangle RGT \cong$ _____.
2. What postulate allows you to make the congruence statement in 1, or, if it is not possible to make a congruence statement explain why.
3. Given the marked congruent parts in the triangles above, what *other* congruence statements do you now know based on your answers to 1 and 2?

Use the following diagram below for exercises 4-6 .

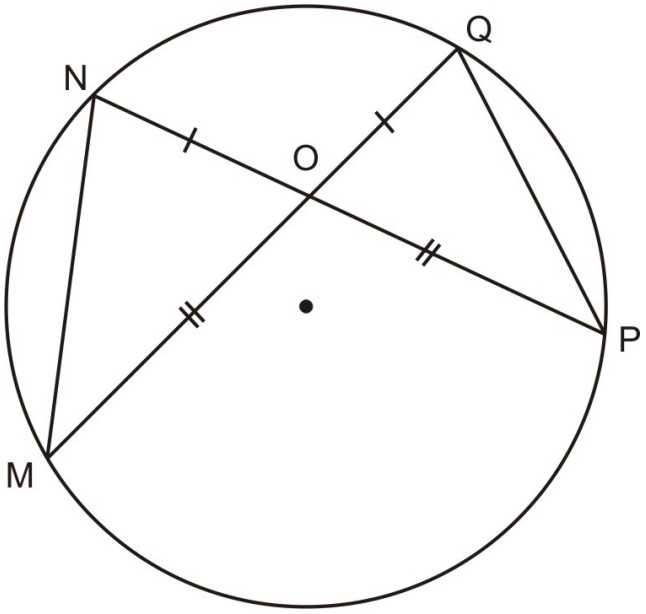


4. Complete the following congruence statement, if possible $\triangle TAR \cong$ _____.
5. What postulate allows you to make the congruence statement in 4, or, if it is not possible to make a congruence statement explain why.
6. Given the marked congruent parts in the triangles above, what *other* congruence statements do you now know based on your answers to 4 and 5?

Use the following diagram below for exercises 7-9.



7. Complete the following congruence statement, if possible $\triangle PET \cong$ _____.
8. What postulate allows you to make the congruence statement in 7, or, if it is not possible to make a congruence statement explain why.
9. Given the marked congruent parts in the triangles above, what *other* congruence statements do you now know based on your answers to 7 and 8?
10. Write one or two sentences and use a diagram to show why AAA is not a triangle congruence postulate.
11. Do the following proof using a two-column format.



Given: $\overline{MQ}$ and $\overline{NP}$ intersect at O.$\overline{NO} \cong \overline{OQ}$, and $\overline{MO} \cong \overline{OP}$

Prove: $\angle NMO \cong \angle OPN$

Table 4.6:

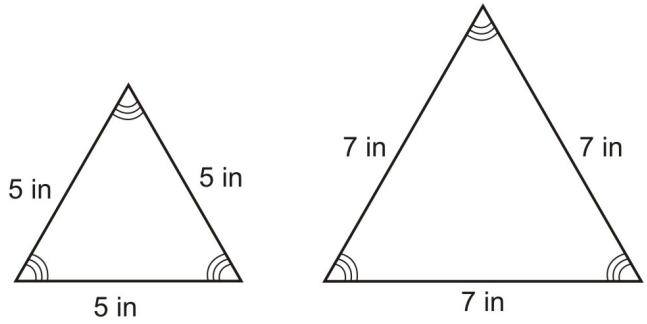
Statement	Reason
1. $\overline{NO} \cong \overline{OQ}$	1. Given
2. (Finish the proof using more steps!)	2.

Table 4.6: (continued)

Statement	Reason

Review Answers

1. $\triangle RGT \cong \triangle NPU$
2. HL triangle congruence postulate
3. $\overline{GR} \cong \overline{PN}, \angle T \cong \angle U$, and $\angle R \cong \angle N$
4. $\triangle TAR \cong \triangle PIM$
5. SAS triangle congruence postulate
6. $\angle T \cong \angle P, \angle A \cong \angle I, \overline{TA} \cong \overline{PI}$
7. No triangle congruence statement is possible
8. SSA is not a valid triangle congruence postulate
9. No other congruence statements are possible
10. One counterexample is to consider two equiangular triangles. If AAA were a valid triangle congruence postulate, than all equiangular (and equilateral) triangles would be congruent. But this is not the case. Below are two equiangular triangles that are not congruent:



These triangles are not congruent.

Table 4.7:

Statement	Reason
1. $\overline{NO} \cong \overline{OQ}$	1. Given
2. $\overline{MO} \cong OP$	2. Given
3. $\overline{MQ}$ and $\overline{NP}$ intersect at O	3. Given
4. $\angle NOM$ and $\angle QOP$ are vertical angles	4. Definition of vertical angles

Table 4.7: (continued)

Statement	Reason
5. $\angle NOM \cong \angle QOP$	5. Vertical angles theorem
6. $\triangle NOM \cong \triangle QOP$	6. SAS triangle congruence postulate
7. $\angle NMO \cong \angle OPN$	7. Definition of congruent triangles (CPCTC)

11.

4.6 Using Congruent Triangles

Learning Objectives

- Apply various triangles congruence postulates and theorems.
- Know the ways in which you can prove parts of a triangle congruent.
- Find distances using congruent triangles.
- Use construction techniques to create congruent triangles.

Introduction

As you can see, there are many different ways to prove that two triangles are congruent. It is important to know all of the different way that can prove congruence, and it is important to know which combinations of sides and angles do *not* prove congruence. When you prove properties of polygons in later chapters you will frequently use

Congruence Theorem Review

As you have studied in the previous lessons, there are five theorems and postulates that provide different ways in which you can prove two triangles congruent without checking all of the angles and all of the sides. It is important to know these five rules well so that you can use them in practical applications.

Table 4.8:

Name	Corresponding congruent parts	Does it prove congruence?
SSS	Three sides	Yes

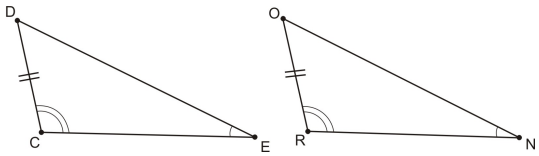
Table 4.8: (continued)

Name	Corresponding congruent parts	Does it prove congruence?
SAS	Two sides and the angle between them	Yes
ASA	Two angles and the side between them	Yes
AAS	Two angles and a side not between them	Yes
HL	A hypotenuse and a leg in a right triangle	Yes
AAA	Three angles	No—it will create a similar triangle, but not of the same size
SSA	Two sides and an angle not between them	No—this can create more than one distinct triangle

When in doubt, think about the models we created. If you can construct only one possible triangle given the constraints, then you can prove congruence. If you can create more than one triangle within the given information, you cannot prove congruence.

Example 1

What rule can prove that the triangles below are congruent?



A. SSS

B. SSA

C. ASA

D. AAS

The two triangles in the picture have two pairs of congruent angles and one pair of corresponding congruent sides. So, the triangle congruence postulate you choose must have two $A's$ (for the angles) and one S (for the side). You can eliminate choices A and B for this reason. Now that you are deciding between choices C and D, you need to identify where the side is located in relation to the given angles. It is adjacent to one angle, but it is not in between them. Therefore, you can prove congruence using AAS. The correct answer is D.

Proving Parts Congruent

It is one thing to identify congruence when all of the important identifying information is provided, but sometimes you will have to identify congruent parts on your own. You have already practiced this in a few different ways. When you were testing SSS congruence, you used the distance formula to find the lengths of sides on a coordinate grid. As a review, the distance formula is shown below.

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

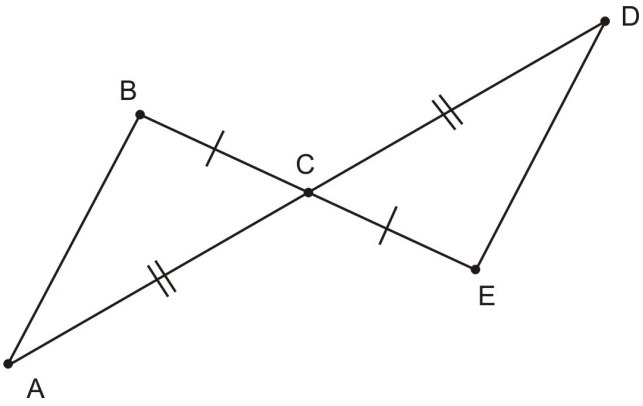
You can use the distance formula whenever you are examining shapes on a coordinate grid.

When you were creating two-column and flow proofs, you also used the **reflexive property of congruence**. This property states that any segment or angle is congruent to itself. While this may sound obvious, it can be very helpful in proofs, as you saw in those examples.

You may be tempted to use your ruler and protractor to check whether two triangles are congruent. However, this method **does not** necessarily work because not all pictures are drawn to scale.

Example 2

How could you prove $\triangle ABC \cong \triangle DEC$ in the diagram below?



We can already see that $\overline{BC} \cong \overline{CE}$ and $\overline{AC} \cong \overline{CD}$. We may be able to use SSS or SAS to show the triangles are congruent. However, to use SSS, we would need $\overline{AB} \cong \overline{DE}$ and there is no obvious way to prove this. Can we show that two of the angles are congruent? Notice that $\angle BCA$ and $\angle ECD$ are *vertical angles* (nonadjacent angles made by the intersection of two lines—i.e., angles on the opposite sides of the intersection).

The Vertical Angle Theorem states that all vertical angles are also congruent. So, this tells us that $\angle BCA \cong \angle ECD$. Finally, by putting all the information together, you can confirm that $\triangle ABC \cong \triangle DEC$ by the SAS Postulate.

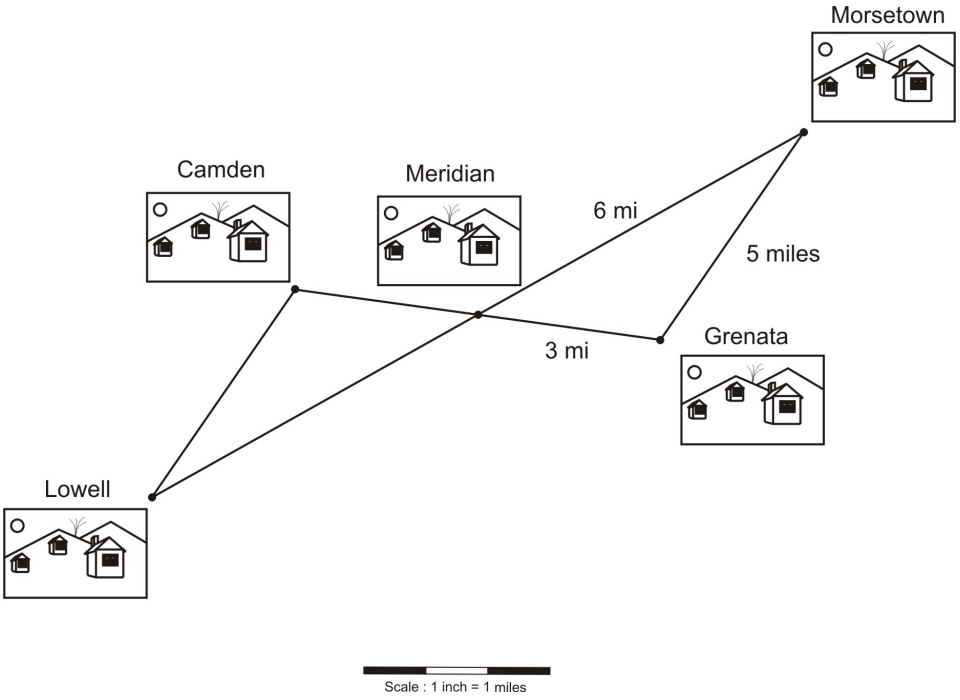
Finding Distances

One way to use congruent triangles is to help you find distances in real life—usually using a map or a diagram as a model.

When using congruent triangles to identify distances, be sure you always match up corresponding sides. The most common error on this type of problem involves matching two sides that are not corresponding.

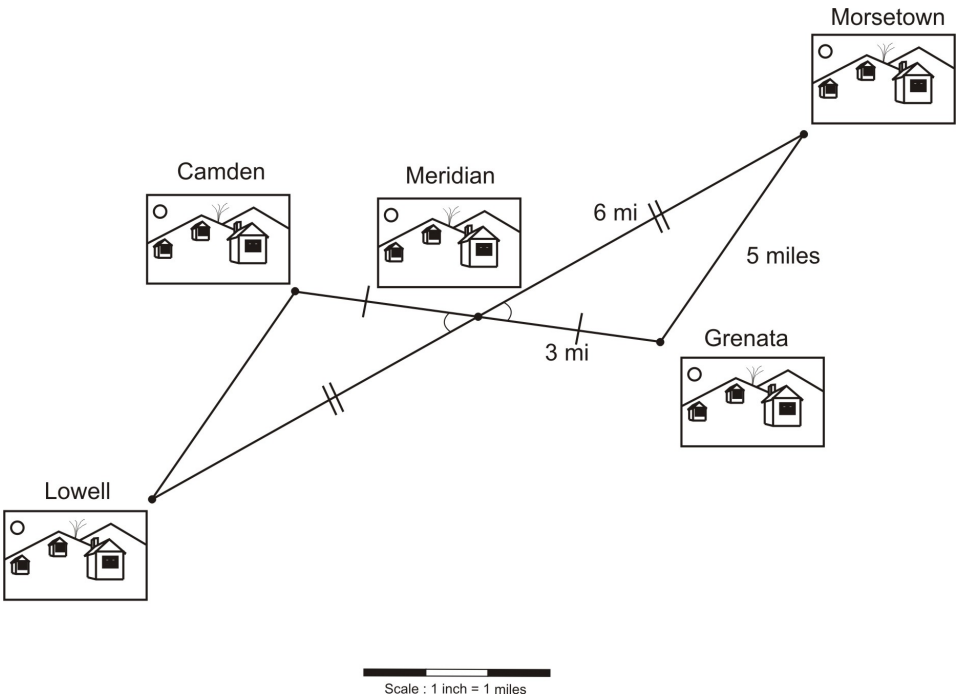
Example 3

The map below shows five different towns. The town of Meridian was given its name because it lies exactly halfway between two pairs of cities: Camden and Grenata, and Lowell and Morsetown.



Using the information in the map, what is the distance between Camden and Lowell?

The first step in this problem is to identify whether or not the marked triangles are congruent. Since you know that the distance from Camden to Meridian is the same as Meridian to Grenata, those two sides are congruent. Similarly, since the distance from Lowell to Meridian is the same as Meridian to Morsetown, those two sides are also a congruent pair. The angles between these lines are also congruent because they are vertical angles.



So, by the SAS postulate, these two triangles are congruent. This allows us to find the distance between Camden and Lowell by identifying its corresponding side on the other triangle. Because they are both opposite the vertical angle, the side connecting Camden and Lowell corresponds to the side connecting Morsetown and Grenata. Since the triangles are congruent, these corresponding sides will also be congruent to each other. Therefore, the distance between Camden and Lowell is five miles.

This use of the definition of congruent triangles is one of the most powerful tools you will use in geometry class. It is often abbreviated as **CPCTC**, meaning **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent.

Constructions

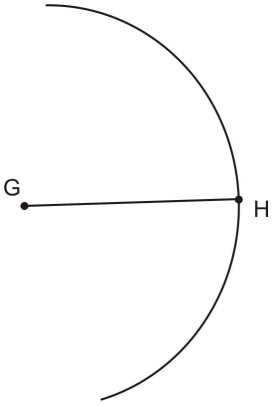
Another important part of geometry is creating geometric figures through **construction.** A construction is a drawing that is made using only a straightedge and a compass—you can think of construction as a special game in geometry in which we make figures using only these tools. You may be surprised how many shapes can be made this way.

Example 4

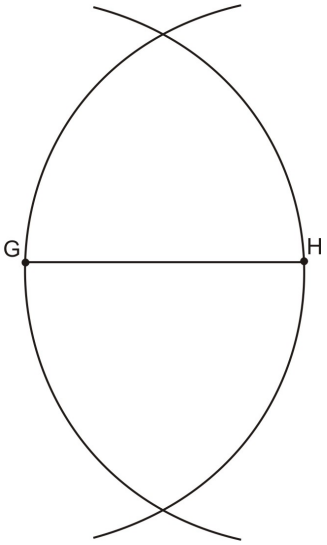
Use a compass and straightedge to construct the perpendicular bisector of the segment below.

Begin by using your compass to create an arc with the same distance from a point as the

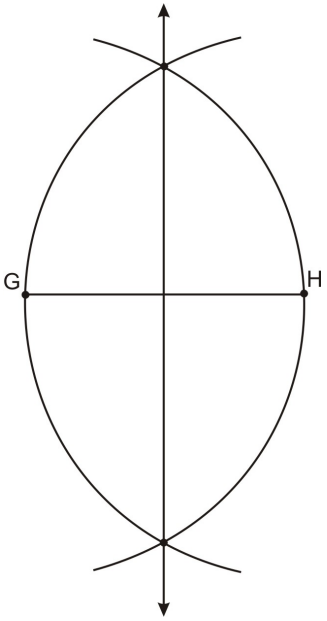
segment.



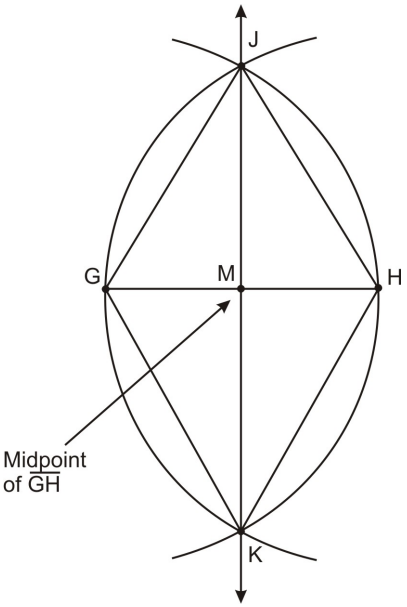
Repeat this process on the opposite side.



Now draw a line through the two points of intersections. This forms the perpendicular bisector.



Draw segments connecting the points on the bisector to the original endpoints.



Knowing that the center point is the midpoint of both line segments and that all angles formed around point M are right angles, you can prove that all four triangles created are congruent by the SAS rule.

Lesson Summary

In this lesson, we explored applications triangle congruence. Specifically, we have learned to:

- Identify various triangles congruence postulates and theorems.
- Use the fact that corresponding parts of congruent triangles are congruent.
- Find distances using congruent triangles.
- Use construction techniques to create congruent triangles.

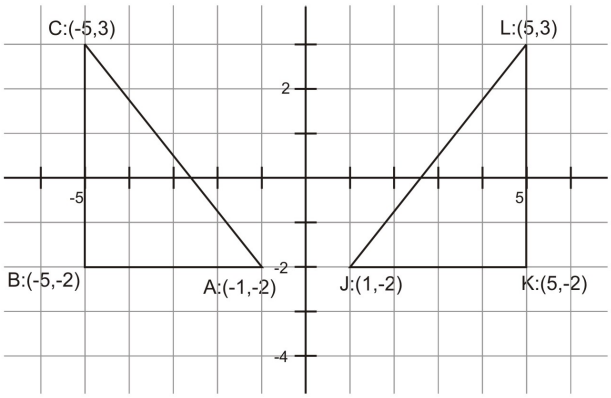
These skills will help you understand issues of congruence involving triangles. Always look for triangles in diagrams, maps, and other mathematical representations.

Points to Consider

You now know all the different ways in which you can prove two triangles congruent. In the next chapter you'll learn more about isosceles and equilateral triangles.

Review Questions

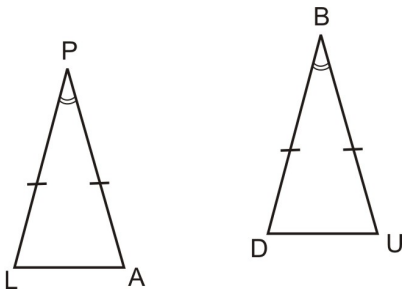
Use the following diagram for exercises 1-5



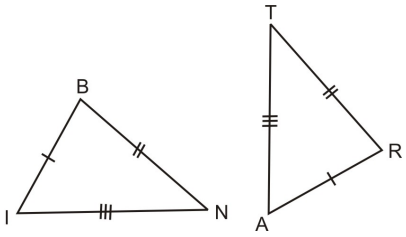
1. Find AB in the diagram above.
2. Find BC in the diagram above.
3. What is $m\angle ABC$? How do you know?
4. What postulate can you use to show $\triangle ABC \cong \triangle JKL$?
5. Use the distance formula to find AC. How can use this to find JL?

6-8: For each pair of triangles, complete the triangle congruence statement, or write "no congruence statement possible." Name the triangle congruence postulate you use, or write a sentence to explain why you can't write a triangle congruence statement.

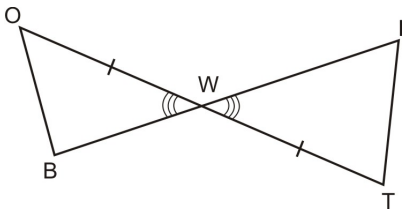
6. $\triangle PAL \cong \triangle$ _____.



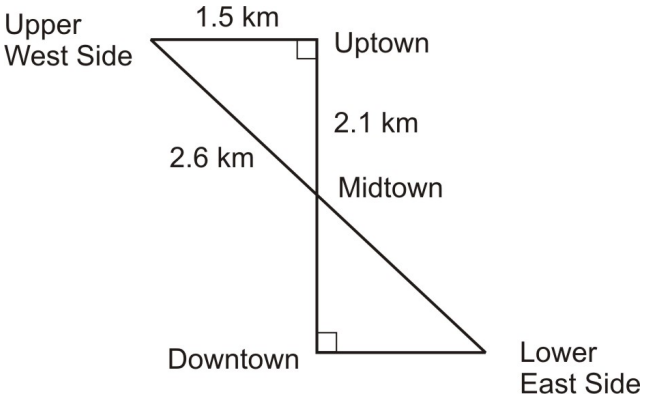
7. $\triangle BIN \cong \triangle$_____.



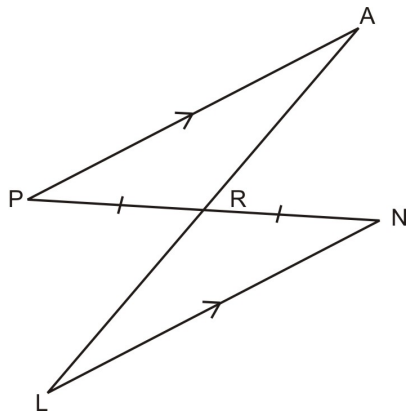
8. $\triangle BOW \cong \triangle$_____.



9. In the following diagram, Midtown is exactly halfway between Uptown and Downtown. What is the distance between Downtown and Lower East Side? How do you know? Write a few sentences to convince a reader your answer is correct.



10. Given: R is the midpoint of $\overline{PN}$ and $\overline{PA}||\overline{LN}$Prove: $\overline{PA} \cong \overline{LN}$



Review Answers

1. $AB = 4$
2. $BC = 5$
3. 90°. Since $\overline{AB}$ is horizontal (parallel to the $x-$axis) and $\overline{BC}$ is vertical (parallel to the $y-$axis), we can conclude that they intersect at a right angle.
4. SAS (other answers are possible)
5. $AC = \sqrt{(-1-(-5))^2+(-2-3)^2} = \sqrt{16+25} = \sqrt{41}$. Since the triangles are congruent, we can conclude $JL = \sqrt{41}$
6. $\triangle PAL \cong \triangle BUD$. SAS triangle congruence postulate
7. $\triangle BIN \cong \triangle RAT$. SSS triangle congruence postulate
8. No congruence statement is possible; we don't have enough information.
9. 1.5 km. Since Midtown is the midpoint of the line connecting Uptown and Downtown, we can use the vertical angle theorem for the angles made by the two lines that meet at Midtown, and then we can conclude that the triangles are congruent using AAS. If the triangles are congruent then all corresponding parts are also congruent

Table 4.9:

Statement	Reason
1. R is the midpoint of $\overline{PN}$	1. Given
2. $\overline{PR} \cong \overline{RN}$	2. Definition of midpoint
3. $\overline{PA} \| \overline{LN}$	3. Given
4. $\angle RNL \cong \angle RPA$	4. Alternate interior angles theorem
5. $\angle PAR \cong \angle NLR$	5. Alternate interior angles theorem

Table 4.9: (continued)

Statement	Reason
6. $\triangle PAR \cong \triangle NLR$	6. AAS triangle congruence postulate
7. $\overline{PA} \cong \overline{LN}$	7. Definition of congruent triangles (corresponding parts are congruent)

10.

4.7 Isosceles and Equilateral Triangles

Learning Objectives

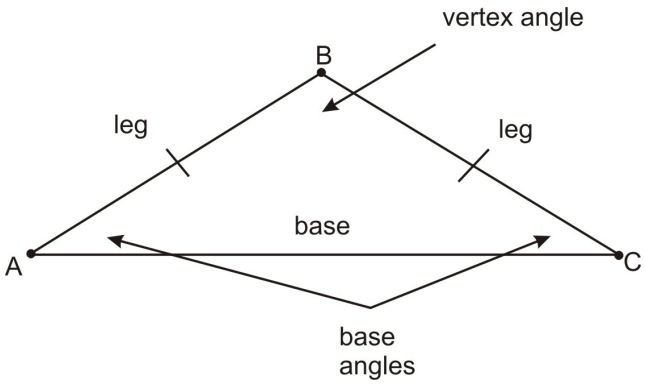
- Prove and use the Base Angles Theorem.
- Prove that an equilateral triangle must also be equiangular.
- Use the converse of the Base Angles Theorem.
- Prove that an equiangular triangle must also be equilateral.

Introduction

As you can imagine, there is more to triangles than proving them congruent. There are many different ways to analyze the angles and sides within a triangle to understand it better. This chapter addresses some of the ways you can find information about two special triangles.

Base Angles Theorem

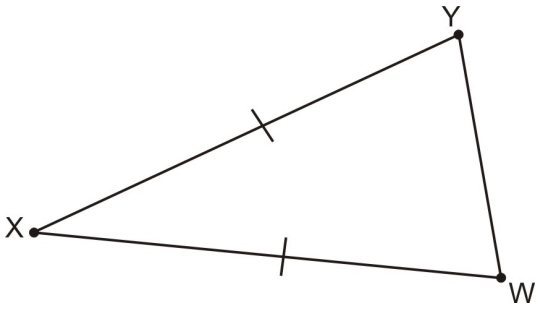
An **isosceles triangle** is defined as a triangle that has at least two congruent sides. In this lesson you will prove that an isosceles triangle also has two congruent angles opposite the two congruent sides. The congruent sides of the isosceles triangle are called the **legs** of the triangle. The other side is called the **base** and the angles between the base and the congruent sides are called **base angles.** The angle made by the two legs of the isosceles triangle is called the **vertex angle.**



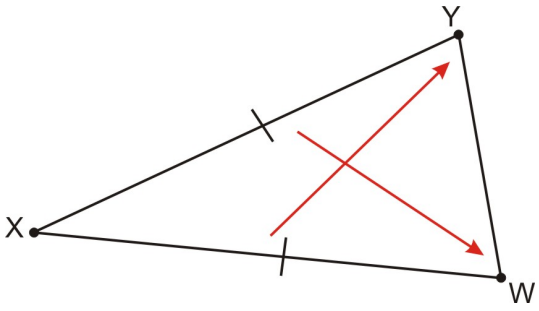
The **Base Angles Theorem** states that if two sides of a triangle are congruent, then their opposite angles are also congruent. In other words, the base angles of an isosceles triangle are congruent. Note, this theorem does not tell us about the vertex angle.

Example 1

Which two angles must be congruent in the diagram below?



The triangle in the diagram is an isosceles triangle. To find the congruent angles, you need to find the angles that are opposite the congruent sides.



This diagram shows the congruent angles. The congruent angles in the triangle are $\angle XYW$ and $\angle XWY$.

So, how do we prove the base angles theorem? Using congruent triangles.

Given: Isosceles $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$

Prove $\angle B \cong \angle C$

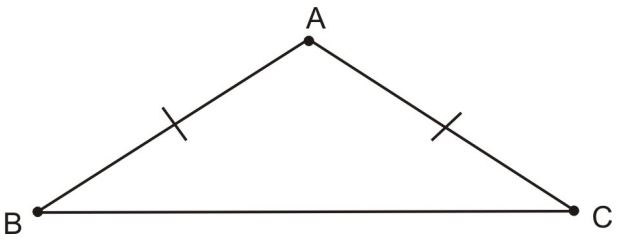
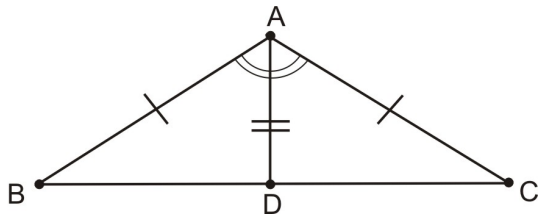
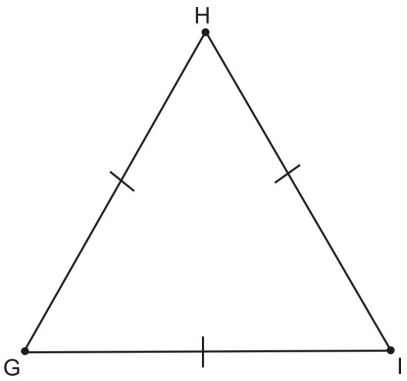


Table 4.10:

Statement	Reason
1. $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$.	1. Given
2. Construct Angle Bisector $\overline{AD}$	2. Angle Bisector Postulate
3. $\angle BAD \cong \angle CAD$	3. Definition of Angle Bisector
4. $\overline{AD} \cong \overline{AD}$	4. Reflexive Property
 5. $\triangle ABD \cong \triangle ACD$	5. SAS Postulate
6. $\angle B \cong \angle C$	6. Definition of congruent triangles (all pairs of corresponding angles are congruent)

Equilateral Triangles

The base angles theorem also applies to **equilateral triangles.** By definition, all sides in an equilateral triangle have exactly the same length.



Because of the base angles theorem, we know that angles opposite congruent sides in an isosceles triangle are congruent. So, if all three sides of the triangle are congruent, then all of the angles are congruent as well.

A triangle that has all angles congruent is called an **equiangular triangle.** So, as a result of the base angles theorem, you can identify that *all equilateral triangles are also equiangular triangles.*

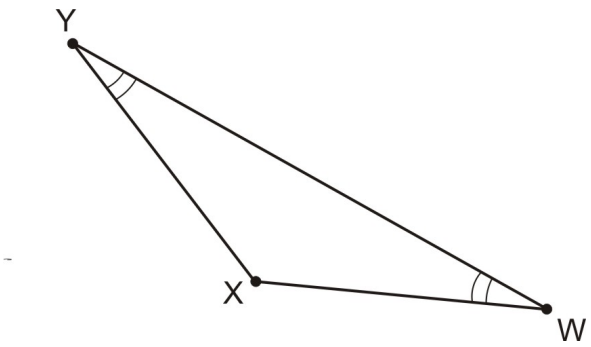
Converse of the Base Angles Theorem

As you know, some theorems have a converse that is also true. Recall that a converse identifies the "backwards," or reverse statement of a theorem. For example, if I say, "If I turn a faucet on, then water comes out," I have made a statement. The converse of that statement is, "If water comes out of a faucet, then I have turned the faucet on." In this case the converse is not true. For example the faucet may have a drip. So, as you can see, converse statements are sometimes true, but not always.

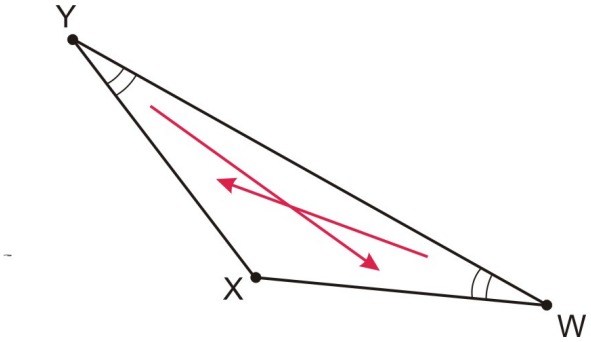
The converse of the base angles theorem is always true. The base angles theorem states that if two sides of a triangle are congruent the angles opposite them are also congruent. The converse of this statement is that if two angles in a triangle are congruent, then the sides opposite them will also be congruent. You can use this information to identify isosceles triangles in many different circumstances.

Example 2

Which two sides must be congruent in the diagram below?



$\triangle WXY$ has two congruent angles. By the converse of the base angles theorem, it is an isosceles triangle. To find the congruent sides, you need to find the sides that are opposite the congruent angles.



This diagram shows arrows pointing to the congruent sides. The congruent sides in this triangle are $\overline{XY}$ and $\overline{XW}$.

The proof of the converse of the base angles theorem will depend on a few more properties of isosceles triangles that we will prove later, so for now we will omit that proof.

Equiangular Triangles

Earlier in this lesson, you extrapolated that all equilateral triangles were also equiangular triangles and proved it using the base angles theorem. Now that you understand that the converse of the base angles theorem is also true, the converse of the equilateral/equiangular relationship will also be true.

If a triangle has three congruent angles, it is be equiangular. Since congruent angles have congruent sides opposite them, all sides in an equiangular triangle will also be congruent. **Therefore, every equiangular triangle is also equilateral**.

Lesson Summary

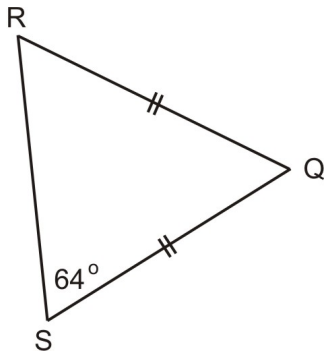
In this lesson, we explored isosceles, equilateral, and equiangular triangles. Specifically, we have learned to:

- Prove and use the Base Angles Theorem.
- Prove that an equilateral triangle must also be equiangular.
- Use the converse of the Base Angles Theorem.
- Prove that an equiangular triangle must also be equilateral.

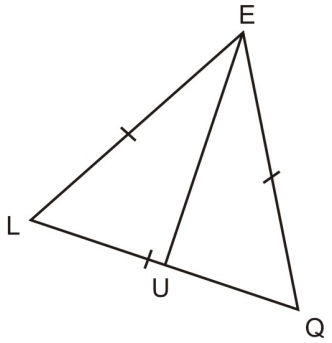
These skills will help you understand issues of analyzing triangles. Always look for triangles in diagrams, maps, and other mathematical representations.

Review Questions

1. Sketch and label an isosceles $\triangle ABC$ with legs $\overline{AB}$ and $\overline{BC}$ that has a vertex angle measuring 118°.
2. What is the measure of each base angle in $\triangle ABC$ from 1?
3. Find the measure of each angle in the triangle below:



4. $\triangle EQL$ below is equilateral. If $\overline{EU}$ bisects $\angle E$, find:



   (a) $m\angle EUL$
   (b) $m\angle UEL$

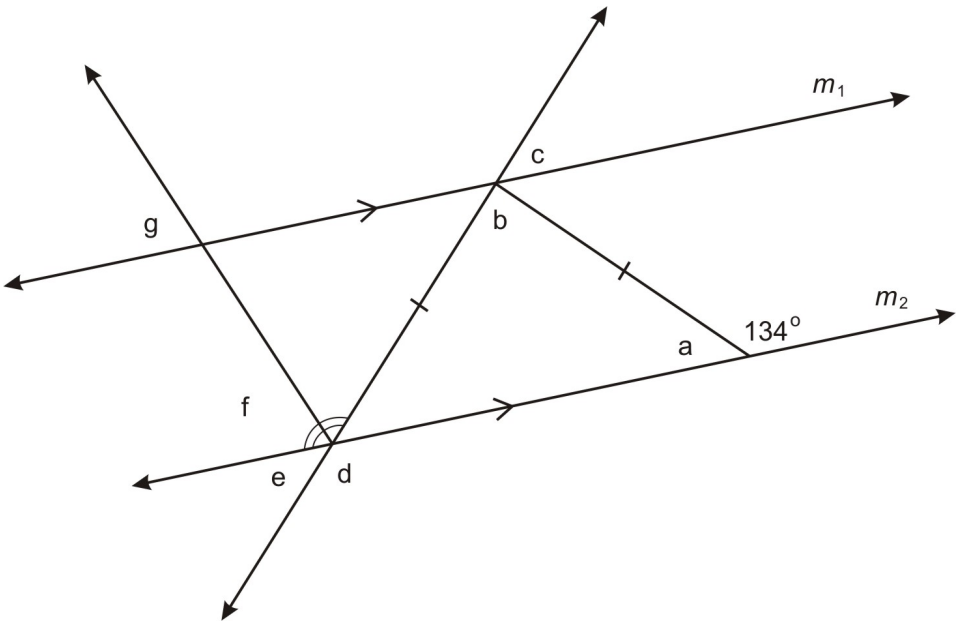
(c) $m\angle ELQ$

5. Which of the following statements *must* be true about the base angles of an isosceles triangle ?

   (a) The base angles are congruent.
   (b) The base angles are complementary.
   (c) The base angles are acute.
   (d) The base angles can be right angles.

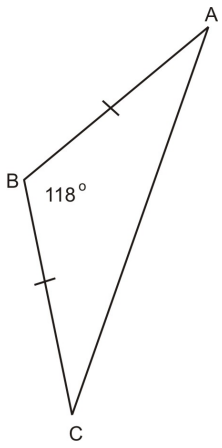
6. One of the statements in 5 is possible (i.e., sometimes true), but not necessarily always true. Which one is it? For the statement that is always false draw a sketch to show why.

7-13: In the diagram below, $m_1 \| m_2$. Use the given angle measure and the geometric markings to find each of the following angles.

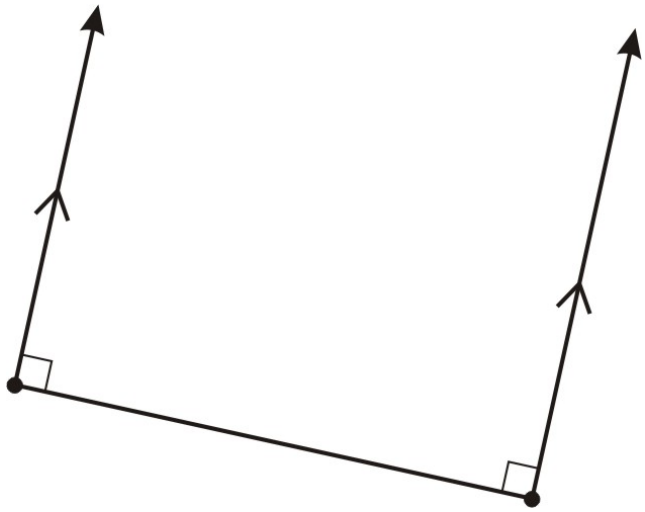


7. $a =$ _____
8. $b =$ _____
9. $c =$ _____
10. $d =$ _____
11. $e =$ _____
12. $f =$ _____
13. $g =$ _____

Review Answers

1. 

2. Each base angle in $\triangle ABC$ measures 31°
3. $m\angle R = 64^\circ$ and $m\angle Q = 52^\circ$
4. (a) $m\angle EUL = 90^\circ$,
   (b) $m\angle UEL = 30^\circ$,
   (c) $m\angle ELQ = 60^\circ$
5. a. and c. only.
6. b. is possible if the base angles are 45°. When this happens, the vertex angle is 90°.
   d. is impossible because if the base angles are right angles, then the "sides" will be parallel and you won't have a triangle.



7. $a = 46^\circ$
8. $b = 88^\circ$
9. $c = 46^\circ$
10. $d = 134^\circ$
11. $e = 46^\circ$
12. $f = 67^\circ$

13. $g = 67°$

4.8 Congruence Transformations

Learning Objectives

- Identify and verify congruence transformations.
- Identify coordinate notation for translations.
- Identify coordinate notation for reflections over the axes.
- Identify coordinate notation for rotations about the origin.

Introduction

Transformations are ways to move and manipulate geometric figures. Some transformations result in congruent shapes, and some don't. This lesson helps you explore the effect of transformations on congruence and find location of the resulting figures. In this section we will work with figures in the coordinate grid.

Congruence Transformations

Congruent shapes have exactly the same size and shape. Many types of transformations will keep shapes congruent, but not all. A quick review of transformations follows.

Table 4.11:

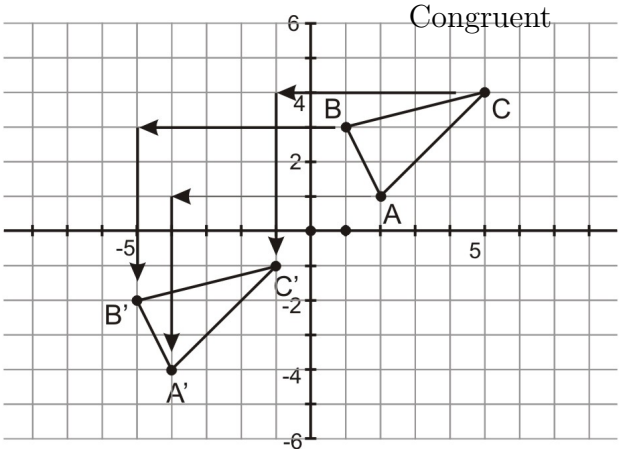
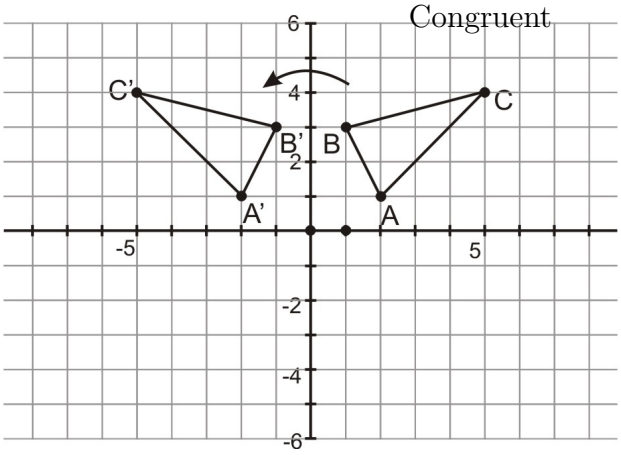
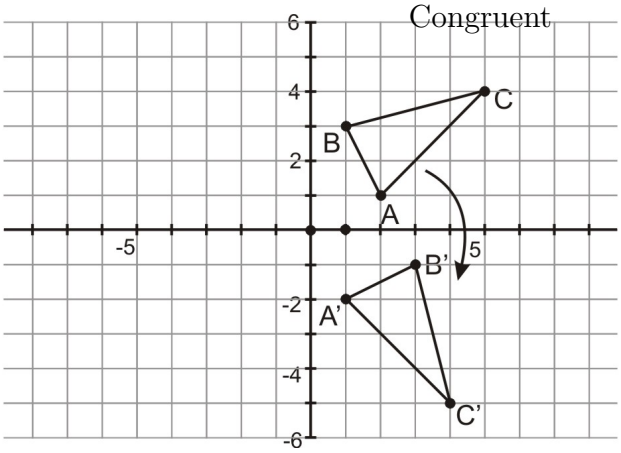
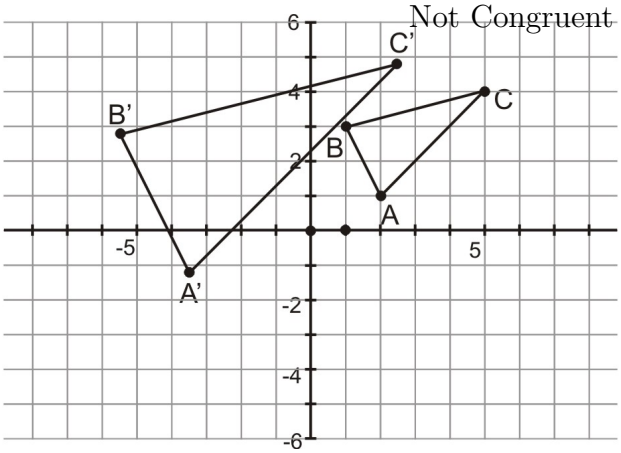
Transformation	Diagram	Congruent or Not?
Translation (Slide)		Congruent

Table 4.11: (continued)

Transformation	Diagram	Congruent or Not?
Reflection (Flip)		Congruent
Rotation (Turn)		Congruent
Dilation (Enlarge or Shrink)		Not Congruent

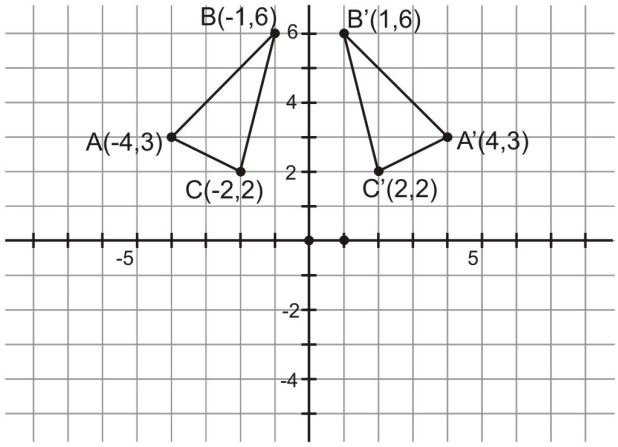
As you can see, the only transformation in this list that interferes with the congruence of the shapes is **dilation**. Dilated figures (whether larger or smaller) have the same shape, but

not the same size. So, these shapes will be similar, but not congruent.

When in doubt, check the length of each side of a triangle in the coordinate grid by using the distance formula. Remember that if triangles have three pairs of congruent sides, the triangles are congruent by the SSS triangle congruence postulate.

Example 1

Use the distance formula to prove that the reflected image below is congruent to the original triangle ABC.



Begin with triangle ABC. First write the coordinates.

A is $(-4, 3)$

B is $(-1, 6)$

C is $(-2, 2)$

Now use the coordinates and the distance formula to find the lengths of each segment in the triangle.

$$\begin{aligned} AB &= \sqrt{((-4)-(-1))^2+(3-6)^2} \\ &= \sqrt{(-4+1)^2+(-3)^2} \\ &= \sqrt{(-3)^2+(-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{((-1)-(-2))^2+(6-2)^2} \\ &= \sqrt{(-1+2)^2+(4)^2} \\ &= \sqrt{(1)^2+(4)^2} \\ &= \sqrt{1+16} \\ &= \sqrt{17} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{((-4)-(-2))^2+(3-2)^2} \\ &= \sqrt{(-4+2)^2+(1)^2} \\ &= \sqrt{(-2)^2+(1)^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \end{aligned}$$

The lengths are as follows.

$AB = \sqrt{18}$, $BC = \sqrt{17}$, and $AC = \sqrt{5}$

Next find the lengths in triangle $A'B'C'$. First write down the coordinates.

A' is $(4, 3)$
B' is $(1, 6)$
C' is $(2, 2)$

Now use the coordinates to find the lengths of each segment in the triangle.

$$\begin{aligned} A'B' &= \sqrt{(4-1)^2+(3-6)^2} \\ &= \sqrt{(3)^2+(-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \end{aligned}$$

$$\begin{aligned} B'C' &= \sqrt{(1-2)^2+(6-2)^2} \\ &= \sqrt{(-1)^2+(4)^2} \\ &= \sqrt{1+16} \\ &= \sqrt{17} \end{aligned}$$

$$\begin{aligned} A'C' &= \sqrt{(4-2)^2+(3-2)^2} \\ &= \sqrt{(2)^2+(1)^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \end{aligned}$$

The lengths are as follows.

$A'B' = \sqrt{18}$, $B'C' = \sqrt{17}$, and $A'C' = \sqrt{5}$

Using the distance formula, we demonstrated that the corresponding sides of the two triangles have the same lengths. Therefore, by the SSS congruence postulate, these triangles are congruent. This example shows that reflected figures are congruent.

Translations

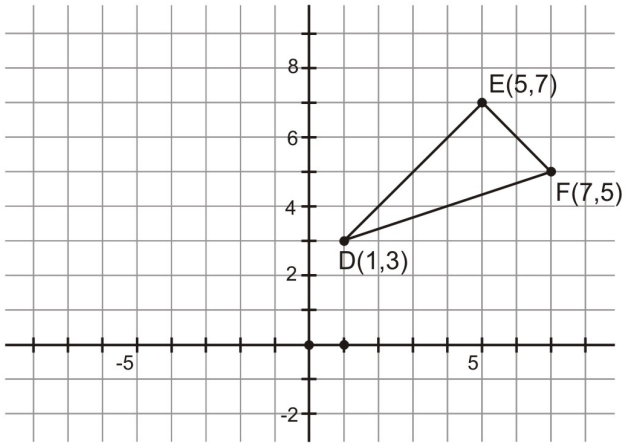
The transformation you saw above is called a reflection. **Translations** are another type of transformation. You translate a figure by moving it right or left and up or down. It is important to know how a transformation of a figure affects the coordinates of its vertices. You'll now have the opportunity to practice translating images and changing the coordinates.

For each unit a figure is translated to the **right, add** 1 unit to each $x-$coordinate in the vertices. For each unit a figure is translated to the **left, subtract** 1 from the $x-$coordinates. Always remember that moving a figure left and right only affects the $x-$coordinates.

If a figure is translated up or down, it affects the $y-$coordinate. So, if you move a figure **up** by 1 unit, then **add** 1 unit to each of the $y-$coordinates in the vertices. Similarly, if you translate a figure **down** by 1 unit, **subtract** 1 unit from the $y-$coordinates.

Example 2

$\triangle DEF$ is shown on the coordinate grid below. What would be the coordinates of $\triangle D'E'F'$ if it has been translated 4 units *to the left and* 2 units *up*?



Analyze the change and think about how that will affect the coordinates of the vertices. The translation moves the figure 4 units to the *left*. That means you will *subtract* 4 from each of the $x-$coordinates. It also says you will move the figure *up* 2 units, which means that you will *add* 2 to each of the $y-$coordinates. So, the coordinate change can be expressed as follows.

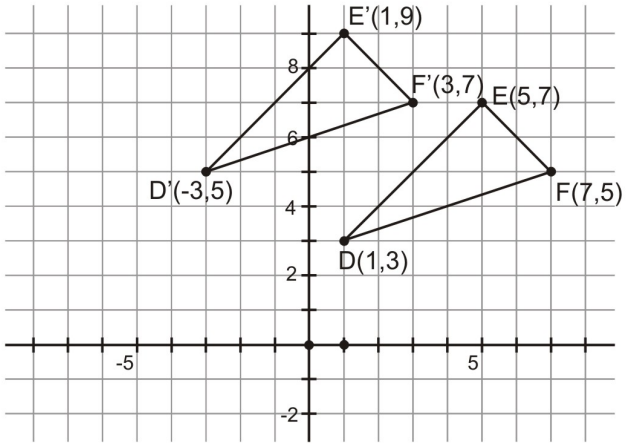
$$(x, y) \rightarrow (x - 4, y + 2)$$

Carefully adjust each coordinate using the formula above.

$$D(1, 3) \rightarrow D'(1 - 4, 3 + 2)$$
$$E(5, 7) \rightarrow E'(5 - 4, 7 + 2)$$
$$F(7, 5) \rightarrow F'(7 - 4, 5 + 2)$$

This gives us the new coordinates $D'(-3, 5)$, $E'(1, 9)$, and $F'(3, 7)$.

Finally, draw the translated triangle to verify that your answer is correct.



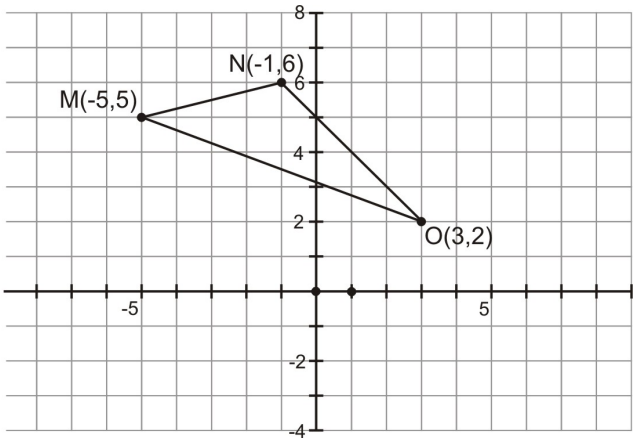
Reflections

Reflections are another form of transformation that also result in congruent figures. When you "flip" a figure over the $x-$axis or $y-$axis, you don't actually change the shape at all. To find the coordinates of a reflected figure, use the opposite of one of the coordinates.

1. If you **reflect an image over the $x-$axis,** the **new $y-$coordinates will be the opposite of the old $y-$coordinates.** The $x-$coordinates remain the same.
2. If you **reflect an image over the $y-$axis**, take the **opposite of the $x-$coordinates.** The $y-$coordinates remain the same.

Example 3

Triangle MNO is shown on the coordinate grid below. What would be the coordinates of $M'N'O'$ if it has been reflected over the $x-$axis?



Since you are finding the reflection of the image over the $x-$axis, you will find the opposite of the $y-$coordinates. The $x-$coordinates will remain the same. So, the coordinate change can be expressed as follows.

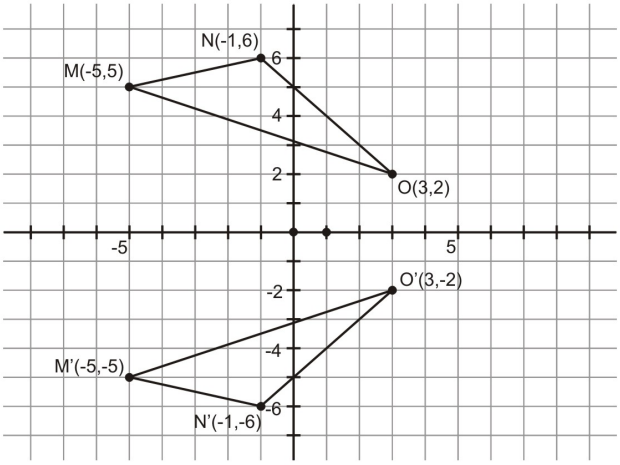
$$(x, y) \rightarrow (x, -y)$$

Carefully adjust each coordinate using the formula above.

$$M(-5, 5) \rightarrow M'(-5, -(5))$$
$$N(-1, 6) \rightarrow N'(-1, -(6))$$
$$O(3, 2) \rightarrow O'(3, -(2))$$

This gives new coordinates $M'(-5, -5)$, $N'(-1, -6)$ and $O'(3, -2)$.

Draw the translated triangle to verify that your answer is correct.



Rotations

The most complicated of the congruence transformations is rotations. To simplify rotations, we will only be concerned with rotations of 90° or 180° about the origin $(0, 0)$. The rules describe how coordinates change under rotations.

180° **rotations: Take the opposite of both coordinates.**

(x, y) becomes $(-x, -y)$

$90°$ clockwise rotations: Find the opposite of the $x-$coordinate, and reverse the coordinates.

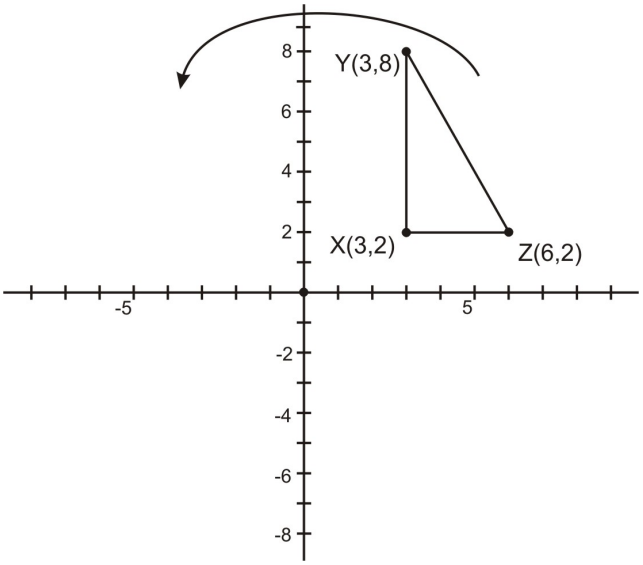
(x, y) becomes $(y, -x)$

$90°$ counterclockwise rotations: Find the opposite of the. $y-$coordinate, and reverse the coordinates.

(x, y) becomes $(-y, x)$

Example 4

Triangle XYZ is shown on the following coordinate grid. What would be the coordinates of $X'Y'Z'$ if it has been rotated $90°$ counterclockwise about the origin?



Since you are finding the rotation of the image $90°$ counterclockwise about the origin, you will find the opposite of the $y-$coordinates and then reverse the order. So, the coordinate change can be expressed as follows.

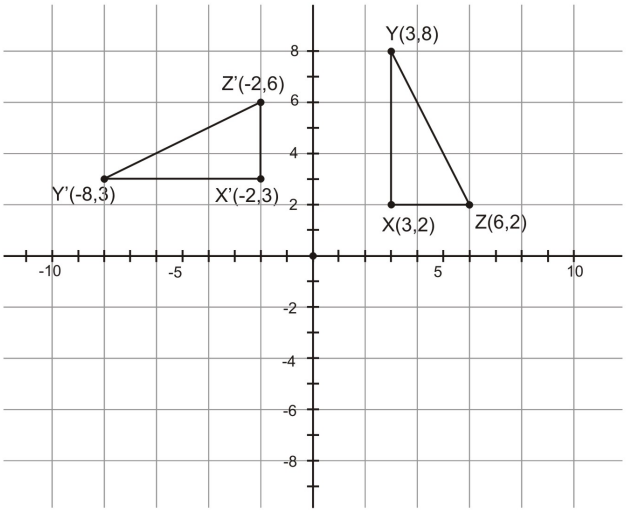
$$(x, y) \rightarrow (-y, x)$$

Carefully adjust each coordinate using the formula above.

$$X(3, 2) \rightarrow X'(-(2), 3)$$
$$Y(3, 8) \rightarrow Y'(-(8), 3)$$
$$Z(6, 2) \rightarrow Z'(-(2), 6)$$

This gives new coordinates $X'(-2, 3)$, $Y'(-8, 3)$, and $Z'(-2, 6)$.

Finally, we draw the rotated triangle to verify that your answer is correct.



Lesson Summary

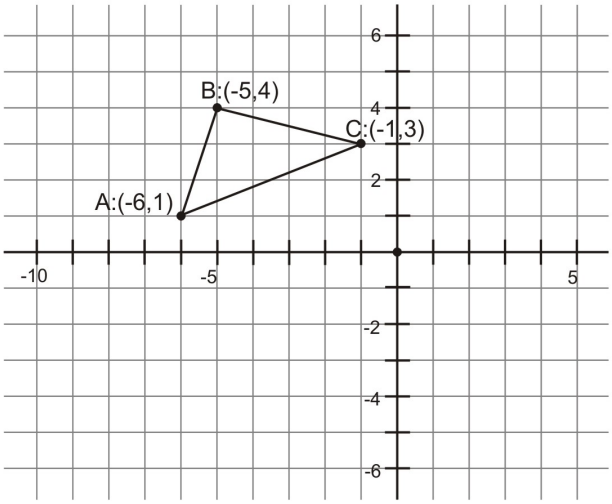
In this lesson, we explored transformations with triangles. Specifically, we have learned to:

- Identify and verify congruence transformations.
- Identify coordinate notation for translations.
- Identify coordinate notation for reflections over the axes.
- Identify coordinate notation for rotations about the origin.

These skills will help you understand many different situations involving coordinate grids. Always look for triangles in diagrams, maps, and other mathematical representations.

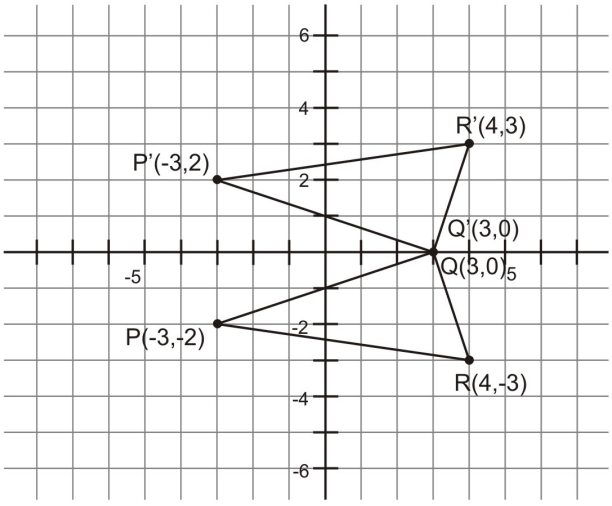
Review Questions

Use the following diagram of $\triangle ABC$ for exercise 1-4. Given the coordinates $A(-6, 1)$, $B(-5, 4)$, and $C(-1, 3)$, find the new coordinates of A', B', C' after each transformation.



1. Slide down three units.
2. Slide up 2 units and to the right 5 units.
3. Reflect across the $y-$axis.
4. Rotate 90° clockwise about the origin. Draw a sketch to help visualize what this looks like.

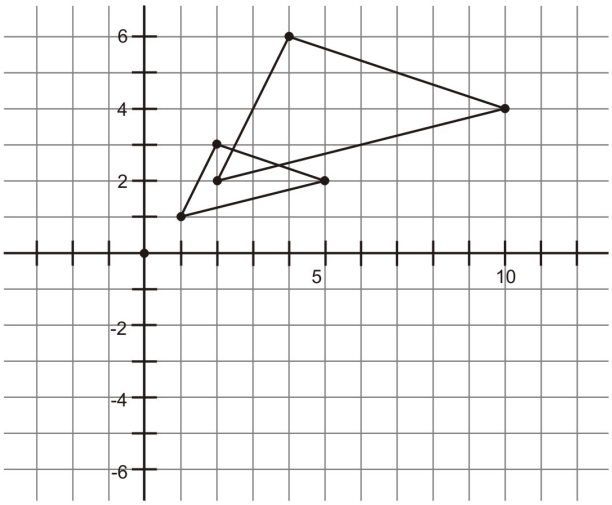
Use the following diagram that shows a transformation of $\triangle PQR$ to $\triangle P'Q'R'$ for exercises 5-7:



5. What kind of transformation was used to go from $\triangle PQR$ to $\triangle P'Q'R'$?
6. Use the distance formula to show that $PQ = P'Q'$.

7. Is the transformation congruence preserving? Justify your answer.

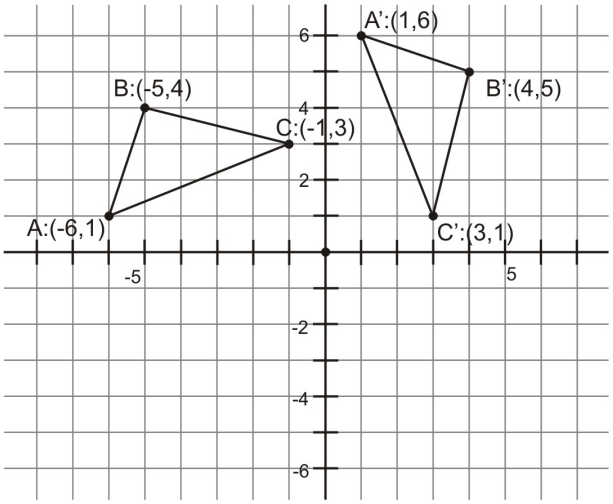
Use the following diagram for exercises 8-9.



8. What kind of transformation is shown above?
9. Is this transformation congruence preserving? Justify your answer.
10. Can a 180° rotation be described in terms of reflections? Justify your answer.

Review Answers

1. $A' : (-6, -2)$, $B' : (-5, 1)$, $C' : (-1, 0)$
2. $A' : (-1, 3)$, $B' : (0, 6)$, $C' : (4, 5)$
3. $A' : (6, 1)$, $B' : (5, 4)$, $C' : (1, 3)$
4. $A' : (1, 6)$, $B' : (4, 5)$, $C' : (3, 1)$



5. Reflection about the $x-$axis

6. $$\begin{aligned} PQ &= \sqrt{(3-(-3))^2+(0-(-2))^2} \\ &= \sqrt{6^2+2^2} \\ &= \sqrt{36+4} = \sqrt{40} \\ &= \sqrt{(3-(-3))^2+(0-(2))^2} \\ &= \sqrt{(6^2+(-2)^2} \\ &= \sqrt{36+4} = \sqrt{40} \end{aligned}$$

7. .
8. This is a dilation.
9. No, we can see that each side of the larger triangle is twice as long as the corresponding side in the original, so this is not length preserving.
10. Yes, a 180° rotation about the origin is the same as two reflections done consecutively, one across the $x-$axis and then one across the $y-$axis. For a 180°; rotation, the rule for transforming coordinates is $(x, y) \rightarrow (-x, -y)$. Now, suppose point (x, y) is reflected twice, first across the $x-$axis, and then across the $y-$axis. After the first transformation, the coordinates are $(x, y) \rightarrow (x, -y)$ Then after reflection on the $y-$axis, we get $(x, -y) \rightarrow (x, -y)$, which is the same coordinates that result from a 180° rotation.

Chapter 5

Relationships Within Triangles

5.1 Midsegments of a triangle

Learning Objectives

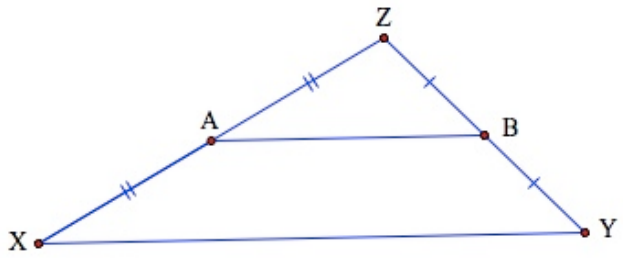
- Identify the midsegment of a triangle.
- Apply the Midsegment Theorem to solve problems involving side lengths and midsegments of triangles.
- Use the Midsegment Theorem to solve problems involving variable side lengths and midsegments of triangles.

Introduction

In previous lessons, we used the parallel postulate to learn new theorems that enabled us to solve a variety of problems about parallel lines:

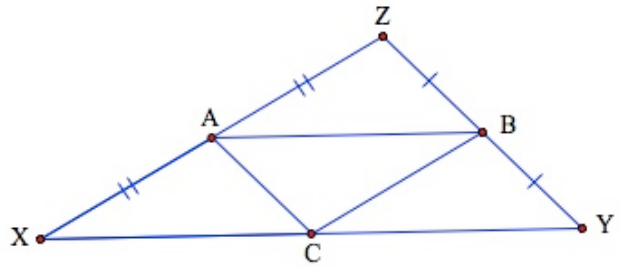
Parallel Postulate: Given: line l and a point P not on l. There is exactly one line through P that is parallel to l.

In this lesson we extend these results to learn about special line segments within triangles. For example, the following triangle contains such a configuration:



Triangle $\triangle XYZ$ is cut by $\overline{AB}$ where A and B are midpoints of sides $\overline{XZ}$ and $\overline{YZ}$ respectively. $\overline{AB}$ is called a ***midsegment*** of $\triangle XYZ$. Note that $\triangle XYZ$ has other midsegments in addition to $\overline{AB}$. Can you see where they are in the figure above?

If we construct the midpoint of side $\overline{XY}$ at point C and construct $\overline{CA}$ and $\overline{CB}$ respectively, we have the following figure and see that segments $\overline{CA}$ and $\overline{CB}$ are midsegments of $\triangle XYZ$.



In this lesson we will investigate properties of these segments and solve a variety of problems.

Properties of midsegments within triangles

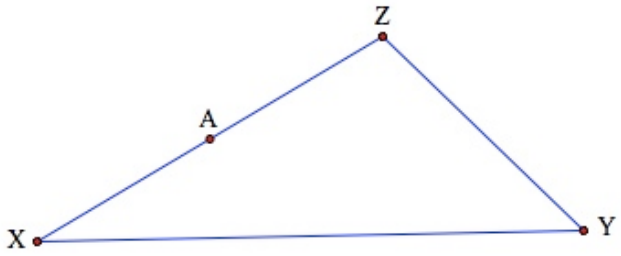
We start with a theorem that we will use to solve problems that involve midsegments of triangles.

Midsegment Theorem: The segment that joins the midpoints of a pair of sides of a triangle is:

1. parallel to the third side.
2. half as long as the third side.

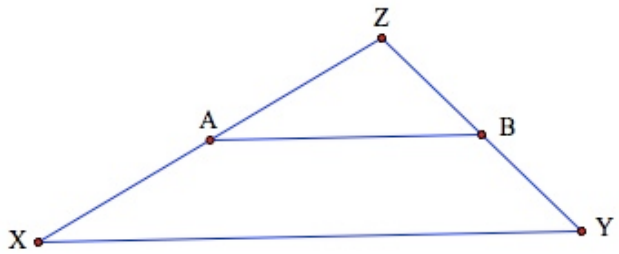
Proof of 1. We need to show that a midsegment is parallel to the third side. We will do this using the Parallel Postulate.

Consider the following triangle $\triangle XYZ$. Construct the midpoint A of side $\overline{XZ}$.



By the Parallel Postulate, there is exactly one line though A that is parallel to side $\overline{XY}$. Let's say that it intersects side $\overline{YZ}$ at point B. We will show that B must be the midpoint

of $\overline{XY}$ and then we can conclude that $\overline{AB}$ is a midsegment of the triangle and is parallel to $\overline{XY}$.

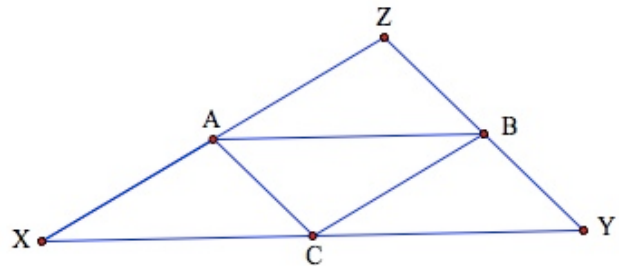


We must show that the line through A and parallel to side $\overline{XY}$ will intersect side $\overline{YZ}$ at its midpoint. If a parallel line cuts off congruent segments on one transversal, then it cuts off congruent segments on every transversal. This ensures that point B is the midpoint of side $\overline{YZ}$.

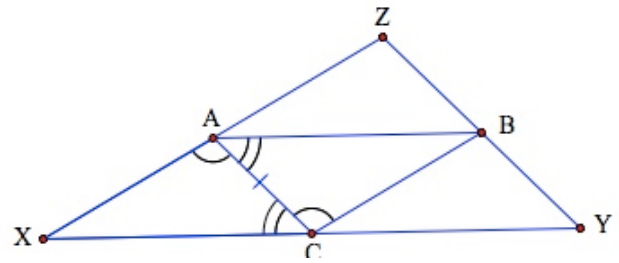
Since $\overline{XA} \cong \overline{AZ}$, we have $\overline{BZ} \cong \overline{BY}$. Hence, by the definition of midpoint, point B is the midpoint of side $\overline{YZ}$. $\overline{AB}$ is a midsegment of the triangle and is also parallel to $\overline{XY}$.

Proof of 2. We must show that $AB = \frac{1}{2}XY$.

In $\triangle XYZ$, construct the midpoint of side $\overline{XY}$ at point C and midsegments $\overline{CA}$ and $\overline{CB}$ as follows:



First note that $\overline{CB} \parallel \overline{XZ}$ by part one of the theorem. Since $\overline{CB} \parallel \overline{XZ}$ and $\overline{AB} \parallel \overline{XY}$, then $\angle XAC \cong \angle BCA$ and $\angle CAB \cong \angle ACX$ since alternate interior angles are congruent. In addition, $\overline{AC} \cong \overline{CA}$.

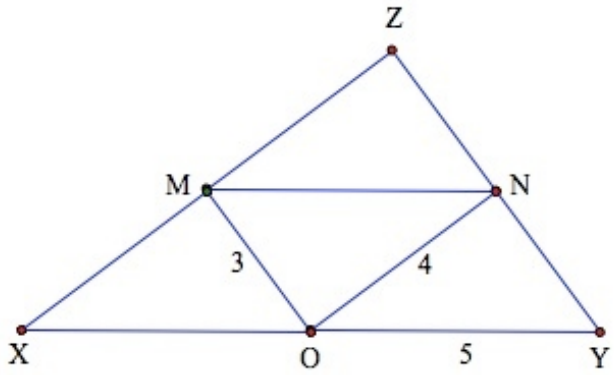


Hence, $\triangle AXC \cong \triangle CBA$ by The ASA Congruence Postulate. $\overline{AB} \cong \overline{XC}$ since corresponding parts of congruent triangles are congruent. Since C is the midpoint of $\overline{XY}$, we have $XC = CY$ and $XY = XC + CY = XC + XC = 2AB$ by segment addition and substitution.

So, $2AB = XY$ and $AB = \frac{1}{2}XY$. ♦

Example 1

Use the Midsegment Theorem to solve for the lengths of the midsegments given in the following figure.



M, N and O are midpoints of the sides of the triangle with lengths as indicated. Use the Midsegment Theorem to find

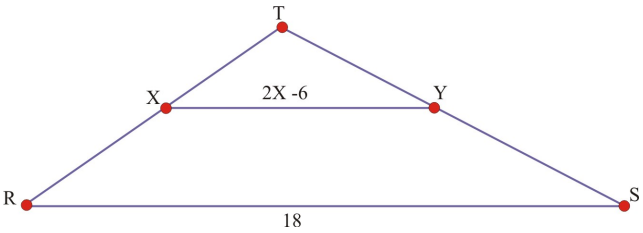
A. MN.
B. *The perimeter of the triangle* $\triangle XYZ$.

A. Since O is a midpoint, we have $XO = 5$ and $XY = 10$. By the theorem, we must have $MN = 5$.
B. By the Midsegment Theorem, $OM = 3$ implies that $ZY = 6$; similarly, $XZ = 8$, and $XY = 10$. Hence, the perimeter is $6 + 8 + 10 = 24$.

We can also examine triangles where one or more of the sides are unknown.

Example 2

Use the Midsegment Theorem to find the value of x in the following triangle having lengths as indicated and midsegment $\overline{XY}$.



By the Midsegment Theorem we have $2x - 6 = \frac{1}{2}(18)$. Solving for x, we have $x = \frac{15}{2}$.

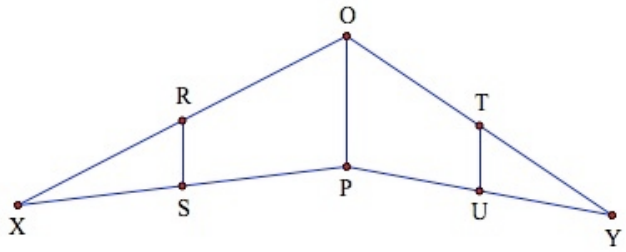
Lesson Summary

In this lesson we:

- Introduced the definition of the midsegment of a triangle and examined examples.
- Stated and proved the Midsegment Theorem.
- Solved problems using the Midsegment Theorem.

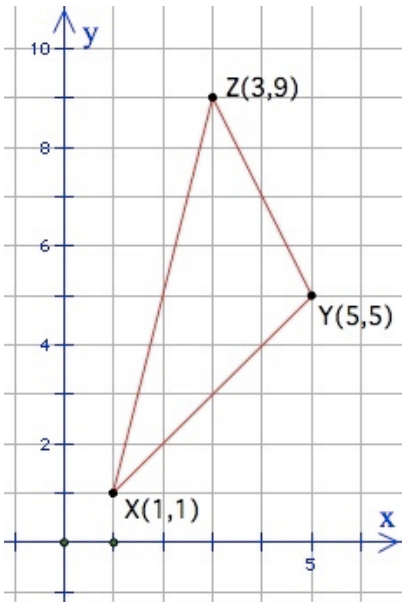
Review Questions

R, S, T, U are midpoints of sides of triangles $\triangle XPO$ and $\triangle YPO$.



Complete the following:

1. If $OP = 12$, then $RS =$ _____ and $TU =$ _____.
2. If $RS = 8$, then $TU =$ _____.
3. If $RS = 2x$ and $OP = 18$, then $x =$_____and $TU =$ _____.
4. If $OP = 4x$ and $RS = 6x - 8$, then $x =$ _____.
5. Consider triangle $\triangle XYZ$ with vertices $X(1, 1), Y(5, 5), Z(3, 9)$ and midpoint M on $\overline{XZ}$.

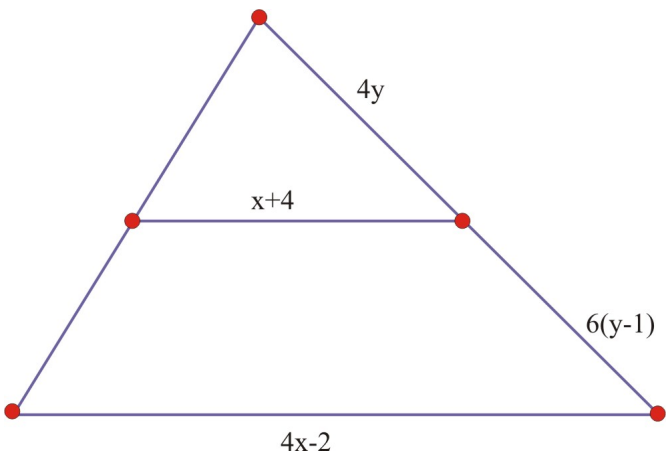


(a) Find the coordinates of point M.
(b) Use the Midsegment Theorem to find the coordinates of the point N on side $\overline{YZ}$ that makes $\overline{MN}$ the midsegment.

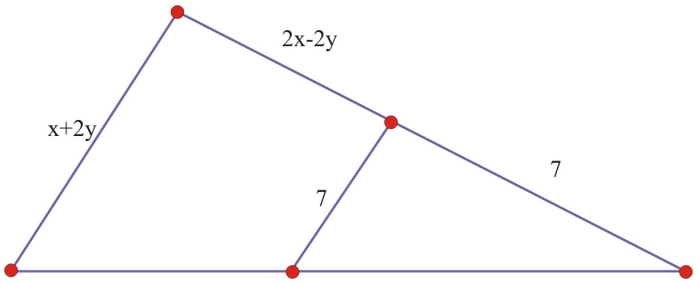
6. For problem 5, describe another way to find the coordinates of point N that does not use the Midsegment Theorem.

In problems 7-8, the segments join the midpoints of two sides of the triangle. Find the values of x and y for each problem.

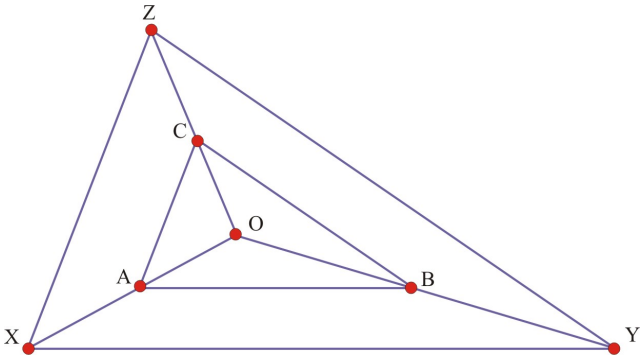
7.



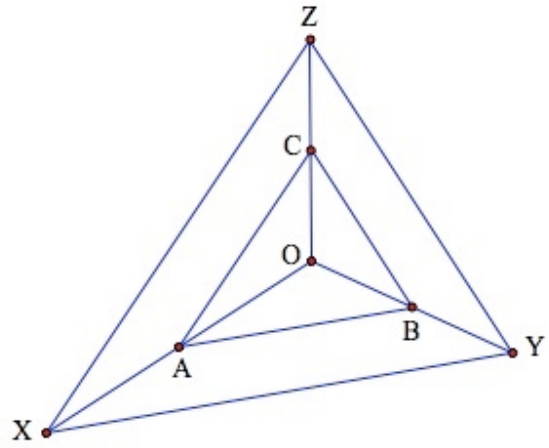
8.



9. In triangle $\triangle XYZ$, sides $\overline{XY}, \overline{YZ}$, and $\overline{ZX}$ have lengths $26, 38$ and 42 respectively. Triangle $\triangle RST$ is formed by joining the midpoints of $\triangle XYZ$. Find the perimeter of $\triangle RST$.
10. (a) For the original triangle $\triangle XYZ$ of 9, find its perimeter and compare to the perimeter of $\triangle RST$.
    (b) Can you state a relationship between a triangle's perimeter and the perimeter of the triangle formed by connecting its midsegments?
11. Given: A is the midpoint of $\overline{OX}, \overline{AB} \parallel \overline{XY}, \overline{BC} \parallel \overline{YZ}$. Prove: $\overline{AC} \parallel \overline{XZ}$.



12. Given: A is the midpoint of $\overline{OX}$, $\overline{AC} \parallel \overline{XZ}, \overline{AB} \parallel \overline{XY}$, and $\triangle OAC \cong \triangle OBA$.



Can you conclude that $\triangle OXZ \cong \triangle OYX$? If true, prove the assertion. If false, provide a counterexample.

Review Answers

1. $RS = 6$ and $TU = 6$
2. $TU = 8$
3. $x = \frac{9}{2}, TU = 9$
4. $x = 2$
5. (a) $M(2, 5)$
   (b) $N(4, 7)$
6. Find midpoint M and then the slope of $\overline{XY}$. Find the line through M parallel to $\overline{XY}$ (line l_1). Find the equation of the line that includes $\overline{YZ}$ (line l_2). Find the intersection of lines l_1 and l_2.
7. $x = 5$, $y = 3$
8. $x = 7$, $y = \frac{7}{2}$
9. $P = 53$
10. (a) The perimeter of $\triangle XYZ$ is 106. The perimeter of $\triangle RST$ is 53.
    (b) The perimeter of the midsegment triangle will always be half the perimeter of the original triangle.
11. Use the givens and Theorem 5-1 to show that point C is the midpoint of $\overline{OZ}$.
12. The assertion is true. Using Theorem 5-1, it can be shown that the triangles are congruent by SSS postulate.

5.2 Perpendicular Bisectors in Triangles

Learning Objectives

- Construct the perpendicular bisector of a line segment.
- Apply the Perpendicular Bisector Theorem to identify the point of concurrency of the perpendicular bisectors of the sides (the circumcenter).
- Use the Perpendicular Bisector Theorem to solve problems involving the circumcenter of triangles.

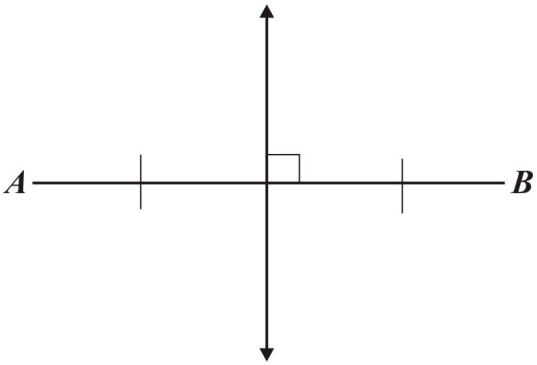
Introduction

In our last lesson we examined midsegments of triangles. In this lesson we will examine another construction that can occur within triangles, called *perpendicular bisectors*.

The perpendicular bisector of a line segment is the line that:

1. divides the line segment into two congruent sub-segments.
2. intersects the line segment at a right angle.

Here is an example of a perpendicular bisector to line segment $\overline{AB}$.

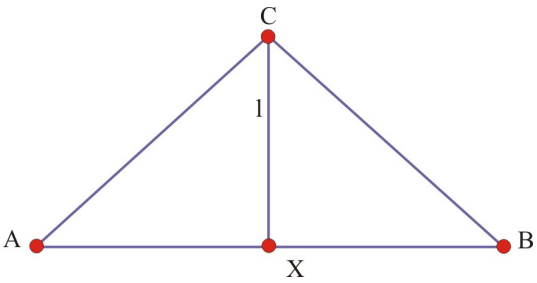


Perpendicular Bisector Theorem and its Converse

We can prove the following pair of theorems about perpendicular bisectors.

Perpendicular Bisector Theorem: If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Proof. Consider $\overline{AB}$ with perpendicular bisector l with points C and X on line l as follows:



We must show that $\overline{AC} \cong \overline{BC}$.

1. Since l is the perpendicular bisector of $\overline{AB}$, it follows that $\overline{AX} \cong \overline{XB}$ and angles $\angle CXA$ and $\angle CXB$ are congruent and are right angles.

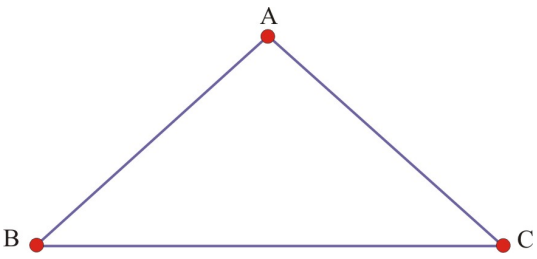
By the SAS postulate, we have $\triangle AXC \cong \triangle BXC$.

So $\overline{AC} \cong \overline{BC}$ by CPCTC (corresponding parts of congruent triangles are congruent). ♦

It turns out that we can also prove the converse of this theorem.

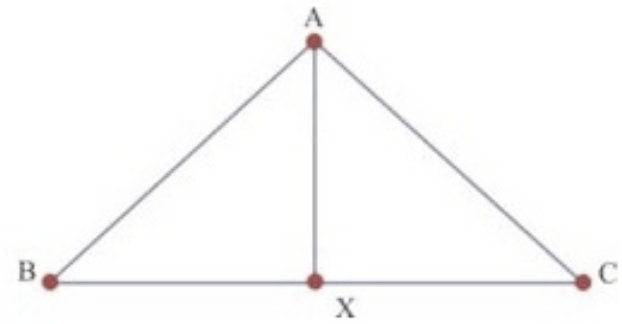
Converse of the Perpendicular Bisector Theorem: If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Proof. Consider $\triangle ABC$ as follows with $\overline{AB} \cong \overline{AC}$.



We will construct the midpoint X of $\overline{BC}$ and show that $\overline{AX}$ is the perpendicular bisector to $\overline{BC}$.

1. Construct the midpoint of $\overline{BC}$ at point X. Construct $\overline{XA}$.



2. Consider $\triangle ABX$ and $\triangle ACX$. These are congruent triangles by postulate SSS.

3. So by CPCTC, we have $\angle AXB \cong \angle AXC$.

4. Since $\angle AXB$ and $\angle AXC$ form a straight angle and are also congruent, then $m\angle AXB = m\angle AXC = 90°$. Hence, X is on the perpendicular bisector to $\overline{BC}$. ♦

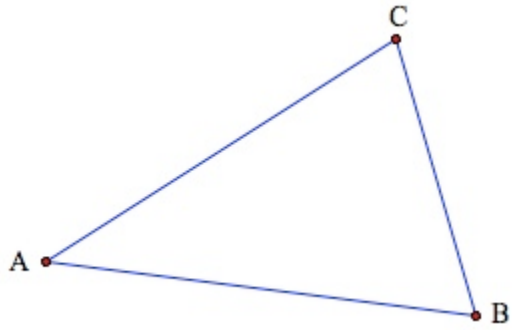
Notice that we just proved the Perpendicular Bisector Theorem and we also proved the Converse of the Perpendicular Bisector Theorem. When you prove a theorem and its converse you have proven a biconditional statement. We can state the Perpendicular Bisector Theorem and its converse in one step: *A point is on the perpendicular bisector of a segment if and only if that point is equidistant from the endpoints of the segment.*

We will now use these theorems to prove an interesting result about the perpendicular bisectors of the sides of a triangle.

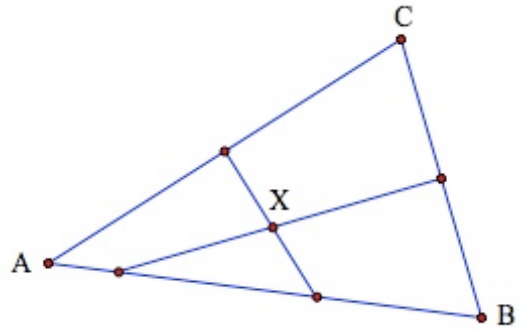
Concurrency of Perpendicular Bisectors: The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the vertices.

Proof. We will use the previous two theorems to establish the proof.

1. Consider $\triangle ABC$

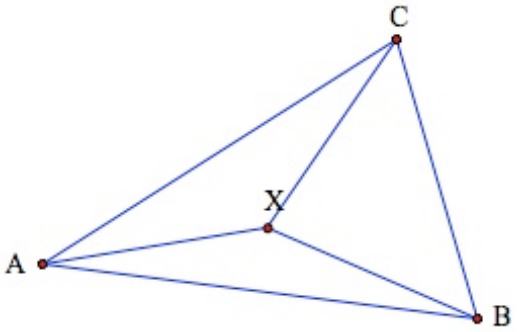


2. We can construct the perpendicular bisectors of sides $\overline{AC}$ and $\overline{BC}$ intersecting at point X as follows.



3. We will show that point X also lies on the perpendicular bisector of $\overline{AB}$ and thus is equidistant from the vertices A, B and C.

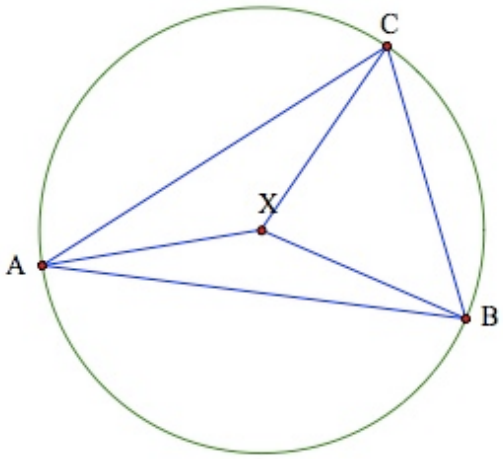
4. Construct line segments $\overline{XA}, \overline{XB}$ and $\overline{XC}$ as follows.



5. Since X is on the perpendicular bisector of $\overline{AC}$, then X is equidistant from A and C by the Perpendicular Bisector Theorem and $\overline{AX} \cong \overline{XC}$. Similarly, X is on the perpendicular bisector of $\overline{CB}$, then X is equidistant from A and B by the Perpendicular Bisector Theorem. Therefore, $\overline{AX} \cong \overline{XB}$.

6. By the transitive law, we have $\overline{AX} \cong \overline{XC}$. By the Converse of the Perpendicular Bisector Theorem, we must have that X is on the perpendicular bisector of $\overline{AB}$. ♦

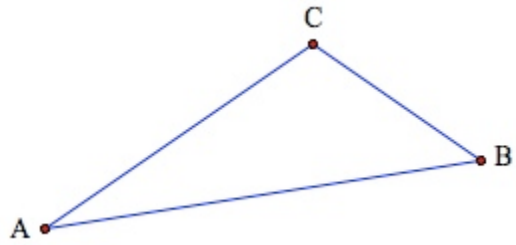
The point X has a special property. Since it is equidistant from each vertex, we can see that X is the center of a circle that circumscribes the triangle. We call X the ***circumcenter*** of the triangle. This is illustrated in the following figure.



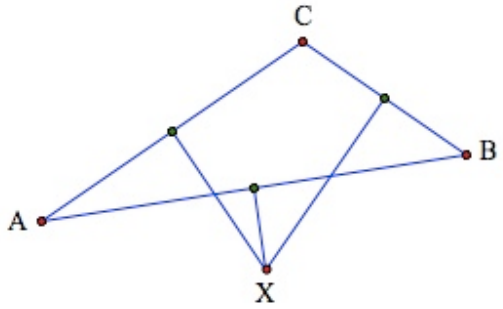
Example 1

Construct a circumscribed triangle using a compass and a straightedge.

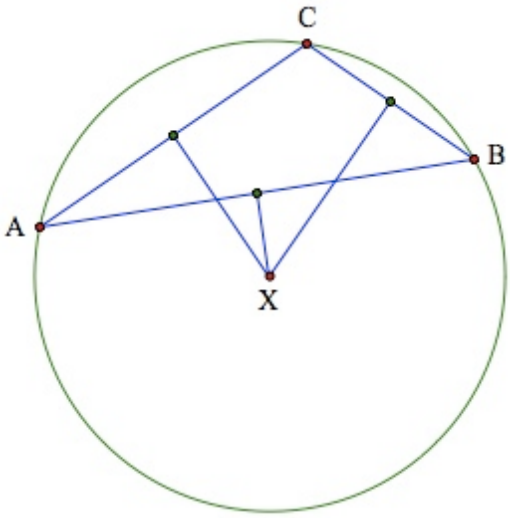
1. Draw triangle $\triangle ABC$ with your straightedge.



2. Use your compass to construct the perpendicular bisectors of the sides and find the **point of concurrency** X.



3. Use your compass to verify that $XA = XB = XC$.
4. Use your compass to construct the circle that circumscribes $\triangle ABC$

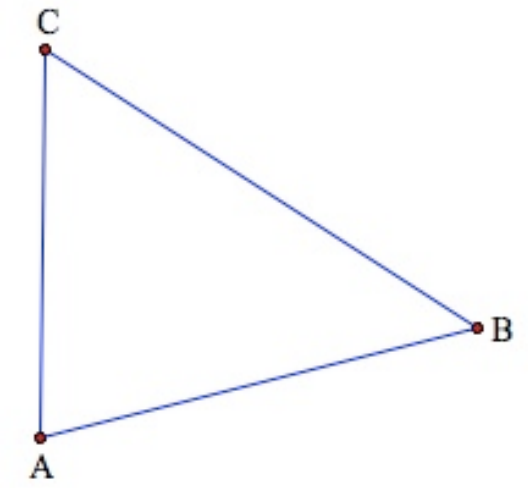


Example 2

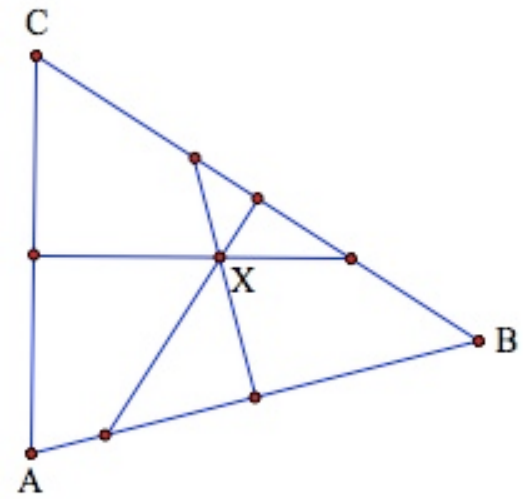
Construct a circumscribed triangle using The Geometer's Sketchpad (GSP)

We can use the commands of GSP to construct the circumcenter and corresponding circle as follows.

1. Open a new sketch and construct triangle $\triangle ABC$ using the Segment Tool.

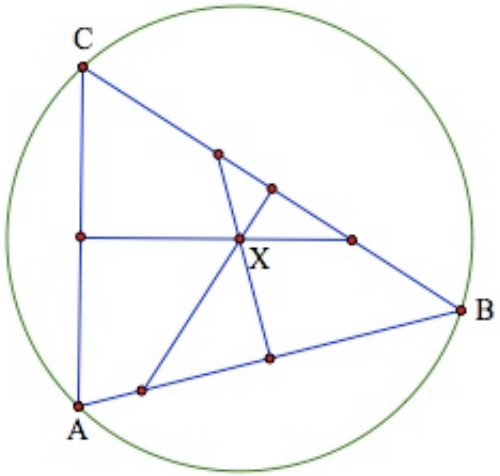


2. You can construct the perpendicular bisectors of the sides by going to the Construct menu and choosing the following options. Select each side and choose Construct Midpoints. Then for each side select the midpoint and the side (and nothing else), and then choose Construct Perpendicular Bisector.



3. Select two of the three bisectors and choose Construct Point of Intersection from the Construct menu. This will provide point X, the circumcenter.

4. Construct the circle having center X and passing through points A, B, and C. Recall that there are two ways to construct the circle: 1. Using the draw tool on the left column, and 2. Using the Construct Menu. For this construction, you will want to use the Construct menu to ensure that the circle passes through the vertices.



As a further exploration, try the following with paper:

1. Cut out any triangle from a sheet of paper.
2. Fold the the triangle over one side so that the side is folded in half.
3. Repeat for the other two sides.
4. What do you notice?

Notice that the folds will cross at the circumcenter, unless the triangle is obtuse. In which case the fold lines will meet outside the triangle if they continued.

Lesson Summary

In this lesson we:

- Defined the perpendicular bisector of a line segment.
- Stated and proved the Perpendicular Bisector Theorem.
- Solved problems using the Perpendicular Bisector Theorem.

Points to Consider

If we think about three non-collinear points in a plane, we can imagine a triangle that has each point as a vertex. Locating the circumcenter, we can draw a circle that all three vertices will be on. What does this tell us about any three non-collinear points in a plane?

There is a unique circle for any three non-collinear points in the same plane.

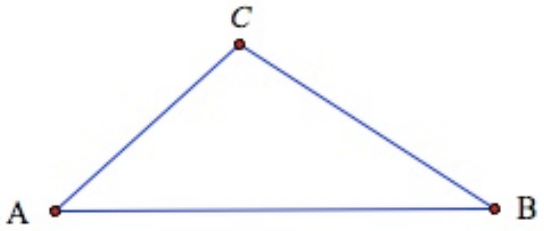
Finding a circle through any three points will also work in coordinate geometry. You can use the circumcenter to find the equation of a circle through any three points. In calculus

this method is used (together with some tools that you have probably not learned yet) to precisely describe the curvature of any curve.

Review Questions

Construct the circumcenter of $\triangle ABC$ and the circumscribed circle for each of the following triangles using a straightedge, compass, and Geometer's Sketchpad.

1.



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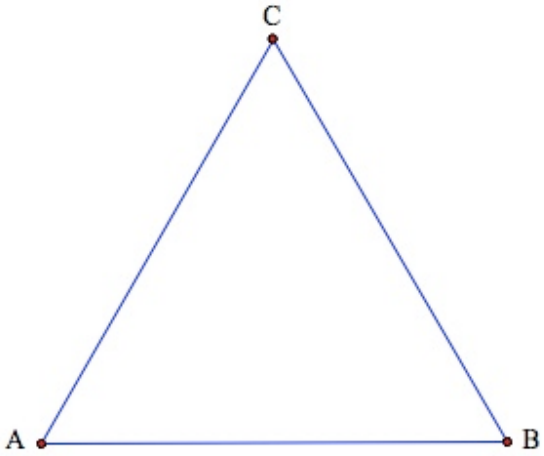
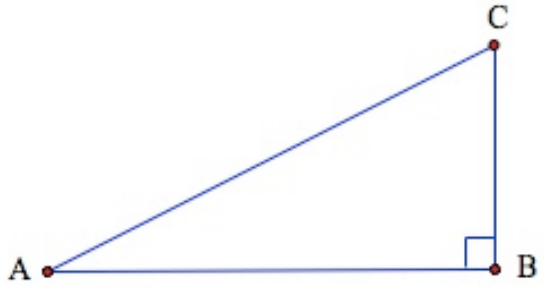


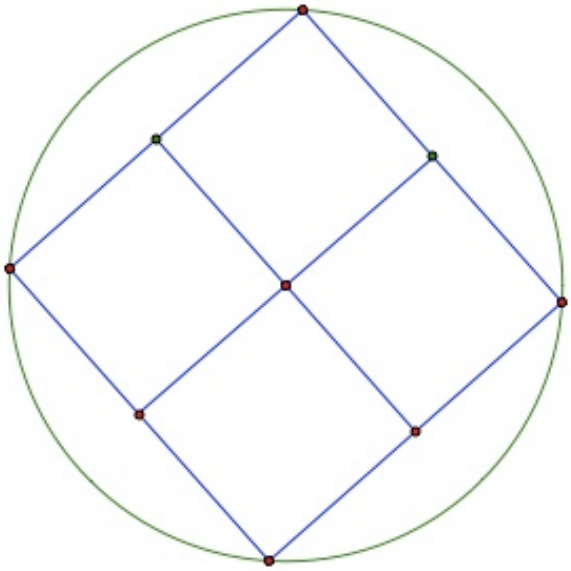
3.



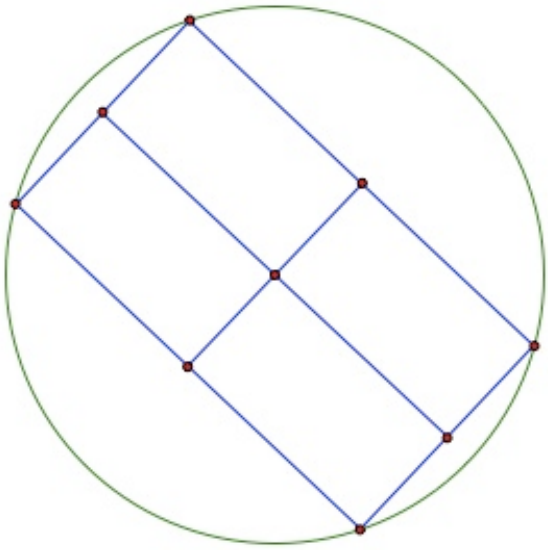
4. Based on your constructions in 1-3, state a conjecture about the relationship between a triangle and the location of its circumcenter.
5. In this lesson we found that we could circumscribe a triangle by finding the point of

concurrency of the perpendicular bisectors of each side. Use Geometer's Sketchpad to see if the method can be used to circumscribe each of the following figures:

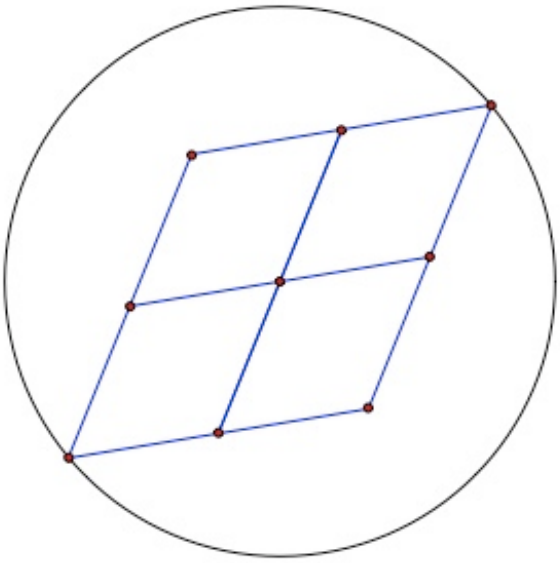
(a) a square



(b) a rectangle

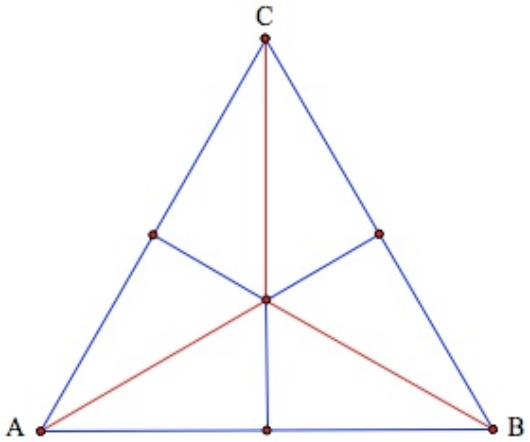


(c) a parallelogram



(d) From your work in a-c, what condition must hold in order to circumscribe a quadrilateral?

6. Consider equilateral triangle $\triangle ABC$. Construct the perpendicular bisectors of the sides of the triangle and the circumcenter X. Connect the circumcenter to each vertex. Your original triangle is now divided into six triangles. What can you conclude about the six triangles?

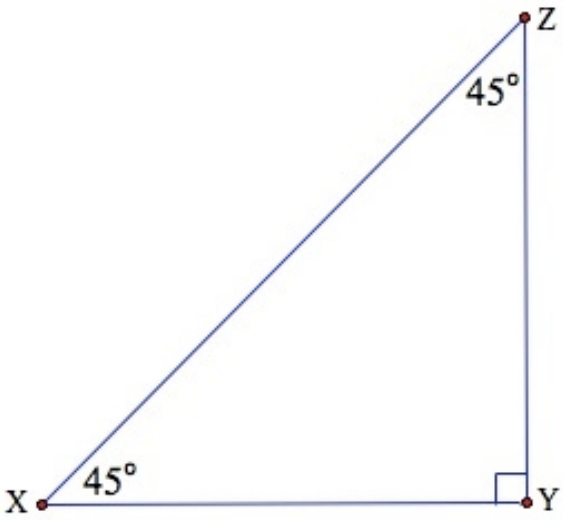


7. Suppose three cities A, B and C are situated as follows.

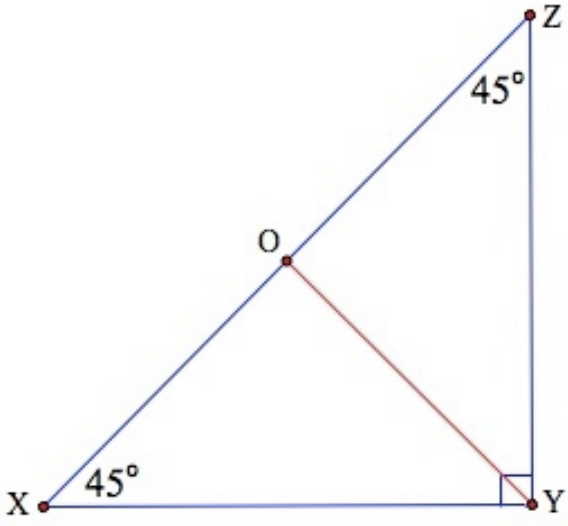
$$C$$
$$A \qquad B$$

The leaders of these cities wish to construct a new health center that is equidistant from each city. Is this a wise plan? Why or why not?

8. True or false: An isosceles triangle will always have its circumcenter located inside the triangle? Give reasons for your answer.
9. True or false: The perpendicular bisectors of an equilateral triangle intersect in the exact center of the triangle's interior. Give reasons for your answer.
10. Consider line segment $\overline{AB}$ with coordinates $A(2,1), B(6,3)$. Suppose that we wish to find point C so that $\triangle ABC$ is equilateral. How can you use perpendicular bisectors to find the location of point C?
11. Suppose that $\triangle XYZ$ is a $45° - 45° - 90°$ right triangle as indicated:



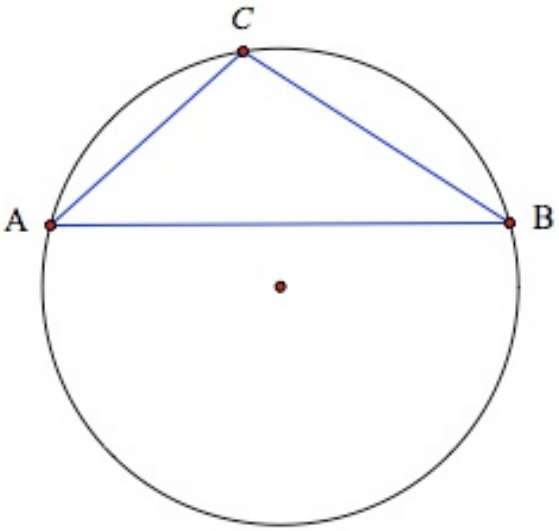
Construct the bisector of $\angle XYZ$.



Prove: $\overline{OY}$ is the perpendicular bisector of $\overline{XZ}$.

Review Answers

1.



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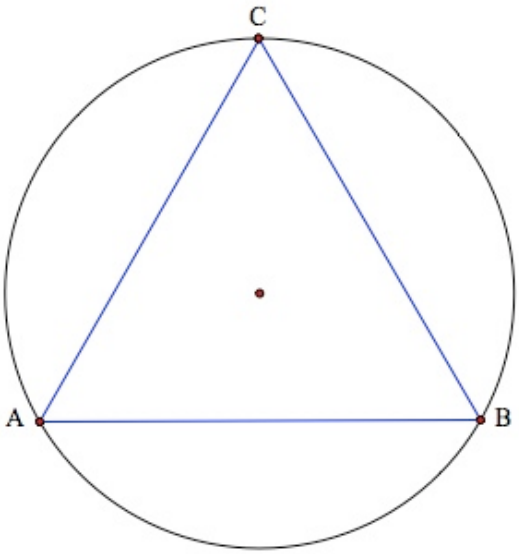

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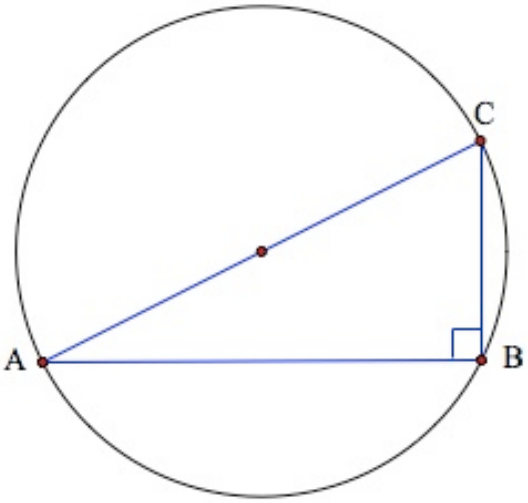
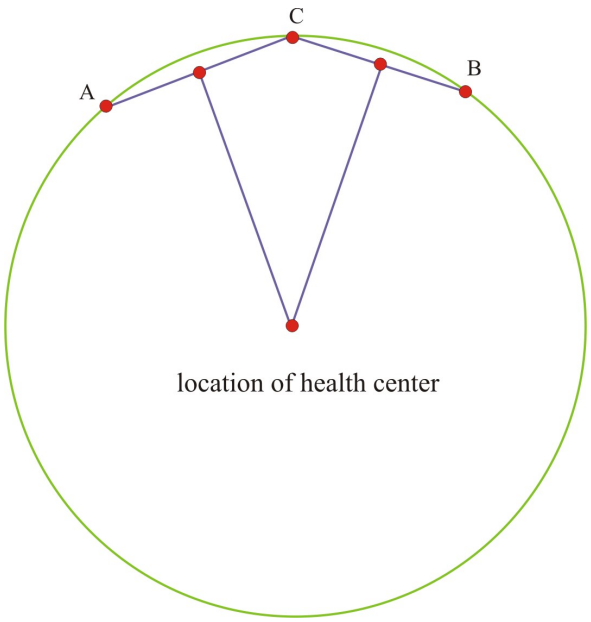


4. If triangle $\triangle ABC$ is acute, then the circumcenter lies inside of the triangle. If triangle $\triangle ABC$ is obtuse, then the circumcenter lies outside of the triangle. If triangle $\triangle ABC$ is a right triangle, then the circumcenter lies on the hypotenuse of the triangle.
5. (a) Yes
   (b) Yes
   (c) No
   (d) Opposite angles must be supplementary.
6. (a) The triangles are congruent to one another and each is a $30^\circ - 60^\circ - 90^\circ$ right triangle.
7. It is not a wise plan. Since A, B and C form an obtuse triangle, the location of the circumcenter would be outside the triangle. Hence, the health center would be located at the circumcenter, which would be a much greater distance from each city than the distance between the cities themselves.



8. False. It is possible to have isosceles triangles that are acute, obtuse, and right. Hence, there are isosceles triangles where the circumcenter could be located outside the triangle (in the case of an obtuse triangle) or on the boundary of the triangle (in the case of a right triangle).
9. True. See the solution to problem 10 to verify this fact.
10. (a) Construct the perpendicular bisector of $\overline{AB}$. Note that the slope of $\overline{AB}$ is $\frac{1}{2}$.
    (b) Point C will be located on the perpendicular bisector. The perpendicular bisector will have slope $m = -2$.
    (c) The perpendicular bisector will pass through the midpoint $(4, 2)$ and have slope $m = -2$. Its equation is $y = -2x + 10$.
    (d) So, the distance from $A(2, 1)$ to $C(x, -2x + 10)$ is equal to AB, which is $2\sqrt{5}$.
    (e) Consider the distance from point A to CA. Solve the following distance equation to find the x coordinate of point C. $\sqrt{(x-2)^2 + (-2x+10-1)^2} = 2\sqrt{5}$
    (f) Since C lies on the line $y = -2x + 10$, use the value of x found from the equation to find the $y-$coordinate.
11. Show that $\triangle YOZ \cong \triangle YOX$ use CPCTC and the definitions of bisector and properties about congruent adjacent angles forming a straight angle.

5.3 Angle Bisectors in Triangles

Learning Objectives

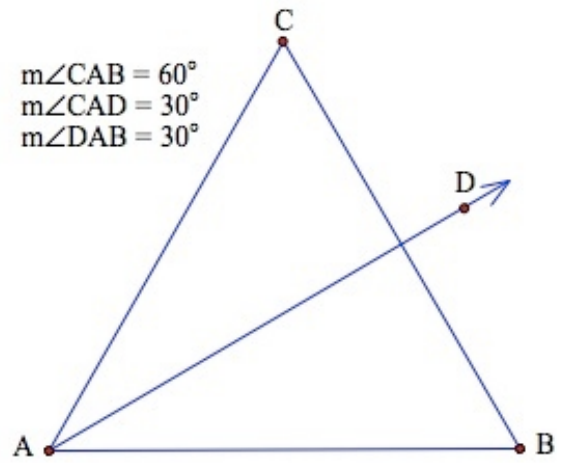
- Construct the bisector of an angle.
- Apply the Angle Bisector Theorem to identify the point of concurrency of the perpen-

dicular bisectors of the sides (the incenter).
- Use the Angle Bisector Theorem to solve problems involving the incenter of triangles.

Introduction

In our last lesson we examined perpendicular bisectors of the sides of triangles. We found that we were able to use perpendicular bisectors to circumscribe triangles. In this lesson we will learn how to inscribe circles in triangles. In order to do this, we need to consider the angle bisectors of the triangle. The bisector of an angle is the ray that divides the angle into two congruent angles.

Here is an example of an angle bisector in an equilateral triangle.

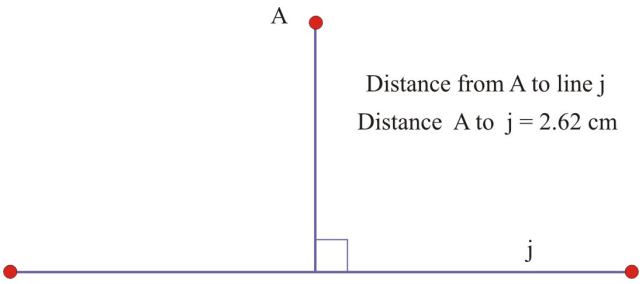


Angle Bisector Theorem and its Converse

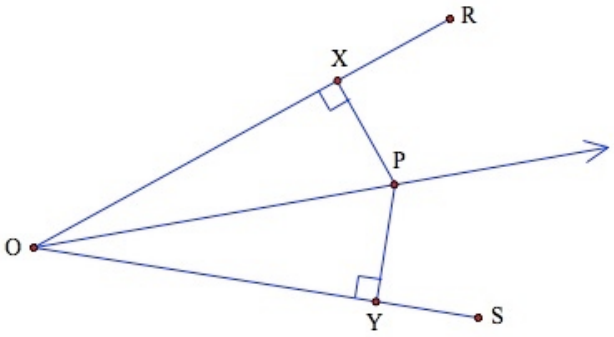
We can prove the following pair of theorems about angle bisectors.

Angle Bisector Theorem: If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

Before we proceed with the proof, let's recall the definition of the distance from a point to a line. The distance from a point to a line is the length of the line segment that passes through the point and is *perpendicular to the original line.*



Proof. Consider $\angle ROS$ with angle bisector $\overline{OP}$, and segments $\overline{PX}$ and $\overline{PY}$, perpendicular to each side through point P as follows:



We will show that $\overline{PX} \cong \overline{PY}$.

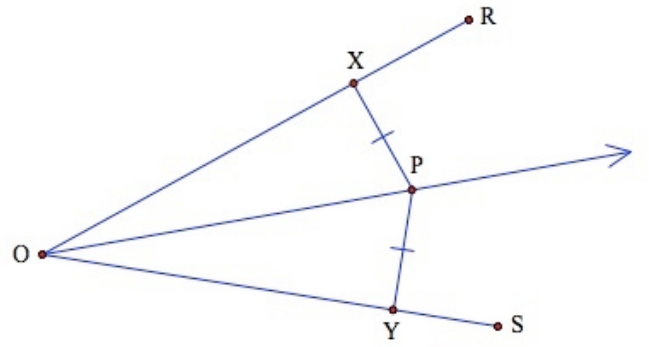
1. Since $\overline{OP}$ is the bisector of $\angle ROS$, then $\angle XOP \cong \angle YOP$ by the definition of angle bisector. In addition, since $\overline{PX}$ and $\overline{PY}$ are perpendicular to the sides of $\angle ROS$, then $\angle PXO$ and $\angle PYO$ are right angles and thus congruent. Finally, $\overline{OP} \cong \overline{OP}$ by the reflexive property.
2. By the AAS postulate, we have $\triangle PXO \cong \triangle PYO$.
3. So $\overline{PX} \cong \overline{PY}$ by CPCTC (corresponding parts of congruent triangles are congruent). ♦

Therefore P is equidistant from each side of the angle. And since P represents any point on the angle bisector, we can say that **every** point on the angle bisector is equidistant from the sides of the angle.

We can also prove the converse of this theorem.

Converse of the Angle Bisector Theorem: If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

Proof. Consider $\angle ROS$ with points X and Y and segment $\overline{OP}$ such that $\overline{PX} \cong \overline{PY}$ as follows:



1. As the distance to each side is given by the lengths of $\overline{PX}$ and $\overline{PY}$ respectively, we have that $\overline{PX}$ and $\overline{PY}$ are perpendicular to sides $\overline{RO}$ and $\overline{SO}$ respectively.
2. Note that $\overline{PO}$ is the hypotenuse of right triangles $\triangle XOP$ and $\triangle YOP$ Hence, since $\overline{PX} \cong \overline{PY}$, and $\overline{PO} \cong \overline{PO}$, and $\angle PXO$ and $\angle PYO$ are right angles, then the triangles are congruent by Theorem 4-6.
3. $\angle POX \cong \angle POY$ by CPCTC.
4. Hence, point P lies on the angle bisector of $\angle ROS$. ♦

Notice that we just proved the Angle Bisector Theorem (If a point is on the angle bisector then it is equidistant from the sides of the angle) and we also proved the converse of the Angle Bisector theorem (If a point is equidistant from the sides of an angle then it is on the angle bisector of the triangle). When we have proven both a theorem and its converse we say that we have proven a **biconditional** statement. We can put the two conditional statements together using *if and only if*: "A point is on the angle bisector of an angle if and only if it is equidistant from the sides of the triangle."

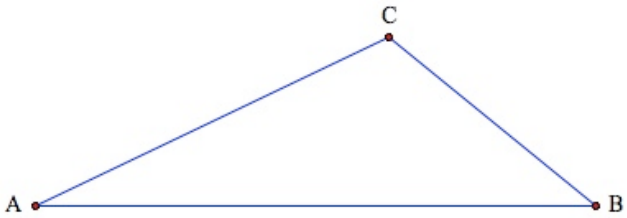
Angle Bisectors in a Triangle

We will now use these theorems to prove an interesting result about the angle bisectors of a triangle.

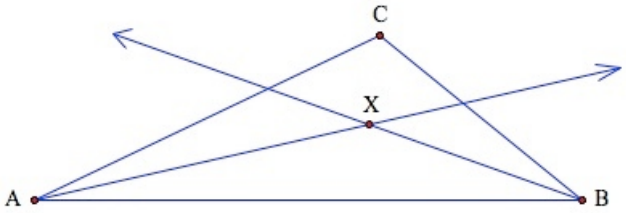
Concurrency of Angle Bisectors Theorem: The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

Proof. We will use the previous two theorems to establish the proof.

1. Consider $\triangle ABC$.

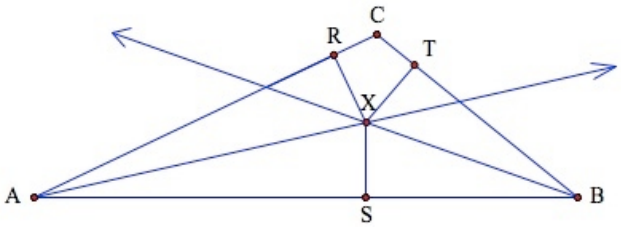


2. We can construct the angle bisectors of $\angle CAB$ and $\angle ABC$ intersecting at point X as follows.



3. We will show that point X is equidistant from sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ and that X is on the bisector of $\angle BCA$.

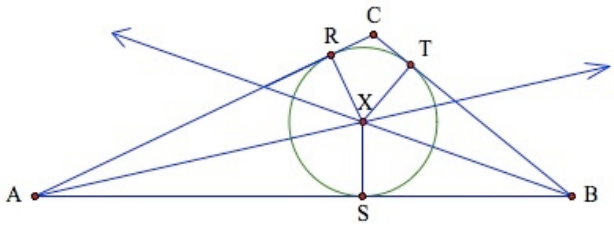
4. Construct perpendicular line segments from point X to sides $\overline{XR}, \overline{XS}$, and $\overline{XT}$ as follows:



5. Since X is on the bisectors of $\angle CAB$ and $\angle ABC$, then by Theorem 5-5, $\overline{XR} \cong \overline{XS} \cong \overline{XT}$. Therefore, X is equidistant from sides $\overline{AB}, \overline{BC}$, and $\overline{CA}$.

6. Since X is equidistant from $\overline{BC}$ and $\overline{CA}$, Theorem 5-6 applies and we must have that X is on the angle bisector of $\angle BCA$. ♦

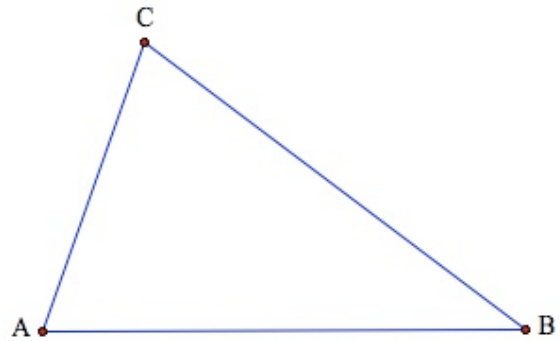
The point X has a special property. Since it is equidistant from each side of the triangle, we can see that X is the center of a circle that lies within the triangle. We say that the circle is **inscribed within** the triangle and the point X is called the **incenter** of the triangle. This is illustrated in the following figure.



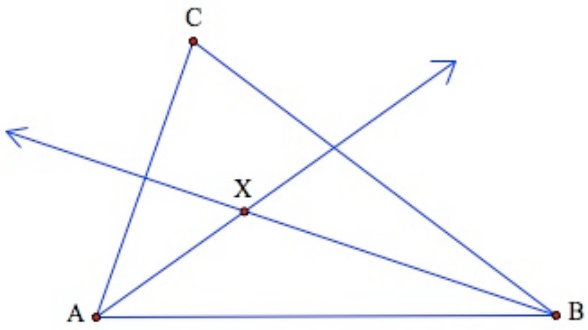
Example 1

Inscribe the following triangle using a compass and a straightedge.

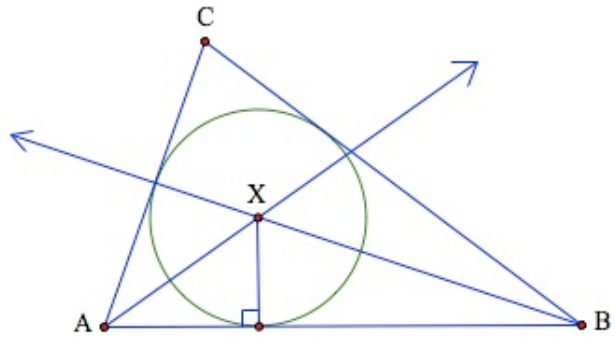
1. Draw triangle $\triangle ABC$ with your straightedge.



2. Use your compass to construct the angle bisectors and find the point of concurrency X.



3. Use your compass to construct the circle that inscribes $\triangle ABC$.

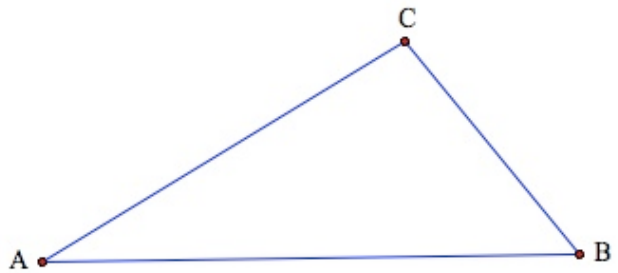


Example 2

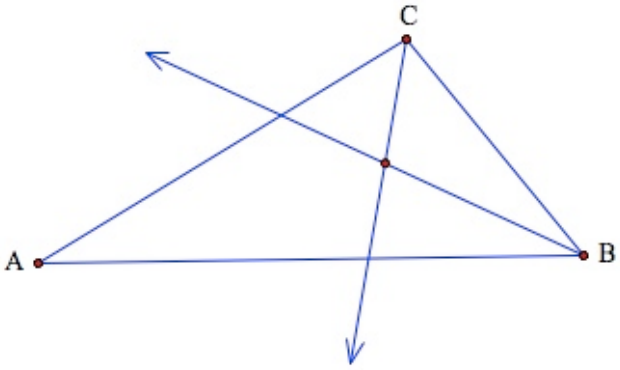
Inscribe a circle within the following triangle using The Geometer's Sketchpad.

We can use the commands of GSP to construct the incenter and corresponding circle as follows:

1. Open a new sketch and construct triangle $\triangle ABC$ using the Segment Tool.

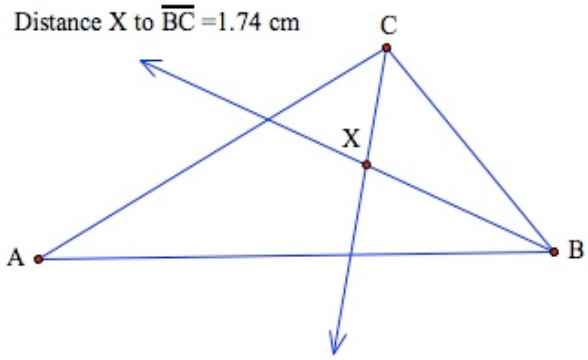


2. You can construct the angle bisectors of the angles by first designating the angle by selecting the appropriate vertices (e.g., to select the angle at vertex A, select points B, A and C in order) and then choosing Construct Angle Bisector from the Construct menu. After bisecting two angles, construct the point of intersection by selecting each angle bisector and choosing Intersection from the Construct menu. (Recall from our proof of the concurrency of angle bisectors theorem that we only need to bisect two of the angles to find the incenter.)

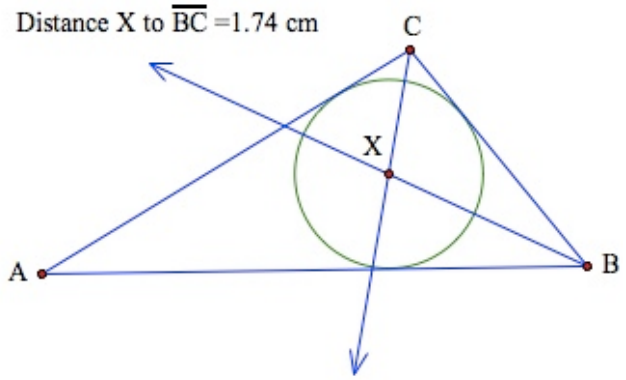


3. You are now ready to construct the circle. Recall that the radius of the circle must be the distance from X to each side – our figure above does not include that segment. However, we do not need to construct the perpendicular line segments as we did to prove Theorem 5-7. Sketchpad will measure the distance for us.

4. To measure the distance from X to each side, select point X and one side of the triangle. Choose Distance from the Measure menu. This will give you the radius of your circle.



5. We are now ready to construct the circle. Select point X and the distance from X to the side of the triangle. Select "Construct circle by center + radius" from the Construct menu. This will give the inscribed circle within the triangle.



Lesson Summary

In this lesson we:

- Defined the angle bisector of an angle.
- Stated and proved the Angle Bisector Theorem.
- Solved problems using the Angle Bisector Theorem.
- Constructed angle bisectors and the inscribed circle with compass and straightedge, and with Geometer's Sketchpad.

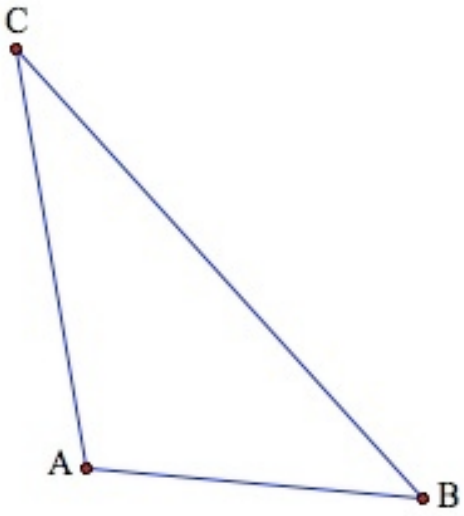
Points to Consider

How are circles related to triangles, and how are triangles related to circles? If we draw a circle first, what are the possibilities for the triangles we can circumscribe? In later chapters we will more carefully define and work with the properties of circles.

Review Questions

Construct the incenter of $\triangle ABC$ and the inscribed circle for each of the following triangles using a straightedge, compass, and Geometer's Sketchpad.

1.



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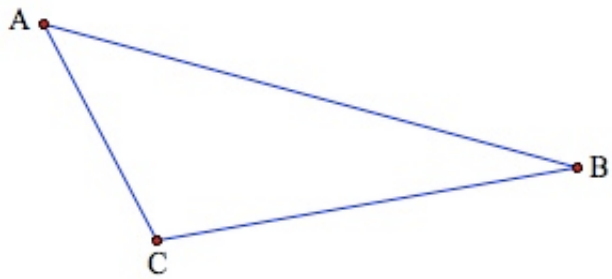
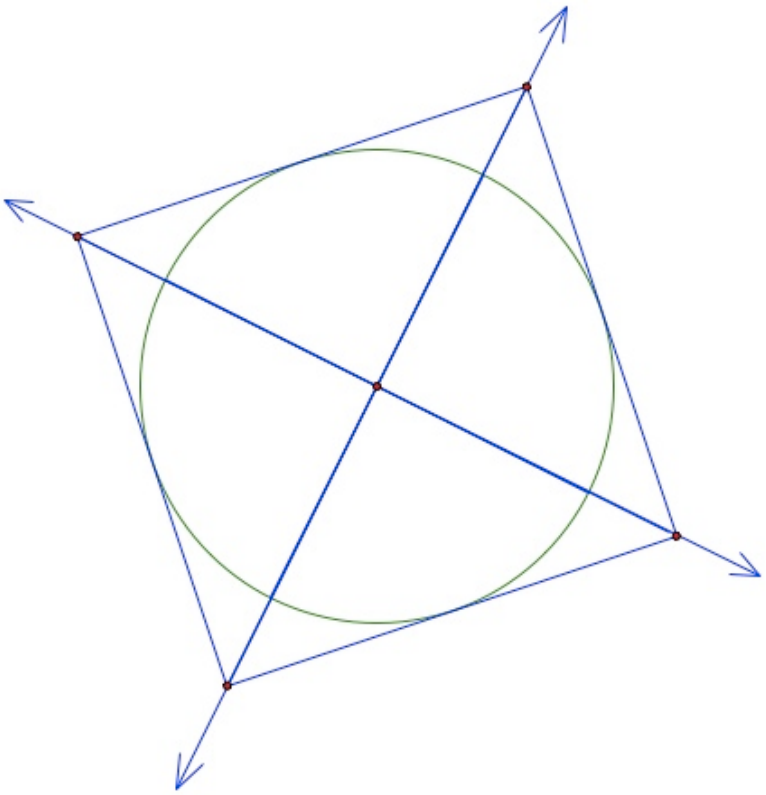
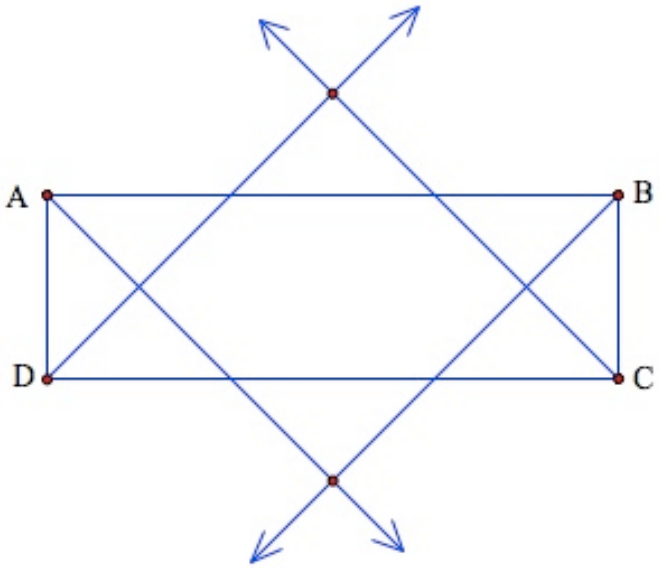


3. In the last lesson we found that we could circumscribe some kinds of quadrilaterals as long as opposite angles were supplementary. Use Geometer's Sketchpad to explore the following quadrilaterals and see if you can inscribe them by the angle bisector method.

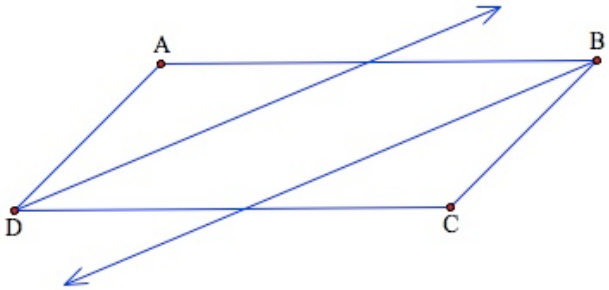
(a) a square



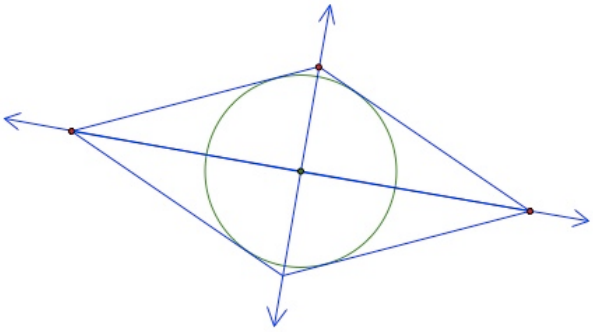
(b) a rectangle



(c) a parallelogram

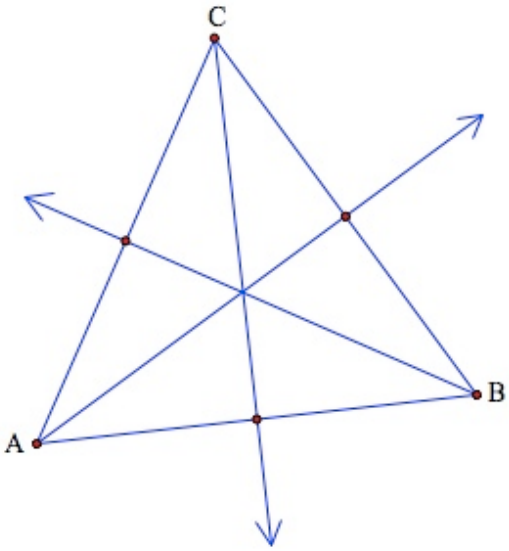


(d) a rhombus



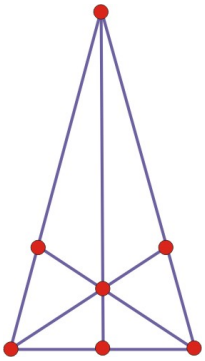
(e) From your work in a-d, what condition must hold in order to circumscribe a quadrilateral?

4. Consider equilateral triangle $\triangle ABC$. Construct the angle bisectors of the triangle and the incenter X. Connect the incenter to each vertex so that the line segment intersects the side opposite the angle as follows.

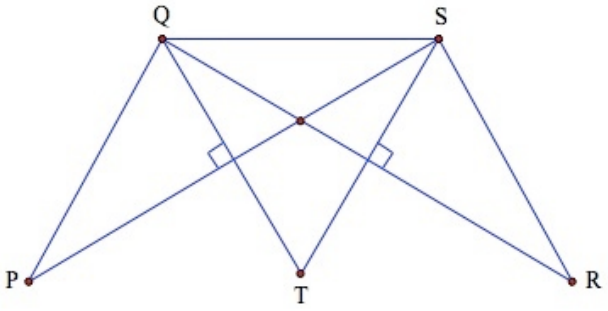


As with circumcenters, we get six congruent $30° - 60° - 90°$ triangles. Now connect the points that intersect the sides. What kind of figure do you get?

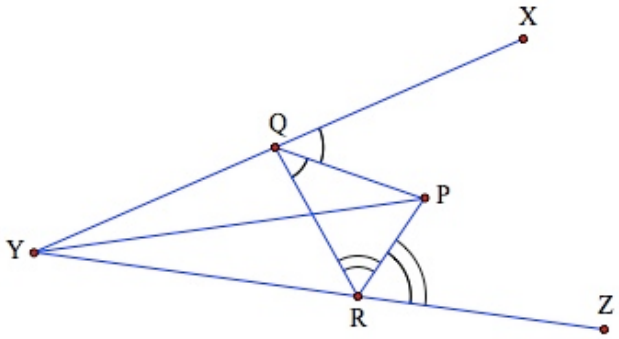
5. True or false: An incenter can also be a circumcenter. Illustrate your reasoning with a drawing.
6. Consider the situation described in exercise 4 for the case of an isosceles triangle. What can you conclude about the six triangles that are formed?



7. Consider line segment $\overline{AB}$ with coordinates $A(4, 4)$, $B(8, 4)$. Suppose that we wish to find points C and D so that the resulting quadrilateral can be either circumscribed or inscribed. What are some possibilities for locating points C and D?
8. Using a piece of tracing or Patty Paper, construct an equilateral triangle. Bisect one angle by folding one side onto another. Unfold the paper. What can you conclude about the fold line?
9. Repeat exercise 8 with an isosceles triangle. What can you conclude about repeating the folds?
10. What are some other kinds of polygons where you could use Patty Paper to bisect an angle into congruent figures?
11. **Given:**$\overline{ST}$ is the perpendicular bisector of $\overline{QR}$.
    $\overline{QT}$ is the perpendicular bisector of $\overline{SP}$.
    Prove:
    $\overline{PQ} \cong \overline{RS}$.

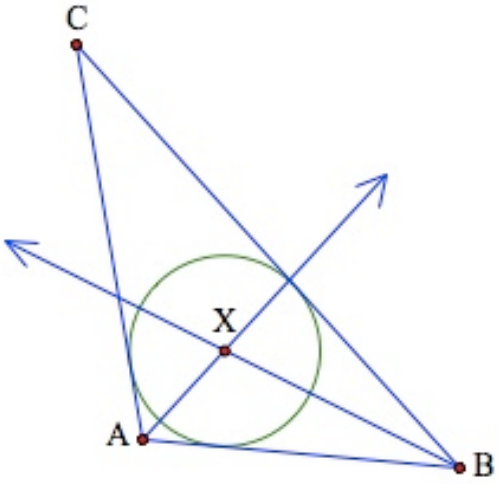


12. **Given:**$\overline{PQ}$ bisects $\angle XQR$.
    $\overline{PR}$ bisects $\angle QRZ$.
    Prove:
    $\overline{PY}$ bisects $\angle XYZ$.

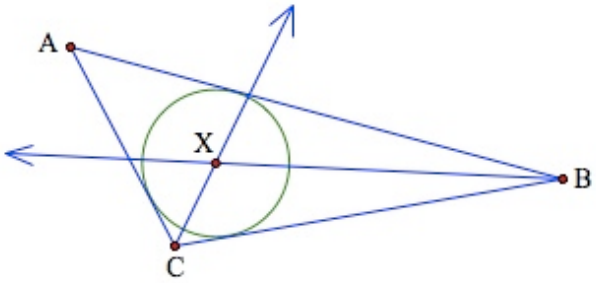


Review Answers

1.

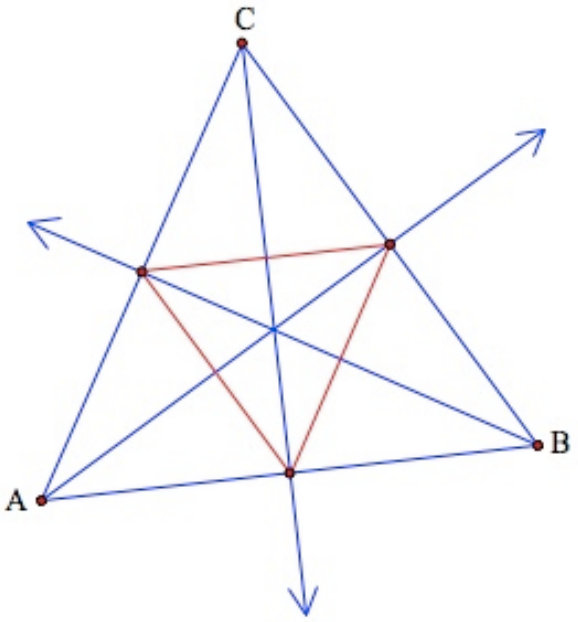


2.

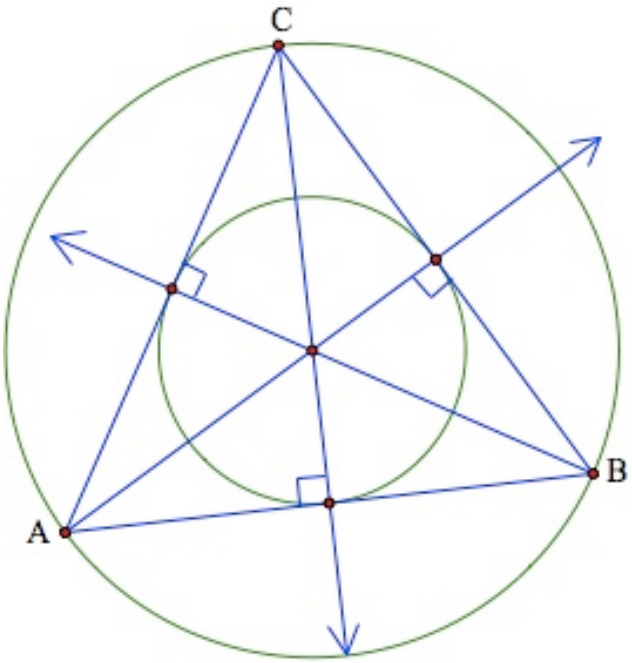


3. (a) Yes
   (b) No: Bisectors are not concurrent at a point.
   (c) No: Bisectors are not concurrent at a point.
   (d) Yes
   (e) Angle bisectors must be concurrent.

4. Equilateral triangle

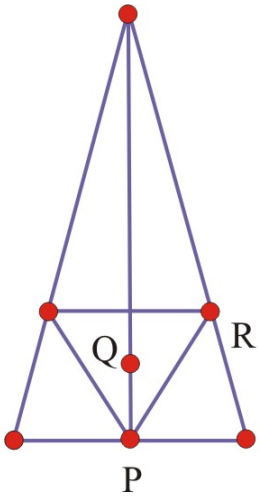


5. The statement is true in the case of an equilateral triangle. In addition, for squares the statement is also true.



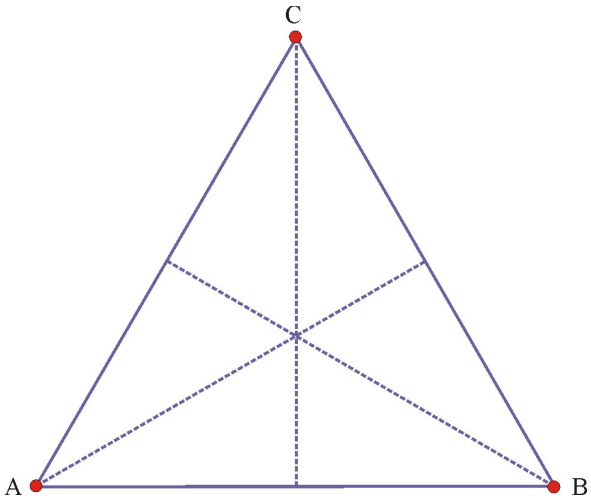
6. We do not get six congruent triangles as before. But we get four congruent triangles and a separate pair of congruent triangles. In addition, if we connect the points where

the bisectors intersect the sides, we get an isosceles triangle.

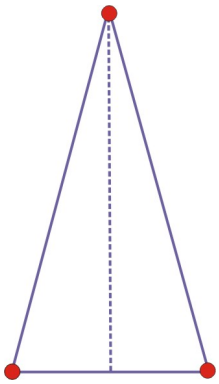


$m\overline{PQ} = 1.26$ cm
$m\overline{QR} = 1.26$ cm
$m\overline{RP} = 1.26$ cm

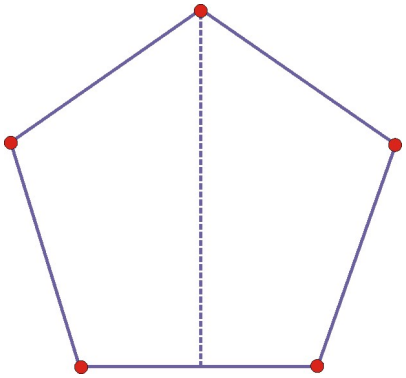
7. From our previous exercises we saw that we could inscribe and circumscribe some but not all types of quadrilaterals. Drawing from those exercises, we see that we could circumscribe and inscribe a square. So, locating the points at $C(4, 8)$ and $D(8, 8)$ is one such possibility. Similarly, we could locate the points at $C(6, 6)$ and $D(6, 2)$ and get a kite that can be inscribed but not circumscribed.
8. The fold line divides the triangle into two congruent triangles and thus is a line of symmetry for the triangle. Note that the same property will hold by folding at each of the remaining angles.



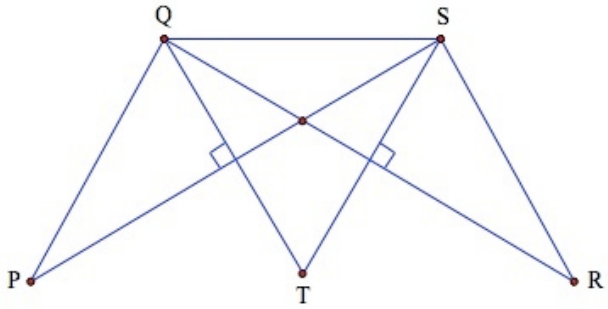
9. There is only one fold line that divides the triangle into congruent triangles, the line that folds the angle formed by the congruent sides.



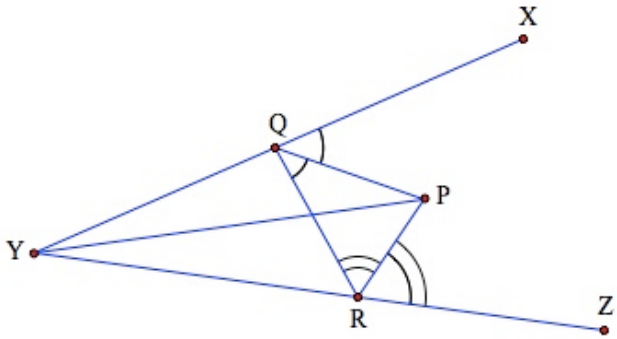
10. Any regular polygon will have this property. For example, a regular pentagon:



11.



12.



5.4 Medians in Triangles

Learning Objectives

- Construct the medians of a triangle.
- Apply the Concurrency of Medians Theorem to identify the point of concurrency of the medians of the triangle (the centroid).
- Use the Concurrency of Medians Theorem to solve problems involving the centroid of triangles.

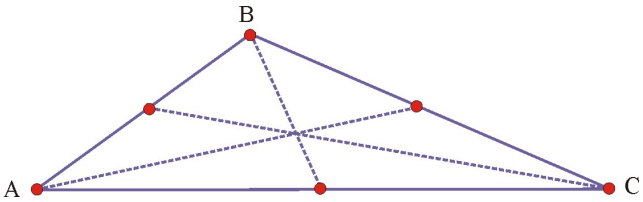
Introduction

In our two last lessons we learned to circumscribe circles about triangles by finding the perpendicular bisectors of the sides and to inscribe circles within triangles by finding the triangle's angle bisectors. In this lesson we will learn how to find the location of a point within the triangle that involves the medians.

Definition of Median of a Triangle

A **median** of a triangle is the line segment that joins a vertex to the midpoint of the opposite side.

Here is an example that shows the medians in an obtuse triangle.



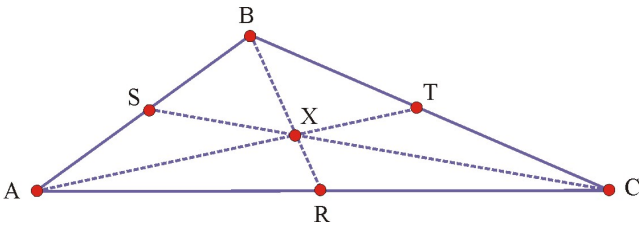
That the three medians appear to intersect in a point is no coincidence. As was true with perpendicular bisectors of the triangle sides and with angle bisectors, the three medians will be concurrent (intersect in a point). We call this point the **centroid** of the triangle. We can prove the following theorem about centroids.

The Centroid of a Triangle

Concurrency of Medians Theorem: The medians of a triangle will intersect in a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side.

Consider $\triangle ABC$ with midpoints of the sides located at R, S, and T and the point of

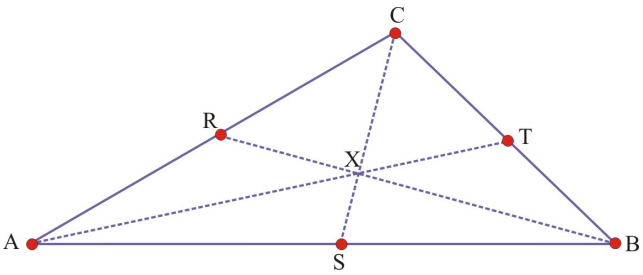
concurrency of the medians at the centroid, X. The theorem states that $CX = \frac{2}{3}CS, AX = \frac{2}{3}AT$, and $BX = \frac{2}{3}BR$.



The theorem can be proved using a coordinate system and the midpoint and distance formulas for line segments. We will leave the proof to you (Homework Exercise #10), but will provide an outline and helpful hints for developing the proof.

Example 1.

Use The Concurrency of Medians Theorem to find the lengths of the indicated segments in the following triangle that has medians $\overline{AT}, \overline{CS}$, and $\overline{BR}$ as indicated.



1. *If $CS = 12$, then $CX =$ _____ and $XS =$ _____.*

$$CX = \frac{2}{3} \cdot 12 = 8$$
$$XS = \frac{1}{3} \cdot 12 = 4$$

2. *If $AX = 6$. then $XT =$ _____ and $AT =$ _____.*

We will start by finding AT.

$$AX = \frac{2}{3}AT$$
$$6 = \frac{2}{3}AT$$
$$9 = AT$$

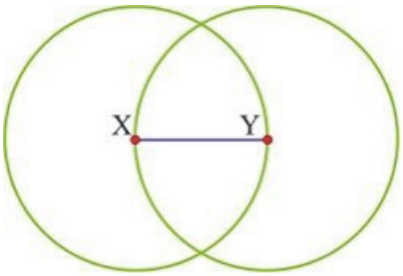
Now for XT,

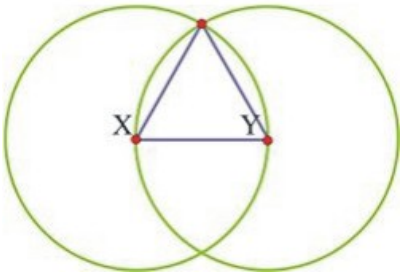
$$\begin{aligned} XT &= \frac{1}{3}AT \\ &= \frac{1}{3}9 \\ &= 3 \end{aligned}$$

Napoleon's Theorem

In the remainder of the lesson we will provide an interesting application of a theorem attributed to Napoleon Bonaparte, Emperor of France from 1804 to 1821, which makes use of equilateral triangles and centroids. We will explore Napoleon's theorem using The Geometer's Sketchpad.

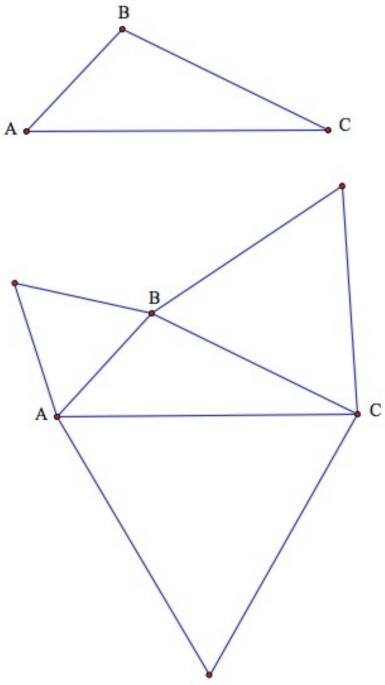
But first we need to review how to construct an equilateral triangle using circles. Consider $\overline{XY}$ and circles having equal radius and centered at X and Y as follows:



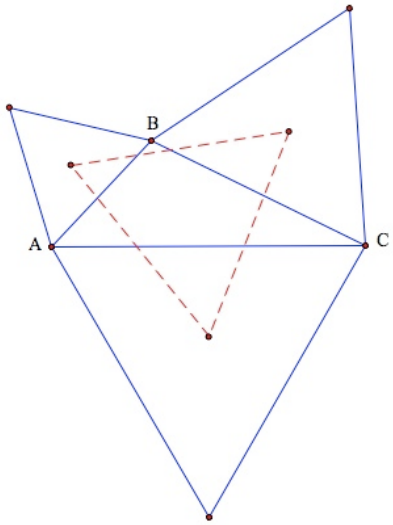


Once you have hidden the circles, you will have an equilateral triangle. You can use the construction any time you need to construct an equilateral triangle by selecting the finished triangle and then making a Tool using the tool menu.

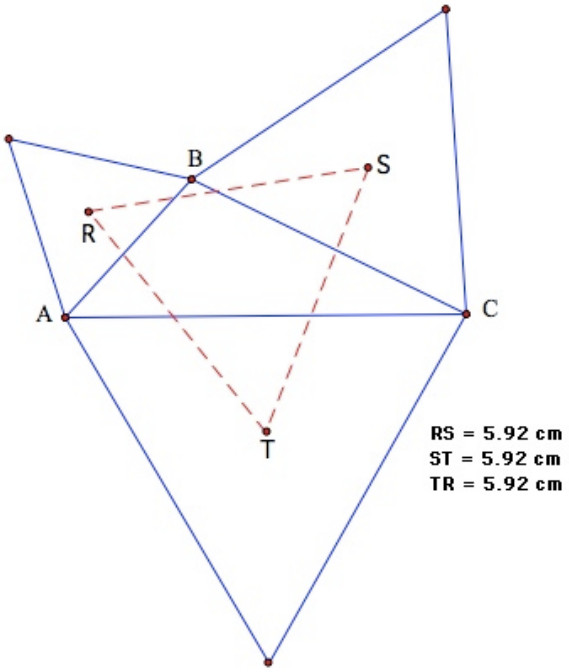
Preliminary construction for Napoleon's Theorem: Construct any triangle $\triangle ABC$. Construct an equilateral triangle on each side.



Find the centroid of each equilateral triangle and connect the centroids to get the Napoleon outer triangle.



Measure the sides of the new triangle using Sketchpad. What can you conclude about the Napoleon outer triangle? (Answer: The triangle is equilateral.)



This result is all the more remarkable since it applies to any triangle $\triangle ABC$. You can verify this fact in GSP by "dragging" a vertex of the original triangle $\triangle ABC$ to form other triangles. The outer triangle will remain equilateral. Homework problem 9 will allow you to further explore this theorem.

Example 2

Try this:

1. Draw a triangle on a sheet of card stock paper (or thin cardboard) and locate the centroid.
2. Carefully cut out the triangle.
3. Hold your pencil point up and place the triangle on it so that the centroid rests on the pencil.
4. What do you notice?

The triangle balances on the pencil. Why does the triangle balance?

Lesson Summary

In this lesson we:

- Defined the centroid of a triangle.

- Stated and proved the Concurrency of Medians Theorem.
- Solved problems using the Concurrency of Medians Theorem.
- Demonstrated Napoleon's Theorem.

Points to Consider

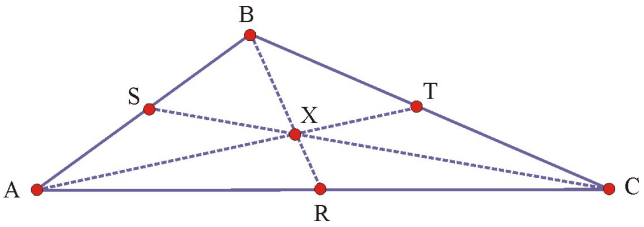
So far we have been looking at relationships within triangles. In later chapters we will review the area of a triangle. When we draw the medians of the triangle, six smaller triangles are created. Think about the area of these triangles, and how that might relate to example 1 above.

Review Questions

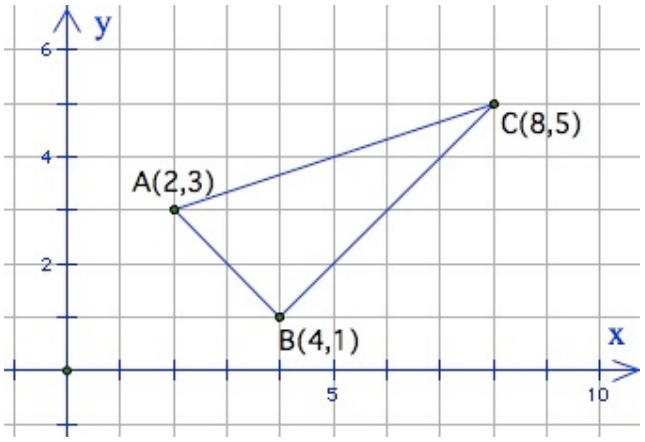
1. Find the centroid of $\triangle ABC$ for each of the following triangles using Geometer's Sketchpad. For each triangle, measure the lengths of the medians and the distances from the centroid to each of the vertices. What can you conclude for each of the triangles?

   (a) an equilateral triangle
   (b) an isosceles triangle
   (c) A scalene triangle

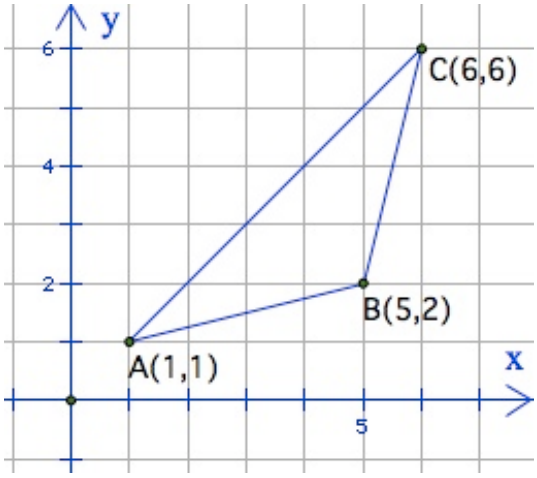
2. $\triangle ABC$ has points R, S, T as midpoints of sides and the centroid located at point X as follows.



   Find the following lengths if $XS = 10, XC =$ _____, $CS =$ _____.
3. True or false: A median cannot be an angle bisector. Illustrate your reasoning with a drawing.
4. Find the coordinates of the centroid X of $\triangle ABC$ with vertices $A(2, 3), B(4, 1)$, and $C(8, 5)$.

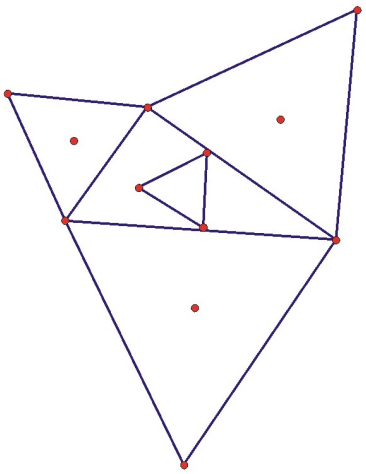


5. Find the coordinates of the centroid X of $\triangle ABC$ with vertices $A(1,1), B(5,2)$, and $C(6,6)$. Also, find XB.



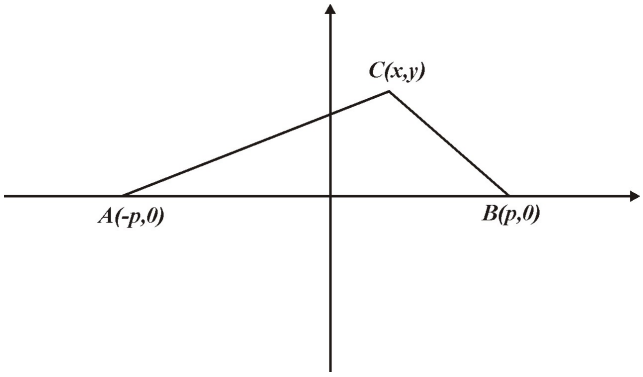
6. Use the example sketch of Napoleon's Theorem to form the following triangle:

   (a) Reflect each of the centroids in the line that is the closest side of the original triangle.
   (b) Connect the points to form a new triangle that is called the inner Napoleon triangle.



(c) What can you conclude about the inner Napoleon triangle?

7. You have been asked to design a triangular metal logo for a club at school. Using the following rectangular coordinates, determine the logo's centroid. $A(1,0), B(1,8), C(10,4)$
8. Prove Theorem 5-8. An outline of the proof together with some helpful hints is provided here. **Proof**. Consider $\triangle ABC$ with $A(-p,0), B(p,0), C(x,y)$ as follows:



Hints: Note that the midpoint of side $\overline{AB}$ is located at the origin. Construct the median from vertex C to the origin, and call it $\overline{CO}$. The point of concurrency of the three medians will be located on $\overline{CO}$ at point P that is two-thirds of the way from C to the origin.

9. *Prove:* Each median of an equilateral triangle divides the triangle into two congruent triangles.

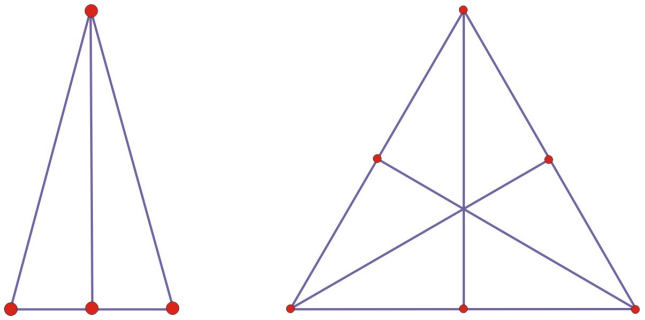
Review Answers

1. (a) Medians all have same length; distances from vertices to centroid – all are same; they are two-thirds the lengths of the medians.
   (b) Two of the medians have same length; distances from vertices to centroid are same

for these two; all are two-thirds the lengths of the medians.

(c) Medians all have different lengths; distances from vertices to centroid; all are different; they are two-thirds the lengths of the medians.

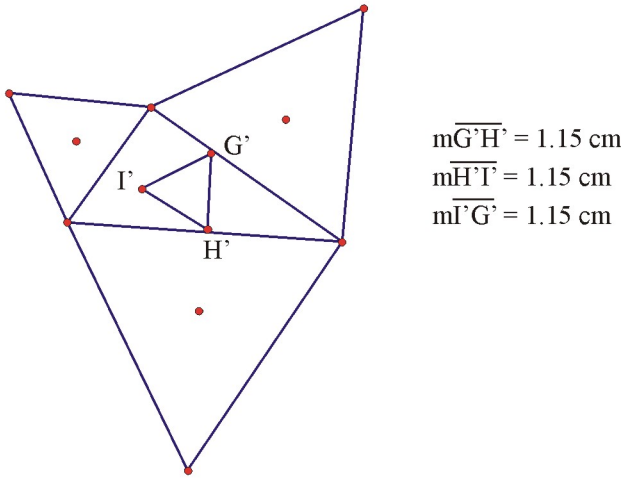
2. $XC = 20, CS = 30$.
3. False. The statement is true in the case of isosceles (vertex angle) and all angles in an equilateral triangle.



4. $X(\frac{14}{3}, 3)$
5. (a) $X(4, 3)$

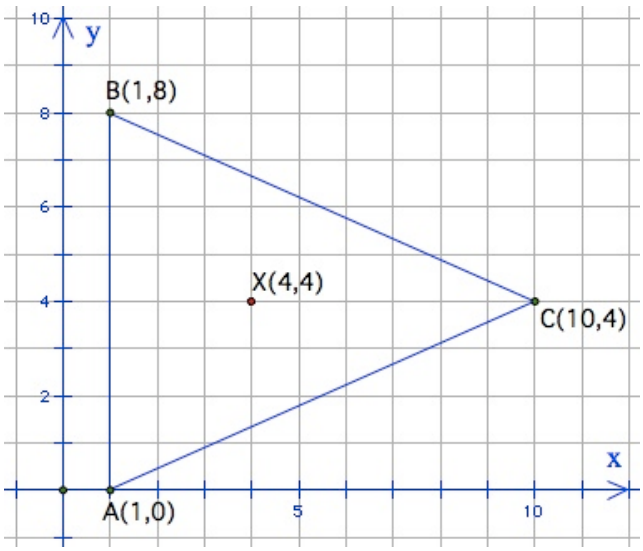
   (b) $XB = \sqrt{2}$.
6. (a) The triangle is equilateral.

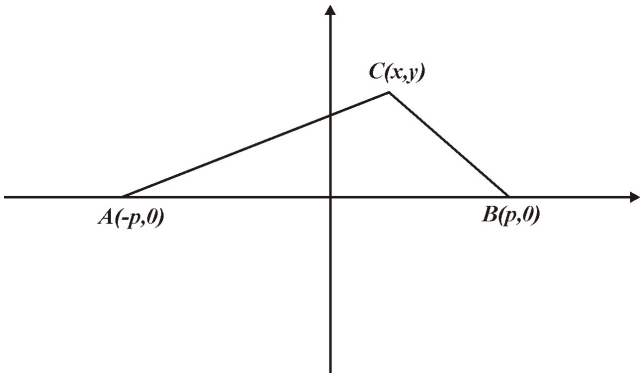
   (b) The difference in the areas of the inner and outer triangles is equal to the area of the original triangle.



7. The centroid will be located at $X(4, 4)$. The midpoint of the vertical side of the triangle is located at $(1, 4)$. Note that $(1, 4)$ is located 9 units from point C and that centroid will be one-third of the distance from $(1, 4)$ point to C.

8.





(a) Note that the midpoint of side $\overline{AB}$ is located at the origin. Construct the median from vertex C to the origin, and call it $\overline{CO}$. The point of concurrency of the three medians will be located on $\overline{CO}$ at point P that is two-thirds of the way from C to the origin.

(b) Using slopes and properties of straight lines, the point can be determined to have coordinates $P(\frac{1}{3}x, \frac{1}{3}y)$.

(c) Use the distance formula to show that point P is two-thirds of the way from each of the other two vertices to the midpoint of the opposite side.

9. Construct a median in an equilateral triangle. The triangles can be shown to be congruent by the SSS postulate.

5.5 Altitudes in Triangles

Learning Objectives

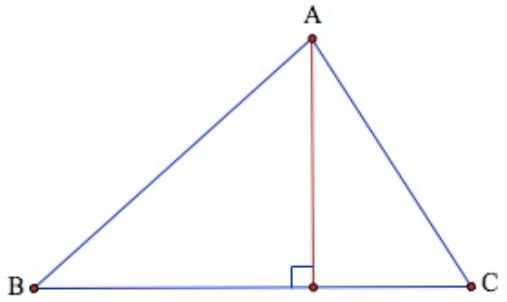
- Construct the altitude of a triangle.
- Apply the Concurrency of Altitudes Theorem to identify the point of concurrency of the altitudes of the triangle (the orthocenter).
- Use the Concurrency of Altitudes Theorem to solve problems involving the orthocenter of triangles.

Introduction

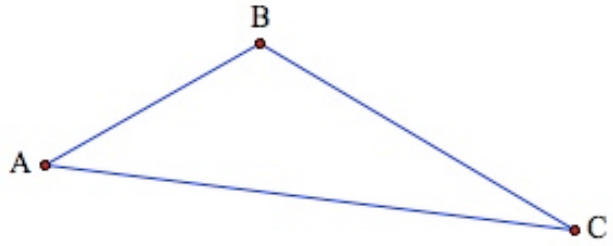
In this lesson we will conclude our discussions about special line segments associated with triangles by examining altitudes of triangles. We will learn how to find the location of a point within the triangle that involves the altitudes.

Definition of Altitude of a Triangle

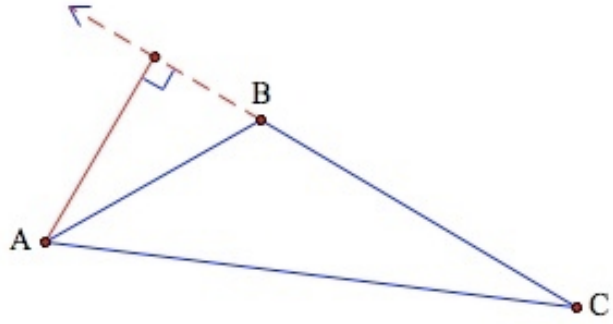
An **altitude** of a triangle is the line segment from a vertex perpendicular to the opposite side. Here is an example that shows the altitude from vertex A in an acute triangle.



We need to be careful with altitudes because they do not always lie inside the triangle. For example, if the triangle is obtuse, then we can easily see how an altitude would lie outside of the triangle. Suppose that we wished to construct the altitude from vertex A in the following obtuse triangle:

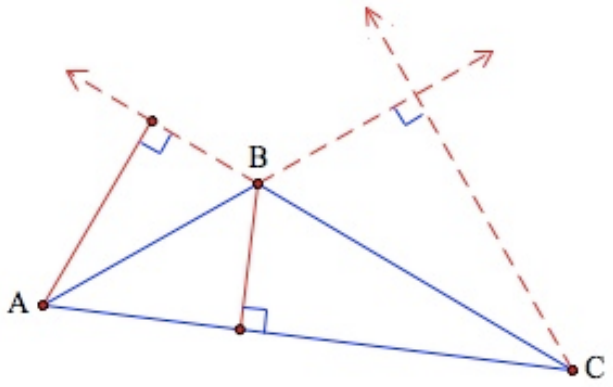


In order to do this, we must extend side $\overline{CB}$ as follows:



Will the remaining altitudes for $\triangle ABC$ (those from vertices B and C) lie inside or outside of the triangle?

Answer: The altitude from vertex B will lie inside the triangle; the altitude from vertex C will lie outside the triangle.

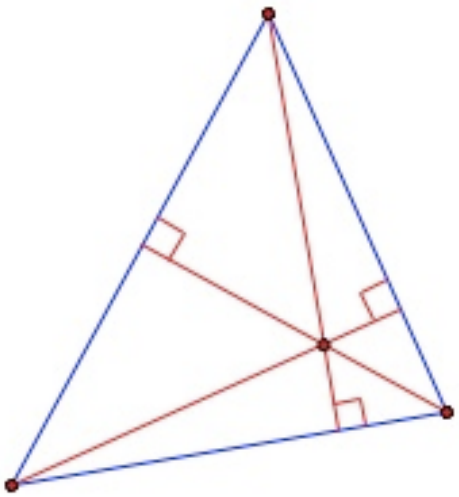


As was true with perpendicular bisectors (which intersect at the circumcenter), and angle bisectors (which intersect at the incenter), and medians (which intersect at the centroid), we can state a theorem about the altitudes of a triangle.

Concurrency of Triangle Altitudes Theorem: The altitudes of a triangle will intersect in a point. We call this point the **orthocenter** of the triangle.

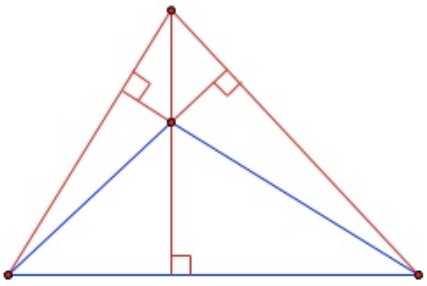
Rather than prove the theorem, we will demonstrate it for the three types of triangles (acute, obtuse, and right) and then illustrate some applications of the theorem.

Acute Triangles



The orthocenter lies within the triangle.

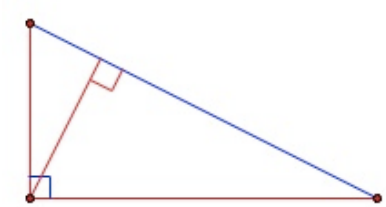
Obtuse Triangles



The orthocenter lies outside of the triangle.

Right Triangles

The legs of the triangle are altitudes. The orthocenter lies at the vertex of the right angle of the triangle.



Even with these three cases, we may still encounter special triangles that exhibit interesting properties.

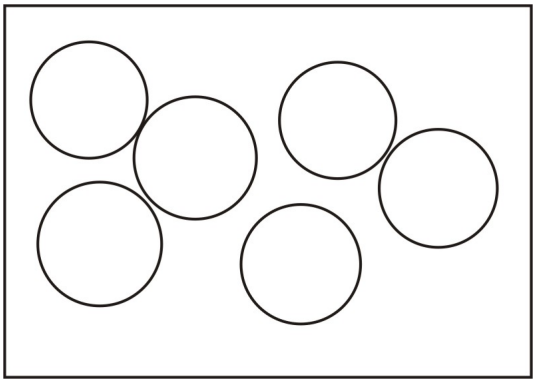
Example 1

Use a piece of Patty Paper ($5'' \times 5''$ tracing paper), or any square piece of paper to explore orthocenters of isosceles $\triangle ABC$. Note: *Patty Paper may be purchased in bulk from many Internet sites.*

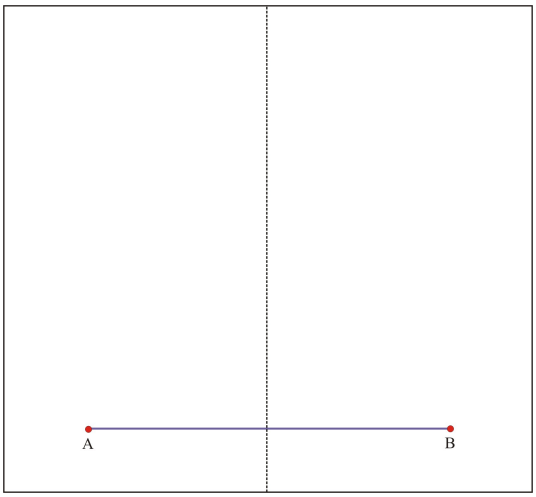
Determine any relationships between the location of the orthocenter and the location of the incenter, circumcenter, and centroid.

First let's recall that you can construct an isosceles triangle with Patty Paper as follows:

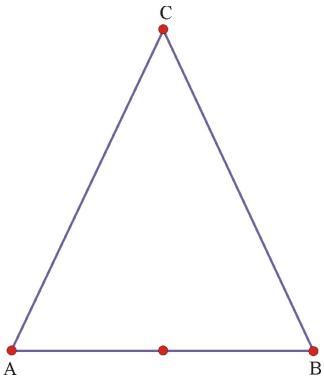
1. Draw line segment $\overline{AB}$.



2. Fold point A onto point B to find the fold line.



3. Locate point C anywhere on the fold line and connect point C to points A and B. (Hint: Locate point C as far away from A and B as possible so that you end up with a good-sized triangle.). Trace three copies of $\triangle ABC$ onto Patty Paper (so that you end up with four sheets of paper, each showing $\triangle ABC$).



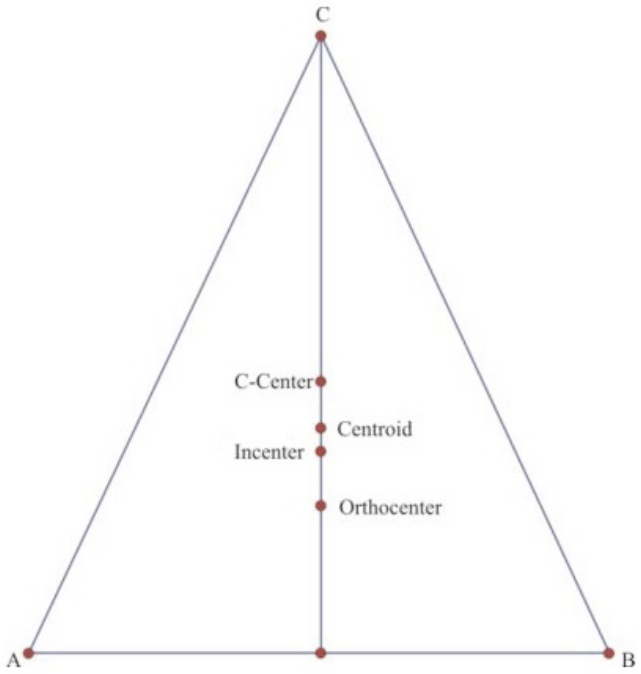
4. For one of the sheets, fold the paper to locate the median, angle bisector, and perpendicular bisector relative to the vertex angle at point C. What do you observe? (Answer: They are the same line segment.) Fold to find another bisector and locate the intersection of the two lines, the incenter.

5. For the second sheet, locate the circumcenter of $\triangle ABC$.

6. For the third sheet, locate the centroid of $\triangle ABC$.

7. For the third sheet, locate the orthocenter of $\triangle ABC$.

8. Trace the location of the circumcenter, centroid, and orthocenter onto the original triangle. What do you observe about the four points? (Answer: The incenter, orthocenter, circumcenter, and centroids are collinear and lie on the median from the vertex angle.)



Do you think that the four points will be collinear for all other kinds of triangles? The answer is pretty interesting! In our homework we will construct the four points for a more general case.

Lesson Summary

In this lesson we:

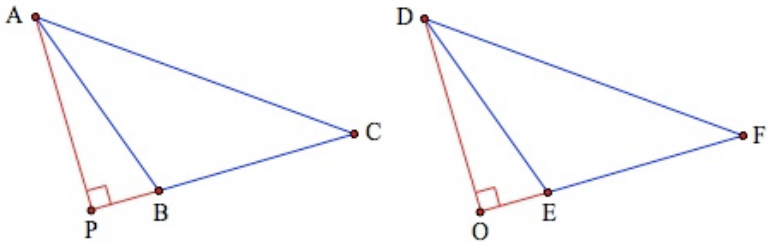
- Defined the orthocenter of a triangle.
- Stated the Concurrency of Altitudes Theorem.
- Solved problems using the Concurrency of Altitudes Theorem.
- Examined the special case of an isosceles triangle and determined relationships about among the incenter, circumcenter, centroid, and orthocenter.

Points to Consider

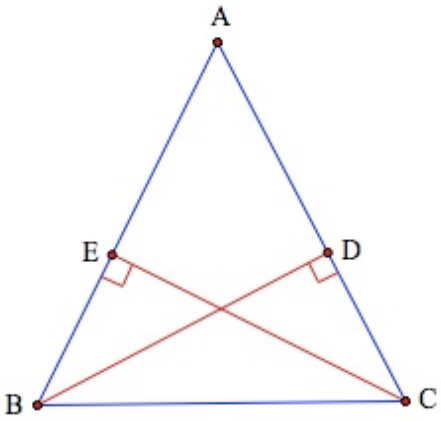
Remember that the altitude of a triangle is also its height and can be used to find the area of the triangle. The altitude is the shortest distance from a vertex to the opposite side.

Review Questions

1. In our lesson we looked at the special case of an isosceles triangle and determined relationships about among the incenter, circumcenter, centroid, and orthocenter. Explore the case of an equilateral triangle $\triangle ABC$ and see which (if any) relationships hold.
2. Perform the same exploration for an acute triangle. What can you conclude?
3. Perform the same exploration for an obtuse triangle. What can you conclude?
4. Perform the same exploration for a right triangle. What can you conclude?
5. What can you conclude about the four points for the general case of $\triangle ABC$?
6. In 3 you found that three of the four points were collinear. The segment joining these three points define the Euler segment. Replicate the exploration of the general triangle case and measure the lengths of the Euler segment and the sub-segments. Drag your drawing so that you can investigate potential relationships for several different triangles. What can you conclude about the lengths?
7. (Found in ***Exploring Geometry, 1999, Key Curriculum Press***) Construct a triangle and find the Euler segment. Construct a circle centered at the midpoint of the Euler segment and passing through the midpoint of one of the sides of the triangle. This circle is called the nine-point circle. The midpoint it passes through is one of the nine points. What are the other eight?
8. Consider $\triangle ABC \cong \triangle DEF$ with, $\overline{AP}$, $\overline{DO}$ altitudes of the triangles as indicated.



   Prove: $\overline{AP} \cong \overline{DO}$.
9. Consider isosceles triangle $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$, and $\overline{BD} \perp \overline{AC}, \overline{CE} \perp \overline{AB}$. Prove: $\overline{BD} \cong \overline{CE}$.



Review Answers

1. All four points are the same.
2. The four points all lie inside the triangle.
3. The four points all lie outside the triangle.
4. The orthocenter lies on the vertex of the right angle and the circumcenter lies on the midpoint of the hypotenuse.
5. The orthocenter, the circumcenter, and the centroid are always collinear.
6. (a) The circumcenter and the orthocenter are the endpoints of the Euler segment.
   (b) The distance from the orthocenter to the centroid is twice the distance from the centroid to the circumcenter.
7. Three of the points are the midpoints of the triangle's sides. Three other points are the points where the altitudes intersect the opposite sides of the triangle. The last three points are the midpoints of the segments connecting the orthocenter with each vertex.
8. The congruence can be proven by showing the congruence of triangles $\triangle APB$ and $\triangle DOE$. This can be done by applying postulate AAS to the two triangles.
9. The proof can be completed by using the AAS postulate to show that triangles $\triangle CEB$ and $\triangle BDC$ are congruent. The conclusion follows from CPCTC.

5.6 Inequalities in Triangles

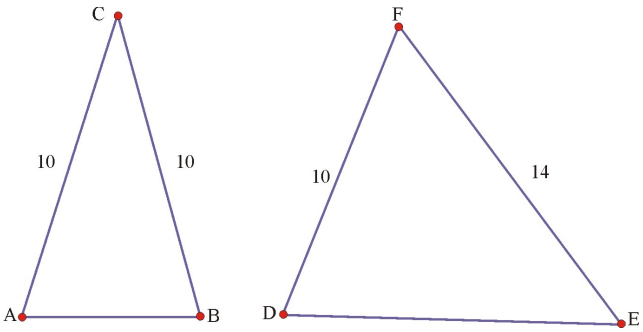
Learning Objectives

- Determine relationships among the angles and sides of a triangle.
- Apply the Triangle Inequality Theorem to solve problems.

Introduction

In this lesson we will examine the various relationships among the measure of the angles and the lengths of the sides of triangles. We will do so by stating and proving a few key theorems that will enable us to determine the types of relationships that hold true.

Look at the following two triangles



We see that the first triangle is isosceles while in the the second triangle, $\overline{DE}$ is longer than $\overline{AB}$. How are the measures of the angles at C and F related to the lengths of $\overline{AB}$ and $\overline{DE}$? It appears (and, in fact it is the case) that the measure of the angle at vertex F is larger than $\angle A$.

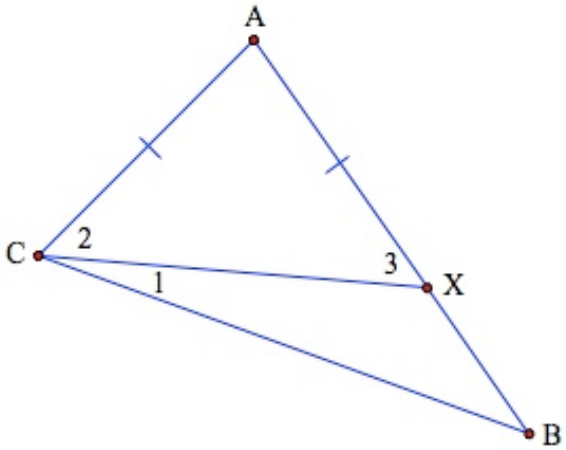
In this section we will formally prove theorems that reveal when such relationships hold. We will start with the following theorem.

Relationship Between the Sides and Angles of a Triangle

Theorem: If two sides of a triangle are of unequal length, then the angles opposite these sides are also unequal. The larger side will have a larger angle opposite it.

Proof. Consider $\triangle ABC$ with $AB > AC$. We must show that $m\angle ACB > m\angle ABC$.

1. By the Ruler Postulate, there is a point X on $\overline{AB}$ such that $AX = AC$. Construct $\overline{CX}$ and label angles 1, 2 and 3 as follows.



2. Since $\triangle AXC$ is isosceles, we have $m\angle 3 = m\angle 2$.

3. By angle addition, we have $m\angle ACB = m\angle 1 + m\angle 2$.

4. So $m\angle ACB > m\angle 2$, and by substitution $m\angle ACB > m\angle 3$.

5. Note that $\angle 3$ is exterior to $\triangle XBC$, so $m\angle 3 > m\angle ABC$.

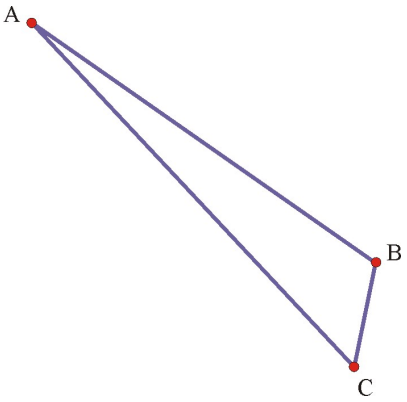
6. Hence, $m\angle ACB > m\angle 3$ and $m\angle 3 > m\angle ABC$, we have $m\angle ACB > m\angle ABC$. ♦

We can also prove a similar theorem about angles.

Larger angle has longer opposite side: If one angle of a triangle has greater measure than a second angle, then the side opposite the first angle is longer than the side opposite the second angle.

Proof. In order to prove the theorem, we will use a method that relies on indirect reasoning, a method that we will explore further. The method relies on starting with the assumption that the conclusion of the theorem is wrong, and then reaching a conclusion that logically contradicts the given statements.

1. Consider $\triangle ABC$ with $m\angle ABC > m\angle ACB$. We must show that $AC > AB$.



2. Assume temporarily that AC **is not greater than** AB. Then either $AC = AB$ or $AC < AB$.

3. If $AC = AB$, then the angles at vertices B and C are congruent. This is a contradiction of our given statements.

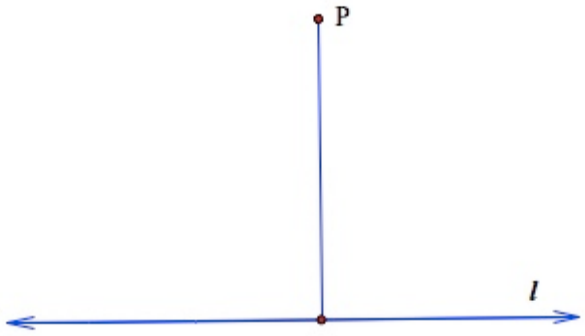
4. If $AC < AB$, then $m\angle ABC < m\angle ACB$ by the fact that the longer side is opposite the largest angle (the theorem we just proved). But this too contradicts our given statements. Hence, we must have $AC > AB$. ♦

With these theorems we can now prove an interesting corollary.

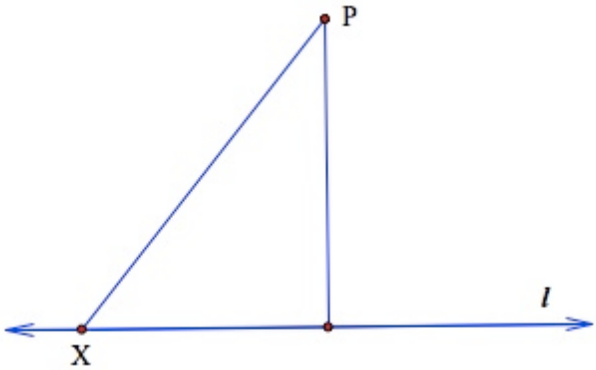
Corollary The perpendicular segment from a point to a line is the shortest segment from the point to the line.

Proof. The proof is routine now that we have proved the major results.

Consider point P and line l and the perpendicular line segment from P to l as follows.



We can draw the segment from P to any point on line l and get the case of a right triangle as follows:



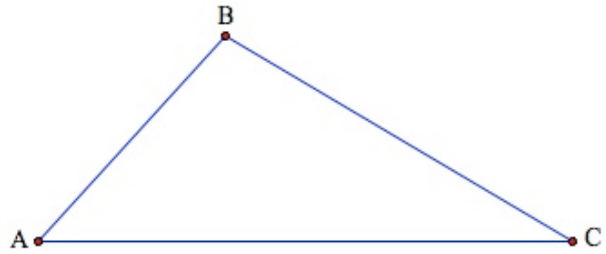
Since the triangle is a right triangle, the side opposite the right angle (the hypotenuse) will always have length greater than the length of the perpendicular segment from P to l, which is opposite an angle of $90°$. ♦

Now we are ready to prove one of the most useful facts in geometry, the triangle inequality theorem.

Triangle Inequality Theorem: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

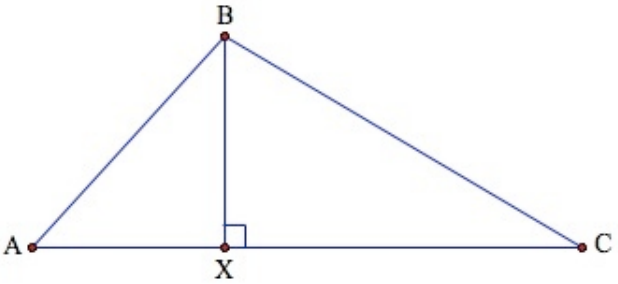
Proof. Consider $\triangle ABC$. We must show the following:

1. $AB + BC > AC$
2. $AC + BC > AB$
3. $AB + AC > BC$



Suppose that $\overline{AC}$ is the longest side. Then statements 2 and 3 above are true.

In order to prove 1. $AB + BC > AC$, construct the perpendicular from point B to X on the opposite side as follows:



Now we have two right triangles and can draw the following conclusions:

Since the perpendicular segment is the shortest path from a point to a line (or segment), we have $\overline{AX}$ is the shortest segment from A to $\overline{XB}$. Also, $\overline{CX}$ is the shortest segment from C to $\overline{XB}$. Therefore $AB > AX$ and $BC > CX$ and by addition we have

$$AB + BC > AX + XC = AC.$$

So, $AB + BC > AC$. ♦

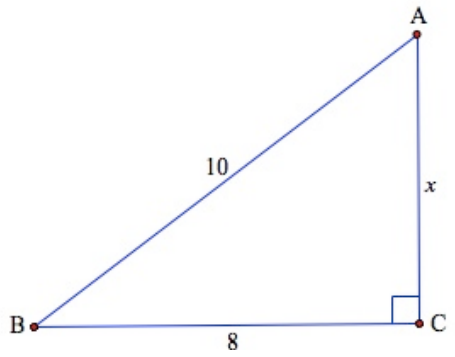
Example 1

Can you have a triangle with sides having lengths $4, 5, 10$?

Without a drawing we can still answer this question—it is an impossible situation, we cannot have such a triangle. By the Triangle Inequality Theorem, we must have that the sum of lengths of any two sides of the triangle must be greater than the length of the third side. In this case, we note that $4 + 5 = 9 < 10$.

Example 2

Find the angle of smallest measure in the following triangle.



$\angle B$ has the smallest measure. Since the triangle is a right triangle, we can find $x = 6$ using the Pythagorean Theorem (which we will prove later).

By the fact that the longest side is opposite the largest angle in a triangle, we can conclude that $m\angle B < m\angle A < \angle C$.

Lesson Summary

In this lesson we:

- Stated and proved theorems that helped us determine relationships among the angles and sides of a triangle.
- Introduced the method of indirect proof.
- Applied the Triangle Inequality Theorem to solve problems.

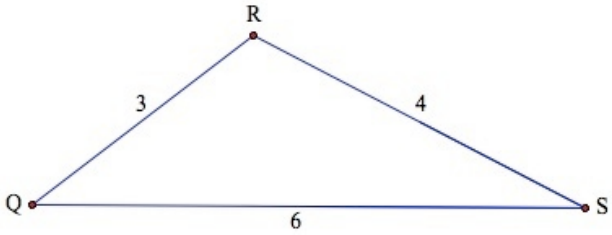
Points to Consider

Knowing these theorems and the relationships between the angles and sides of triangles will be applied when we use trigonometry. Since the size of the angle affects the length of the opposite side, we can show that there are specific angles associated with certain relationships (ratios) between the sides in a right triangle, and vice versa.

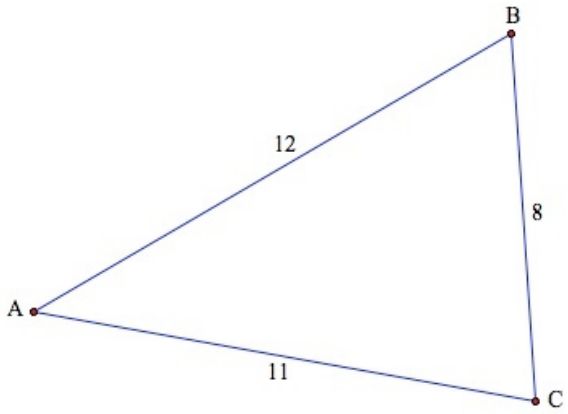
Review Questions

1. Name the largest and smallest angles in the following triangles:

(a)

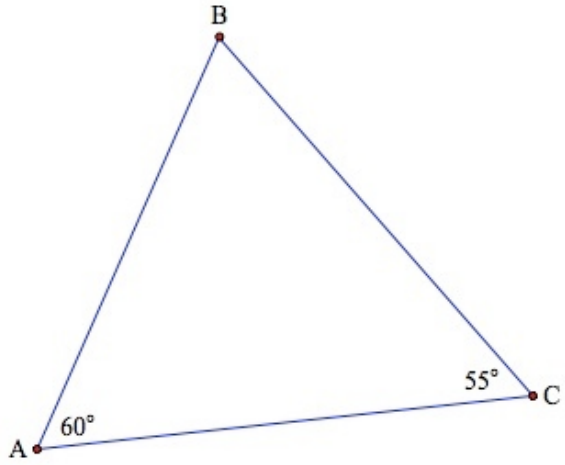


(b)

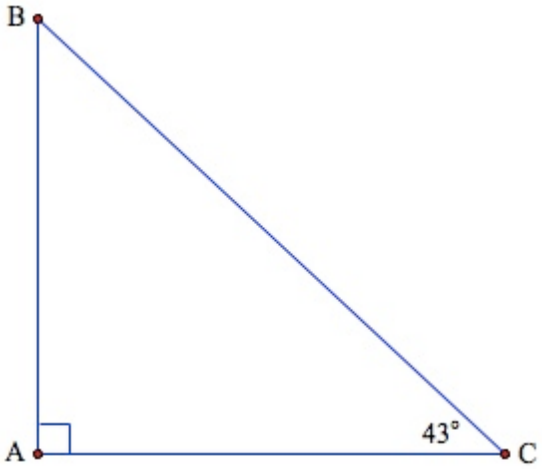


2. Name the longest side and the shortest side of the triangles.

(a)



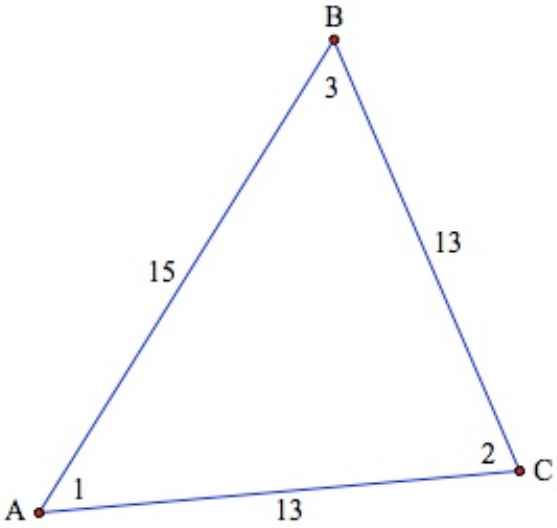
(b)



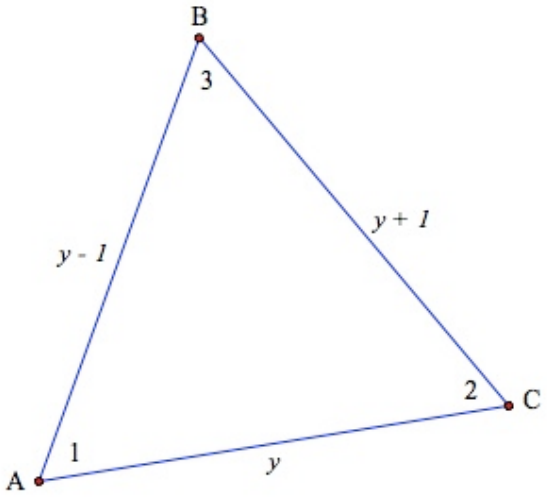
3. Is it possible to have triangles with the following lengths? Give a reason for your answer.
   (a) $6, 13, 6$
   (b) $8, 9, 10$
   (c) $7, 18, 11$
   (d) $3, 4, 5$
4. Two sides of a triangle have lengths 18 and 24. What can you conclude about the length of the third side?
5. The base of an isosceles triangle has length 30. What can you say about the length of each leg?

In exercises 6 and 7, find the numbered angle that has the largest measure of the triangle.

6.

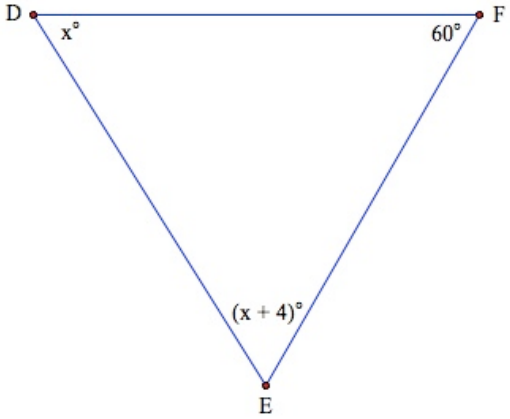


7.



In exercises 8-9, find the longest segment in the diagram.

8.



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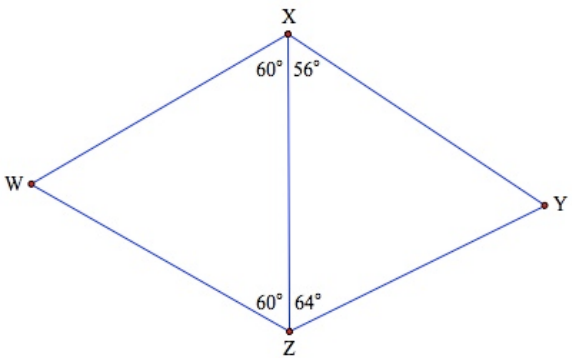
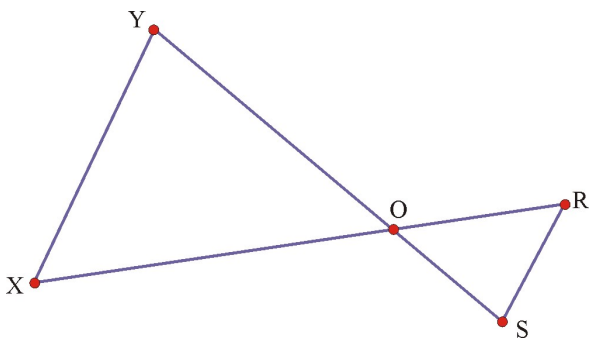


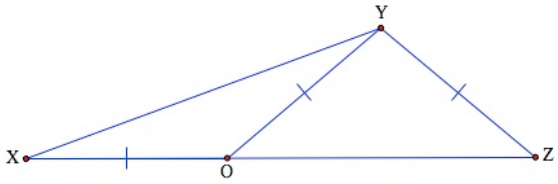
10.



Given: $m\angle Y > m\angle X, m\angle S > m\angle R$

Prove: $XR > YS$

11.



Given: $\overline{XO} \cong \overline{OY} \cong \overline{YZ}$

Prove: $XY > ZY$

Review Answers

1. (a) $\angle R$ is largest and $\angle S$ is smallest.
   (b) $\angle C$ is largest and $\angle A$ is smallest.
2. (a) $\overline{AC}$ is longest and $\overline{AB}$ is the shortest.
   (b) $\overline{BC}$ is longest and $\overline{AB}$ is the shortest.
3. (a) No, $6 + 6 = 12 < 13$.
   (b) Yes
   (c) No, $7 + 11 = 18$.
   (d) Yes
4. The third side must have length x such that $6 < x < 42$.
5. The legs each must have length greater than 15.
6. $\angle 2$
7. $\angle 1$
8. $\overline{DF}$
9. $\overline{XY}$
10. Since the angle opposite each of the two segments that comprise $\overline{XR}$ is greater than the angle opposite the corresponding segments of $\overline{XS}, XR > RS$.

11. Proof

5.7 Inequalities in Two Triangles

Learning Objectives

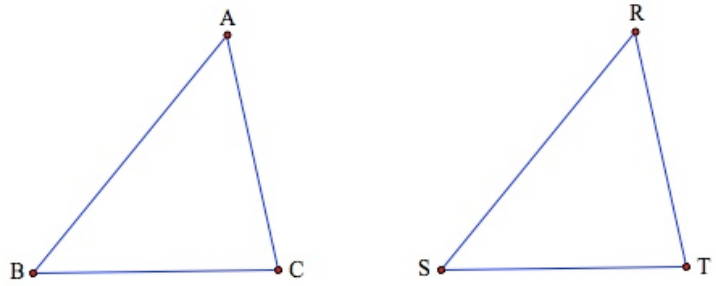
- Determine relationships among the angles and sides of two triangles.
- Apply the SAS and SSS Triangle Inequality Theorems to solve problems.

Introduction

In our last lesson we examined the various relationships among the measure of angles and the lengths of the sides of triangles and proved the Triangle Inequality Theorem that states that the sum of the lengths of two sides of a triangle is greater than the third side. In this lesson we will look at relationships in two triangles.

SAS Inequality Theorem

Let's begin our discussion by looking at the following congruent triangles.

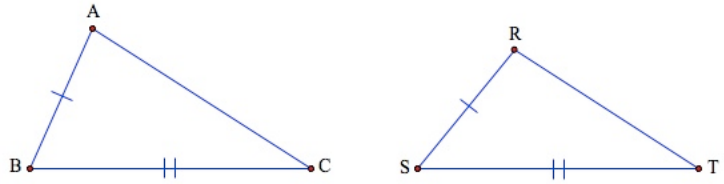


If we think of the sides of the triangle as matchsticks that are "hinged" at B and S respectively then we can increase the measure of the angles by opening up the sticks. If we open them so that $m\angle B > m\angle S$, then we see that $AC > RT$. Conversely, if you open them so that $AC > RT$, then we see that $m\angle B > m\angle S$. We can prove theorems that involve these relationships.

SAS Inequality Theorem (The Hinge Theorem): If two sides of a triangle are congruent to two sides of another triangle, but the included angle of the first triangle has greater measure than the included angle of the second triangle, then the third side of the first triangle is longer than the third side of the second triangle.

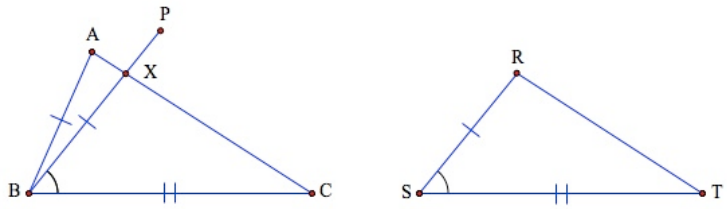
Proof. Consider $\triangle ABC$ and $\triangle RST$ with $\overline{AB} \simeq \overline{RS}, \overline{BC} \cong \overline{ST}, m\angle ABC > m\angle RST$. We

must show that $AC > RT$.



Construct $\overline{BP}$ so that $m\angle PBC = m\angle RST$. On $\overline{BP}$, take point X so that $BX = SR$. Either X is on $\overline{AC}$ or X is not on $\overline{AC}$. In either case, we must have $\triangle XBC \cong \triangle RST$ by SAS postulate and $\overline{XC} \cong \overline{RT}$ by CPCTC.

Case 1: X is on $\overline{AC}$.

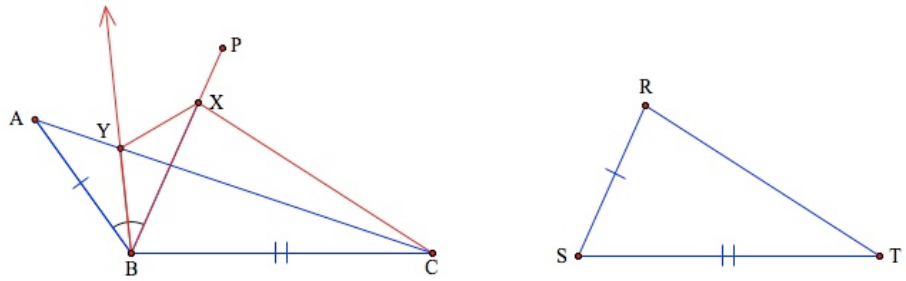


By the Segment Addition Postulate $AC = AX + XC$, so $AC > XC$. But from our congruence above we had $\overline{XC} \cong \overline{RT}$. By substitution we have $AC > RT$ and we have proven case 1.

Case 2: X is not on $\overline{AC}$.

Construct the bisector of $\angle ABX$ so that it intersects $\overline{AC}$ at point Y. Draw $\overline{XY}$ and $\overline{XC}$.

Recall that $AB = RS = BX$.



Note that $\triangle ABY \cong \triangle XBY$ by SAS postulate. Then $\overline{AY} \cong \overline{XY}$.

So, $XY + YC > XC$ by the Triangle Inequality Theorem.

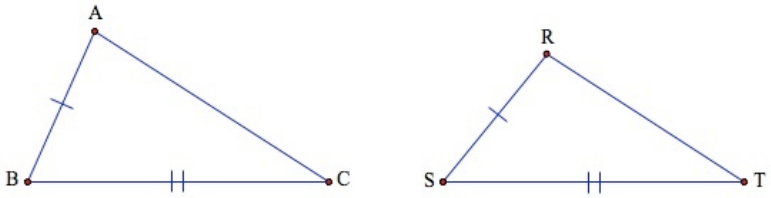
Now $XY + YC = AC$ by the segment addition postulate, and $XC = RT$ by our original construction of $\overline{XC}$, so by substitution we have $AC = XY + YC > XC = RT$ or $AC > RT$ and we have proven case 2. ♦

We can also prove the converse of the Hinge theorem.

SSS Inequality Theorem-Converse of Hinge Theorem: If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.

Proof. In order to prove the theorem, we will again use indirect reasoning as we did in proving Theorem 5-11.

Consider $\triangle ABC$ and $\triangle RST$ with $\overline{AB} \cong \overline{RS}, \overline{BC} \cong \overline{ST}, AC > RT$. We must show that $m\angle ABC > m\angle RST$.



1. Assume that $m\angle ABC$ is not greater than $m\angle RST$. Then either $m\angle ABC = m\angle RST$ or $m\angle ABC < m\angle RST$.

Case 1: If $m\angle ABC = m\angle RST$, then $\triangle ABC$ and $\triangle RST$ are congruent by SAS postulate and we have $\overline{AC} \cong \overline{RT}$. But this contradicts the given condition that $AC > RT$.

2. **Case 2:** If $m\angle ABC < m\angle RST$, then $AC < RT$ by Theorem 5-11. This contradicts the given condition that $AC > RT$.

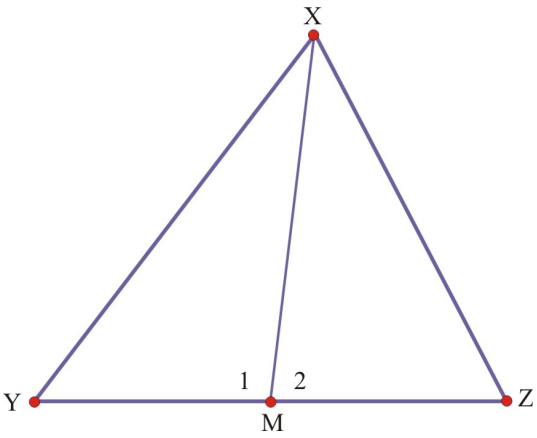
3. Since we get contradictions in both cases, then our original assumption was incorrect and we must have $m\angle ABC > m\angle RST$. ♦

We can now look at some problems that we could solve with these theorems.

Example 1

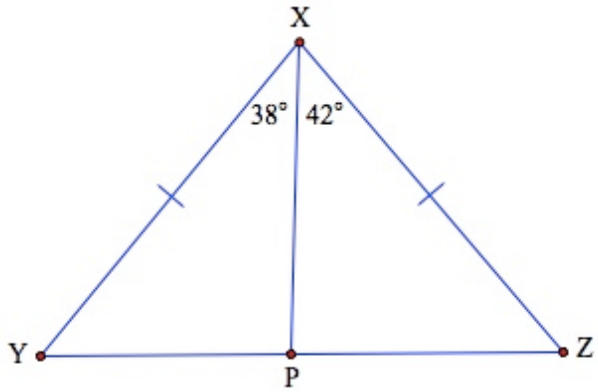
What we can deduce from the following diagrams.

1. *Given:* $\overline{XM}$ *is a median of* $\triangle XYZ$ *with* $XY > XZ$.



Since $XY > XZ$ and $YM = MZ$ then Theorem 5-14 applies and we have $m\angle 1 > m\angle 2$.

2. *Given:* $\triangle XYZ$ *as indicated.*



Since we have two sides of $\triangle XYP$ congruent with two sides of $\triangle XZP$, then Theorem 5-13 applies and we have $PZ > PY$.

Lesson Summary

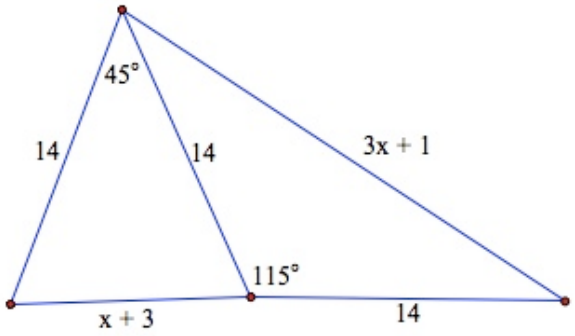
In this lesson we:

- Stated and proved theorems that helped determine relationships among the angles and sides of a pair of triangle.
- Applied the SAS and SSS Inequality Theorems to solve problems.

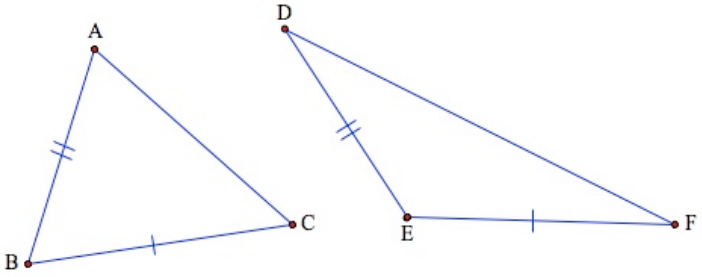
Review Questions

Use the theorems to make deductions in problems 1-5. List any theorems or postulates you use.

1.



2. Suppose that $\angle ABC$ is acute and $\angle DEF$ is obtuse



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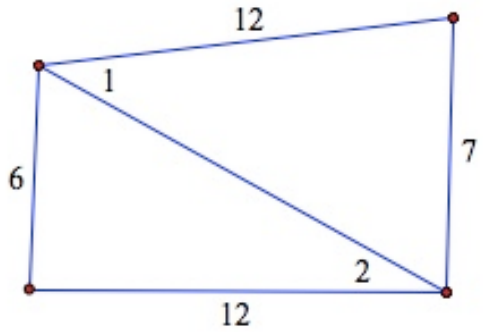

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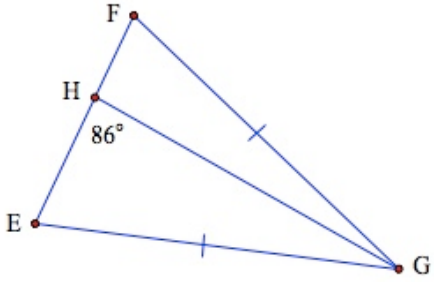

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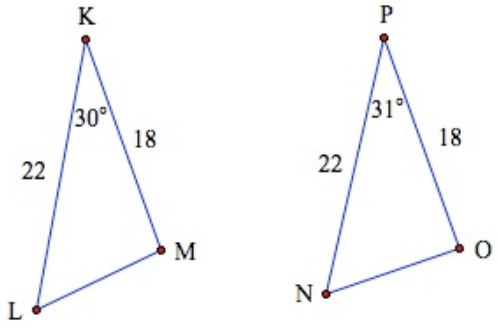
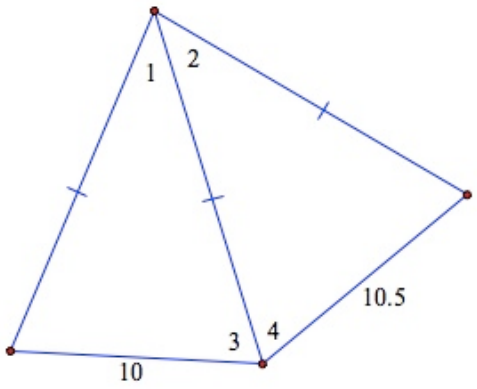
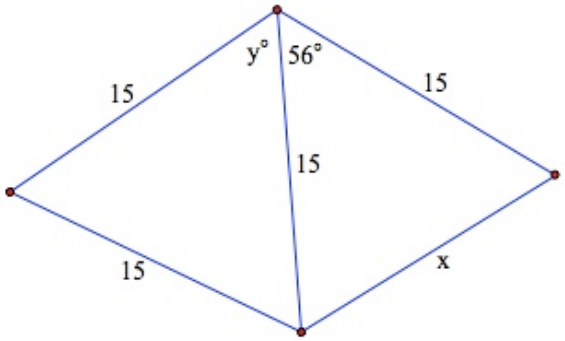


In problems 6-10, determine whether the assertion is true and give reasons to support your answers.

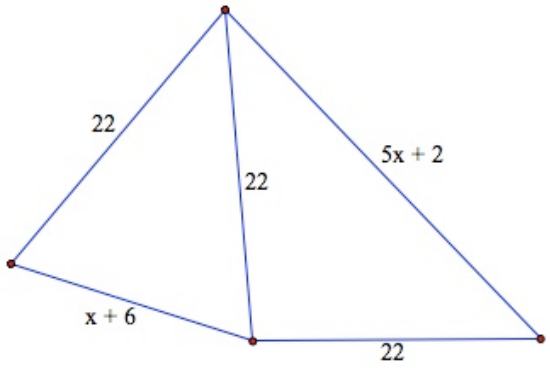
6. Assertion: $m\angle 2 > m\angle 1$ and $m\angle 3 > m\angle 4$.



7. Assertion: $x = 15$ in the figure below.

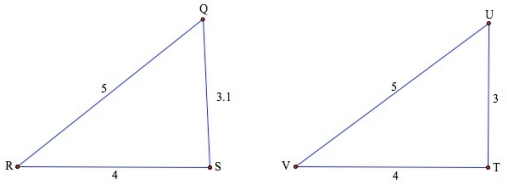


8. Assertion: $x > 1$.

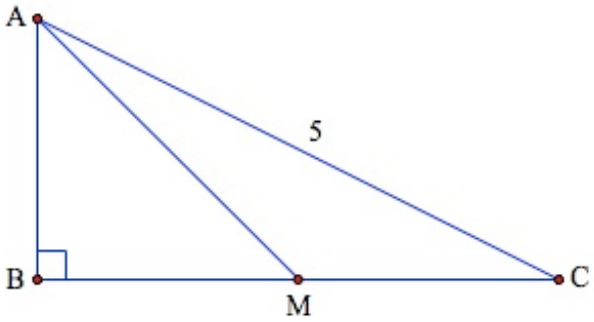


In problems 9-10, is the assertion true or false?

9. Assertion: $\angle R \cong \angle V$.



10. Consider $\triangle ABC$ is a right triangle with median from $\angle A$ as indicated. Assertion: $m\angle CAM > m\angle MCA$.



Review Answers

1. $x > 1$ by Theorem 5-13.
2. $DF > AC$ by Theorem 5-13.
3. $m\angle 1 > m\angle 2$ by Theorem 5-14.
4. We cannot deduce anything as we know nothing about the included angle nor the third side of each triangle.
5. $NO > LM$ by Theorem 5-13.

6. The assertions are true. $m\angle 2 > m\angle 1$ by Theorem 5-14. Since both triangles are isosceles and $m\angle 2 > m\angle 1$, then an implication of the fact that base angles are congruent will imply that $m\angle 3 > m\angle 4$.
7. The assertion is false. The two triangles have two sides congruent. The measure of angle y (adjacent to the 56° angle) is 60° since the triangle is equilateral. Hence, Theorem 5-13 applies and so $x < 15$.
8. The assertion is true by Theorem 5-14: $5x + 2 > x + 6, x > 1$.
9. The assertion is false. Theorem 5-14 applies and we have $m\angle R > m\angle V$.
10. The assertion is false. We do not have enough information to apply the theorems in this example.

5.8 Indirect Proof

Learning Objective

- Reason indirectly to develop proofs of statement

Introduction

Recall that in proving Theorems about the relationship between the sides and angles of triangle we used a method of proof in which we temporarily assumed that the conclusions was false and then reached a contradiction of the given statements. This method of proving something is called **indirect proof**. In this lesson we will practice using indirect proofs with both algebraic and geometric examples.

Indirect Proofs in Algebra

Let's begin our discussion with an algebraic example that we will put into if-then form.

Example 1

If $x = 2$, *then* $3x - 5 \neq 10$

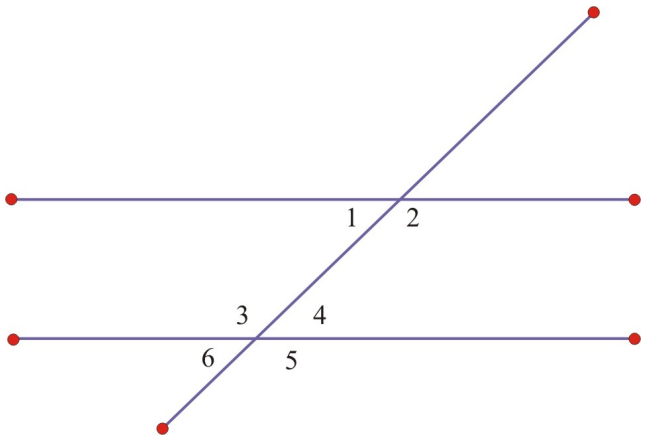
Proof. Let's assume temporarily that the $3x - 5 = 10$. Then we can reach a contradiction by applying our standard algebraic properties of real numbers and equations as follows:

$$\begin{aligned} 3x - 5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

This last statement contradicts the given statement that $x = 2$. Hence, our assumption is incorrect and we must have $3x - 5 \neq 10$. ♦

We can also employ this kind of reasoning in geometric situations. Consider the following theorem which we have previously proven using the Corresponding Angles Postulate:

Theorem: If parallel lines are cut by a transversal, then alternate interior angles are congruent.



Proof. It suffices to prove the theorem for one pair of alternate interior angles. So consider $\angle 1$ and $\angle 4$. We need to show that $m\angle 1 = m\angle 4$.

Assume that we have parallel lines and that $m\angle 1 \neq m\angle 4$. We know that lines are parallel, so we have by postulate 13 that corresponding angles are congruent and $m\angle 1 = m\angle 6$. Since vertical angles are congruent, we have $m\angle 6 = m\angle 4$. So by substitution, we must have $m\angle 1 = m\angle 4$, which is a contradiction.

Lesson Summary

In this lesson we:

- Illustrated some examples of proof by indirect reasoning, from algebra and geometry.

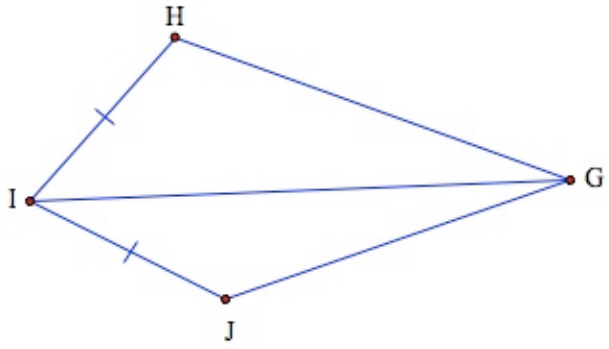
Points to Consider

Indirect reasoning can be a powerful tool in proofs. In the section on logical reasoning we saw that if there are two possibilities for a statement (such as TRUE or FALSE), if we can show one of them is not true (i.e. show that a statement is NOT FALSE), then the opposite possibility is all we have left (i.e. the statement is TRUE).

Review Questions

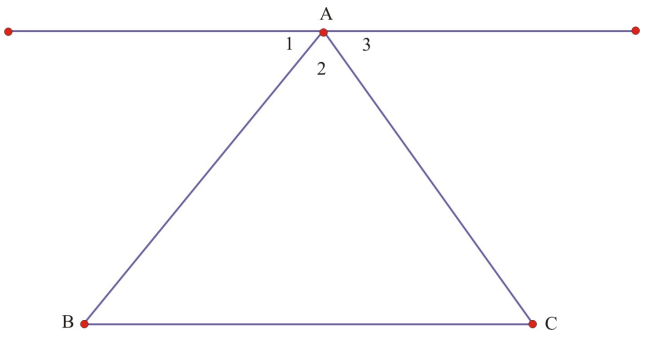
Generate a proof by contradiction for each of the following statements.

1. If n is an integer and n^2 is even, then n is even.
2. If in $\triangle ABC$ we have $m\angle A \neq m\angle B$, then $\triangle ABC$ is not equilateral.
3. If $x > 3$, then $x^2 > 9$.
4. If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.
5. If one angle of a triangle is larger than another angle of a triangle, then the side opposite the larger angle is longer than the side opposite the smaller angle.
6. The base angles of an isosceles triangle are congruent.
7. If n is an integer and n^2 is odd, then n is odd.
8. If we have $\triangle ABC$ with $m\angle A = 110°$, then $\angle C$ is not a right angle.
9. If two angles of a triangle are not congruent, then the sides opposite those angles are not congruent.
10. Consider the triangle the following figure with $\overline{HI} \cong \overline{JI}$, and $HG \neq JG$. Prove that $\overline{GI}$ does not bisect $\angle HIJ$.



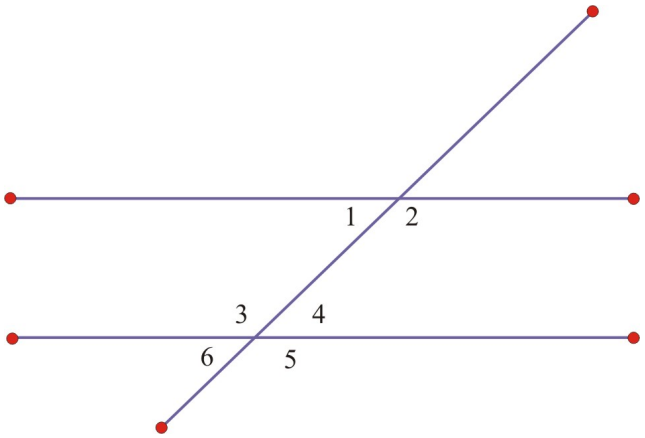
Review Answers

1. Assume n is odd. Then $n = 2a + 1$ for some integer a, and $n^2 = 2 < a^2 + 4a + 1$ which is odd. This contradicts the given statement that n is even.
2. Assume $\triangle ABC$ is equilateral. Then by definition, the sides are congruent. By the parallel postulate, we can construct a line parallel to the base through point A as follows:

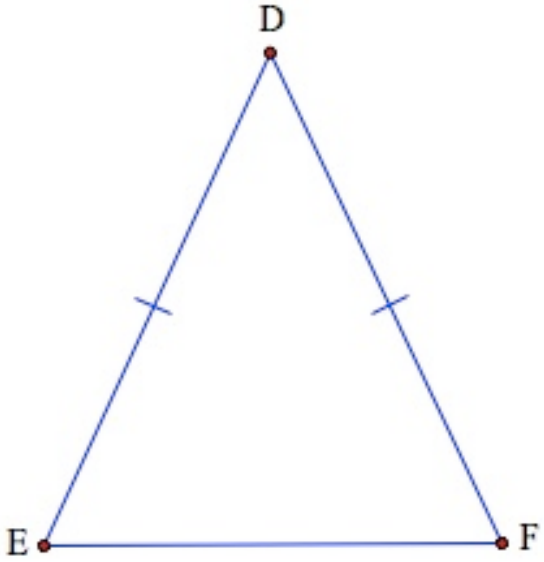


From this we can show with alternate interior angles that the triangle is equiangular so $m\angle A = m\angle B = m\angle C$. This contradicts the given statement that $m\angle A \neq m\angle B$.

3. Assume that x^2 is not greater than 9. Then either $x^2 = 9$ in which case $x = 3$, which is a contradiction, or $x^2 < 9$, in which case we can solve the quadratic inequality to get $-3 < x < 3$, also a contradiction of the fact that $x > 3$.
4. Assume that the lines are not parallel. Then $m\angle 1 \neq m\angle 6$. But $m\angle 6 = m\angle 4$ for vertical angles, so we have $m\angle 4 \neq m\angle 1$. This is a contradiction of the fact that $\angle 4 \cong \angle 1$.



5. Hint: This is theorem 5-11.
6. Assume $m\angle E \neq m\angle F$, say $m\angle E > m\angle F$. By Theorem 5-11 we have $DF > DE$, which contradicts the fact that we have an isosceles triangle.



7. Proof follows the lesson example closely. Assume n is even. Then it can be shown that n^2 must be even, which is a contradiction.
8. In $\triangle ABC$ we have $m\angle A + m\angle B + m\angle C = 180°$. Assume that $\angle C$ is a right angle. Hence, by substitution we have $m\angle A + m\angle B = 90°$ so that $m\angle A = 90° - m\angle B$, which is a contradiction.
9. Suppose we have a triangle in which the sides opposite two angles are congruent. Then it follows that the triangle must be isosceles. By Isosceles Triangle Theorem, the opposite angles are congruent, which is a contradiction.
10. Assume $\overline{GI}$ does bisect $\angle HIJ$. Hence, the two triangles are congruent by SAS and $\overline{HG} \cong \overline{JG}$ by CPCTC, which is a contradiction.

Chapter 6

Quadrilaterals

6.1 Interior Angles

Learning Objectives

- Identify the interior angles of convex polygons.
- Find the sums of interior angles in convex polygons.
- Identify the special properties of interior angles in convex quadrilaterals.

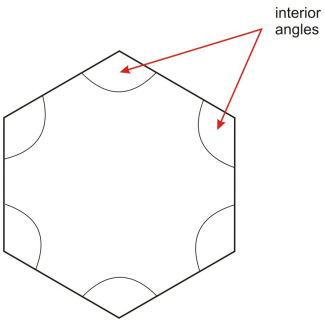
Introduction

By this point, you have studied the basics of geometry and you've spent some time working with triangles. Now you will begin to see some ways to apply your geometric knowledge to other polygons. This chapter focuses on quadrilaterals—polygons with four sides.

Note: Throughout this chapter, any time we talk about polygons, we will assume that we are talking about *convex* polygons.

Interior Angles in Convex Polygons

The interior angles are the angles on the inside of a polygon.

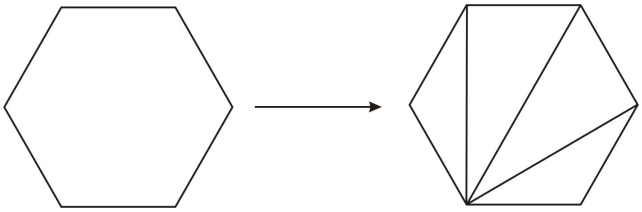


As you can see in the image, a polygon has the same number of interior angles as it does sides.

Summing Interior Angles in Convex Polygons

You have already learned the Triangle Sum Theorem. It states that the sum of the measures of the interior angles in a triangle will always be 180°. What about other polygons? Do they have a similar rule?

We can use the triangle sum theorem to find the sum of the measures of the angles for any polygon. The first step is to cut the polygon into triangles by drawing diagonals from one vertex. When doing this you must make sure none of the triangles overlap.



Notice that the hexagon above is divided into four triangles.

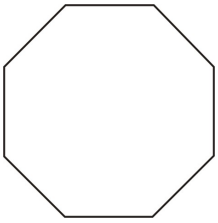
Since each triangle has internal angles that sum to 180°, you can find out the sum of the interior angles in the hexagon. The measure of each angle in the hexagon is a sum of angles from the triangles. Since none of the triangles overlap, we can obtain the TOTAL measure of interior angles in the hexagon by summing all of the triangles' interior angles. Or, multiply the number of triangles by 180°:

$$4(180^\circ) = 720^\circ$$

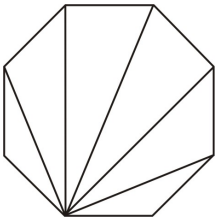
The sum of the interior angles in the hexagon is 720°.

Example 1

What is the sum of the interior angles in the polygon below?



The shape in the diagram is an octagon. Draw triangles on the interior using the same process.



The octagon can be divided into six triangles. So, the sum of the internal angles will be equal to the sum of the angles in the six triangles.

$$6(180^\circ) = 1080^\circ$$

So, the sum of the interior angles is 1080°.

What you may have noticed from these examples is that for any polygon, the number of triangles you can draw will be two less than the number of sides (or the number of vertices). So, you can create an expression for the sum of the interior angles of any polygon using n for the number of sides on the polygon.

The **sum of the interior angles** of a polygon with n sides is

$$\text{Angle Sum} = 180^\circ\ (n-2).$$

Example 2

What is the sum of the interior angles of a nonagon?

To find the sum of the interior angles in a nonagon, use the expression above. Remember that a nonagon has nine sides, so n will be equal to nine.

$$\begin{aligned}\text{Angle sum} &= 180^\circ(n-2)\\ &= 180^\circ(9-2)\\ &= 180^\circ(7)\\ &= 1260^\circ\end{aligned}$$

So, the sum of the interior angles in a nonagon is 1260°.

Interior Angles in Quadrilaterals

A quadrilateral is a polygon with four sides, so you can find out the sum of the interior angles of a convex quadrilateral using our formula.

Example 3

What is the sum of the interior angles in a quadrilateral?

Use the expression to find the value of the interior angles in a quadrilateral. Since a quadrilateral has four sides, the value of n will be 4.

$$\begin{aligned}\text{Angle Sum} &= 180^\circ(n-2)\\ &= 180^\circ(4-2)\\ &= 180^\circ(2)\\ &= 360^\circ\end{aligned}$$

So, the sum of the measures of the interior angles in a quadrilateral is 360°.

This will be true for any type of convex quadrilateral. You'll explore more types later in this chapter, but they will all have interior angles that sum to 360°. Similarly, you can divide any quadrilateral into two triangles. This will be helpful for many different types of proofs as well.

Lesson Summary

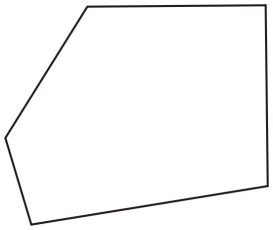
In this lesson, we explored interior angles in polygons. Specifically, we have learned:

- How to identify the interior angles of convex polygons.
- How to find the sums of interior angles in convex polygons.
- How to identify the special properties of interior angles in convex quadrilaterals.

Understanding the angles formed on the inside of polygons is one of the first steps to understanding shapes and figures. Think about how you can apply what you have learned to different problems as you approach them.

Review Questions

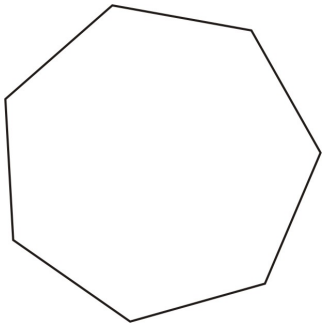
1. Copy the polygon below and show how it can be divided into triangles from one vertex.



2. Using the triangle sum theorem, what is the sum of the interior angles in this pentagon?

3-4: Find the sum of the interior angles of each polygon below.

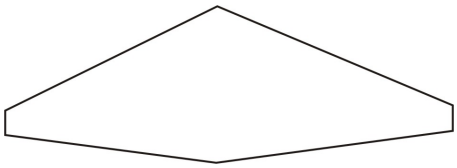
3.



Number of sides =

Sum of interior angles =

4.



Number of sides =

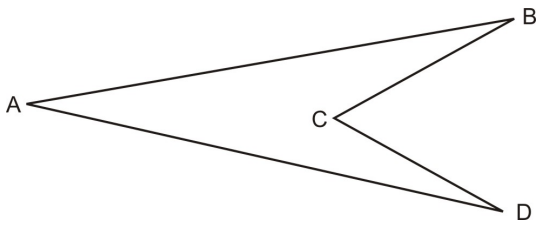
Sum of interior angles =

5. Complete the following table:

Table 6.1:

Polygon name	Number of sides	Sum of measures of interior angles
triangle	4	
	5	
	6	
	7	
octagon		
decagon		
		$1,800^\circ$
	n	

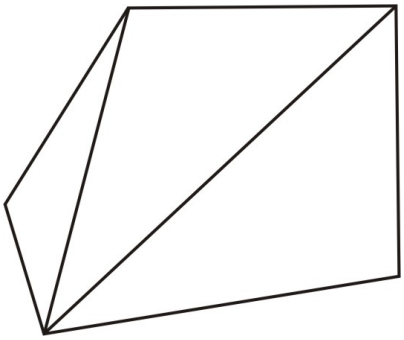
6. A **regular** polygon is a polygon with congruent sides and congruent angles. What is the measure of each angle in a regular pentagon?
7. What is the measure of each angle in a regular octagon?
8. Can you generalize your answer from 6 and 7? What is the measure of each angle in a regular $n-$gon?
9. Can you use the polygon angle sum theorem on a convex polygon? Why or why not? Use the convex quadrilateral $ABCD$ to explain your answer.



10. If we know the sum of the angles in a polygon is 2700°, how many sides does the polygon have? Show the work leading to your answer.

Review Answers

1. One possible answer:



2. $3(180) = 540°$
3. Number of sides = 7, sum of interior angles = 900°
4. Number of sides = 6, sum of interior angles = 720°

Table 6.2:

Polygon name	Number of sides	Sum of measures of interior angles
triangle	3	180°
quadrilateral	4	360°
pentagon	5	540°
hexagon	6	720°
heptagon	7	900°
octagon	8	1,080°
decagon	10	1,440°
dodecagon	12	1,800°
$n-gon$	n	$180(n-2)°$

5.
6. Since the sum of the angles is 540, each angle measures $\frac{540}{5} = 108°$
7. $\frac{1080}{8} = 135°$

8. $\frac{180(n-2)}{n}$
9. Answers will vary. One possibility is no, we cannot use the polygon angle sum theorem because $\angle C$ is an acute angle that does not open inside the polygon. Alternatively, if we allow for angles between 180° and 360°, then we can use the angle sum theorem, but so far we have not seen angles measuring more than 180°
10. Solve the equation:

$$\begin{aligned} 180(n-2) &= 2700 \\ \frac{180(n-2)}{180} &= \frac{2700}{180} \\ n-2 &= 15 \\ n-2+2 &= 15+2 \\ n &= 17 \end{aligned}$$

6.2 Exterior Angles

Learning Objectives

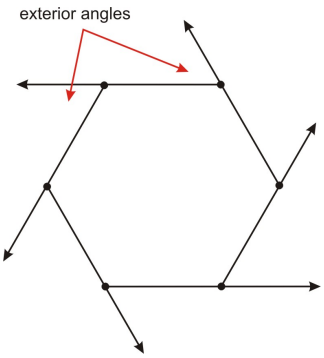
- Identify the exterior angles of convex polygons.
- Find the sums of exterior angles in convex polygons.

Introduction

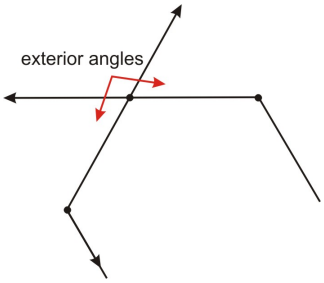
This lesson focuses on the exterior angles in a polygon. There is a surprising feature of the sum of the exterior angles in a polygon that will help you solve problems about regular polygons.

Exterior Angles in Convex Polygons

Recall that *interior* means inside and that *exterior* means *outside*. So, an **exterior angle** is an angle on the outside of a polygon. An exterior angle is formed by extending a side of the polygon.



As you can tell, there are two possible exterior angles for any given vertex on a polygon. In the figure above we only showed one set of exterior angles; the other set would be formed by extending each side in the opposite (clockwise) direction. However, it doesn't matter which exterior angles you use because on each vertex their measurement will be the same. Let's look closely at one vertex, and draw both of the exterior angles that are possible.

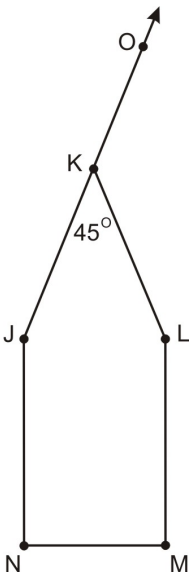


As you can see, the two exterior angles at the same vertex are vertical angles. Since vertical angles are congruent, the two exterior angles possible around a single vertex are congruent.

Additionally, because the exterior angle will be a linear pair with its adjacent interior angle, it will always be supplementary to that interior angle. As a reminder, supplementary angles have a sum of 180°.

Example 1

What is the measure of the exterior angle $\angle OKL$ in the diagram below?



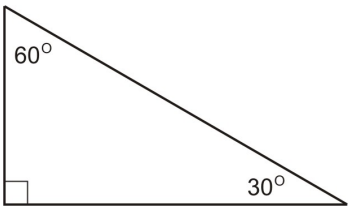
The interior angle is labeled as 45°. Since you need to find the exterior angle, notice that the interior angle and the exterior angle form a linear pair. Therefore the two angles are supplementary—they sum to 180°. So, to find the measure of the exterior angle, subtract 45° from 180°.

$$180 - 45 = 135$$

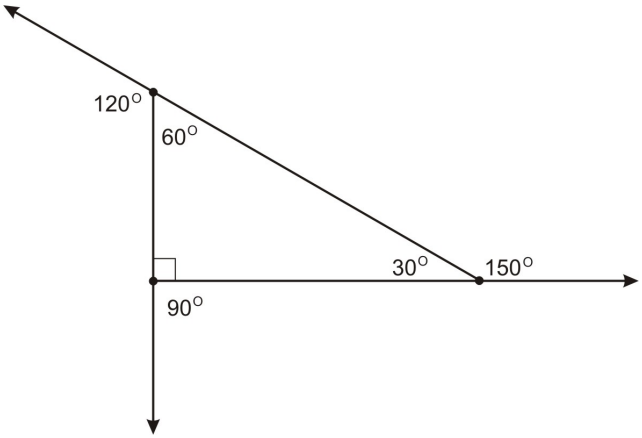
The measure of $\angle OKL$ is 135°.

Summing Exterior Angles in Convex Polygons

By now you might expect that if you add up various angles in polygons, there will be some sort of pattern or rule. For example, you know that the sum of the interior angles of a triangle will always be 180°. From that fact, you have learned that you can find the sums of the interior angles of any polygons with n sides using the expression $180(n-2)$. There is also a rule for exterior angles in a polygon. Let's begin by looking at a triangle.



To find the exterior angles at each vertex, extend the segments and find angles supplementary to the interior angles.

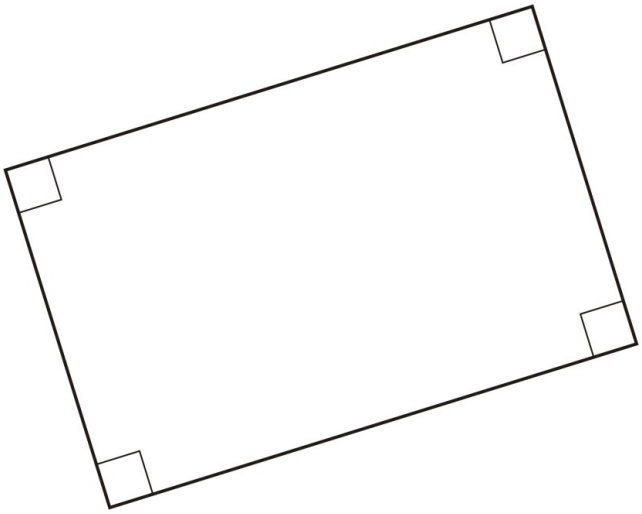


The sum of these three exterior angles is:

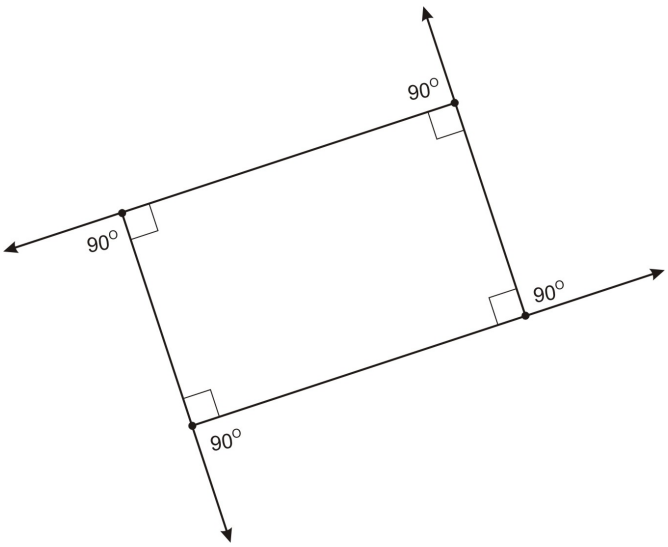
$$150^\circ + 120^\circ + 90^\circ = 360^\circ$$

So, the exterior angles in this triangle will sum to 360°.

To compare, examine the exterior angles of a rectangle.



In a rectangle, each interior angle measures 90°. Since exterior angles are supplementary to interior angles, all exterior angles in a rectangle will also measure 90°.



Find the sum of the four exterior angles in a rectangle.

$$90^\circ + 90^\circ + 90^\circ + 90^\circ = 360^\circ$$

So, the sum of the exterior angles in a rectangle is also 360°.

In fact, the sum of the exterior angles in **any** convex polygon will always be 360°. It doesn't matter how many sides the polygon has, the sum will always be 360°.

We can prove this using algebra as well as the facts that at any vertex the sum of the interior and one of the exterior angles is always 180°, and the sum of all interior angles in a polygon is $180(n-2)$.

Exterior Angle Sum: The sum of the exterior angles of any convex polygon is 360°

Proof. At any vertex of a polygon the exterior angle and the interior angle sum to 180°. So summing all of the exterior angles and interior angles gives a total of 180 degrees times the number of vertices:

$$(\text{Sum of Exterior Angles}) + (\text{Sum of Interior Angles}) = 180^\circ n.$$

On the other hand, we already saw that the sum of the interior angles was:

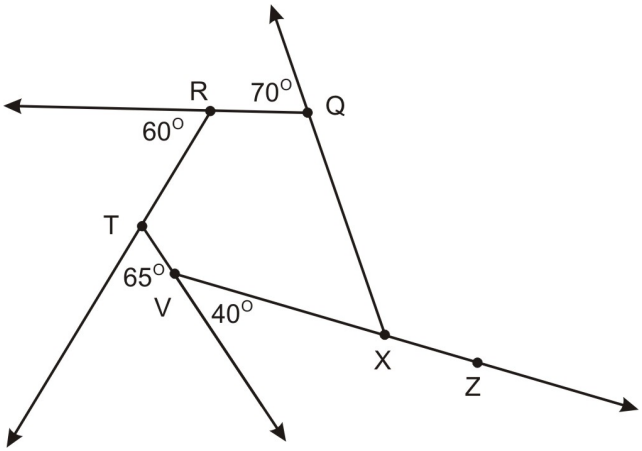
$$(\text{Sum of Interior Angles}) = 180(n-2) = 180^\circ n - 360^\circ.$$

Putting these together we have

$$\begin{aligned} 180n &= (\text{Sum of Exterior Angles}) + (\text{Sum of Interior Angles}) \\ &= (180n - 360) + (\text{Sum of Exterior Angles}) \\ 360 &= (\text{Sum of Exterior Angles}) \end{aligned}$$

Example 2

What is $m\angle ZXQ$ in the diagram below?



$\angle ZXQ$ in the diagram is marked as an exterior angle. So, we need to find the measure of one exterior angle on a polygon given the measures of all of the others. We know that the sum of the exterior angles on a polygon must be equal to 360°, regardless of how many sides the shape has. So, we can set up an equation where we set all of the exterior angles shown (including $m\angle ZXQ$) summed and equal to 180°. Using subtraction, we can find the value of X.

$$\begin{aligned} 70^\circ + 60^\circ + 65^\circ + 40^\circ + m\angle ZXQ &= 360^\circ \\ 235^\circ + m\angle ZXQ &= 360^\circ \\ m\angle ZXQ &= 360^\circ - 235^\circ \\ m\angle ZXQ &= 125^\circ \end{aligned}$$

The measure of the missing exterior angle is 125°.

We can verify that our answer is reasonable by inspecting the diagram and checking whether the angle in question is acute, right, or obtuse. Since the angle should be obtuse, 125° is a reasonable answer (assuming the diagram is accurate).

Lesson Summary

In this lesson, we explored exterior angles in polygons. Specifically, we have learned:

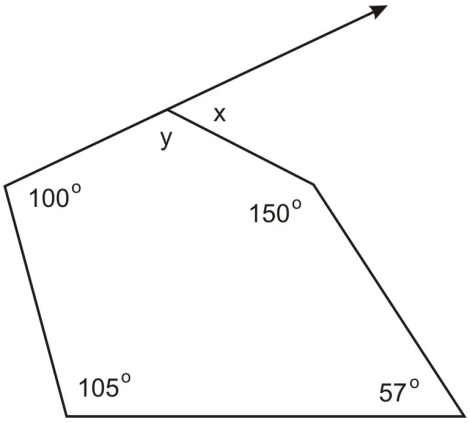
- How to identify the exterior angles of convex polygons.
- How to find the sums of exterior angles in convex polygons.

We have also shown one example of how knowing the sum of the exterior angles can help you find the measure of particular exterior angles.

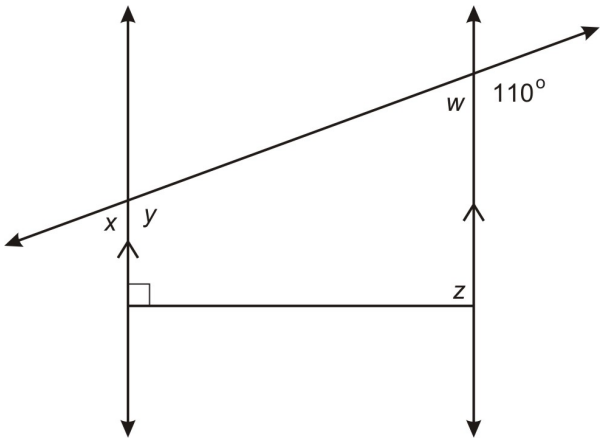
Review Questions

For exercises 1-3, find the measure of each of the labeled angles in the diagram.

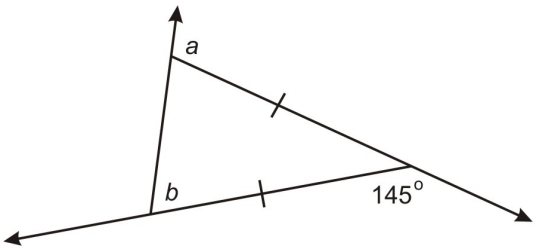
1. $x =$ ____, $y =$ ____



2. $w =$ ____, $x =$ ____, $y =$ ____, $z =$ ____



3. $a =$ ____, $b =$ ____



4. Draw an equilateral triangle with one set of exterior angles highlighted. What is the measure of each exterior angle? What is the sum of the measures of the three exterior angles in an equilateral triangle?
5. Recall that a regular polygon is a polygon with congruent sides and congruent angles. What is the measure of each *interior* angle in a regular octagon?
6. How can you use your answer to 5 to find the measure of each *exterior* angle in a regular octagon? Draw a sketch to justify your answer.
7. Use your answer to 6 to find the sum of the measures of the exterior angles of an octagon.

8. Complete the following table assuming each polygon is a regular polygon. Note: This is similar to a previous exercise with more columns—you can use your answer to that question to help you with this one.

Table 6.3:

Regular Polygon name	Number of sides	Sum of measures of interior angles	Measure of each interior angle	Measure of each exterior angle	Sum of measures of exterior angles
triangle					
	4				
	5				
	6				
	7				
octagon					
decagon					
		$1,800°$			
	n				

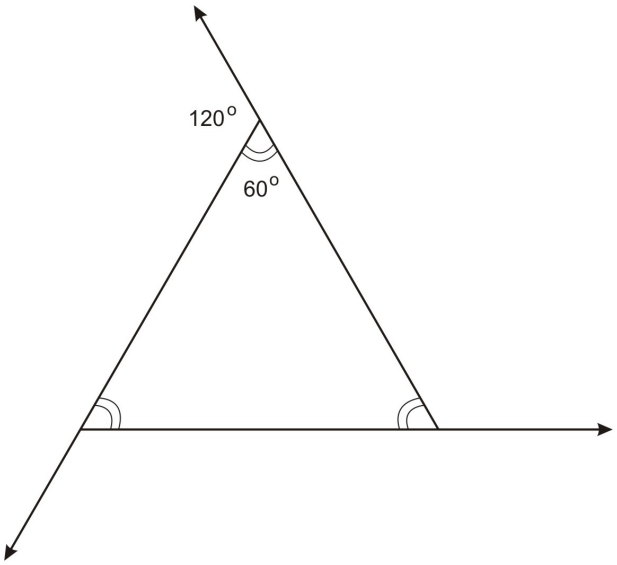
9. Each exterior angle forms a linear pair with its adjacent internal angle. In a regular polygon, you can use two different formulas to find the measure of each exterior angle. One way is to compute $180°-$(measure of each interior angle)... in symbols $180 - \frac{180(n-2)}{n}$.

Alternatively, you can use the fact that all n exterior angles in an $n-$gon sum to 360° and find the measure of each exterior angle with by dividing the sum by n. Again, in symbols this is $\frac{360}{n}$

Use algebra to show these two expressions are equivalent.

Review Answers

1. $x = 52^\circ$, $y = 128^\circ$
2. $w = 70^\circ, x = 70^\circ, y = 110^\circ, z = 90^\circ$
3. $a = 107.5^\circ, b = 72.5^\circ$
4. Below is a sample sketch.



Each exterior angle measures 120°, the sum of the three exterior angles is 360°

5. Sum of the angles is $180(8-2) = 1080^\circ$. So, each angle measures $\frac{1080}{8} = 135^\circ$
6. Since each exterior angle forms a linear pair with its adjacent interior angle, we can find the measure of each exterior angle with $180 - 135 = 45^\circ$
7. $45(8) = 360^\circ$

Table 6.4:

Regular Polygon name	Number of sides	Sum of measures of interior angles	Measure of each interior angle	Measure of each exterior angle	Sum of measures of exterior angles
triangle	3	180°	60°	120°	360°
square	4	360°	90°	90°	360°

Table 6.4: (continued)

Regular Polygon name	Number of sides	Sum of measures of interior angles	Measure of each interior angle	Measure of each exterior angle	Sum of measures of exterior angles
pentagon	5	540°	72°	108°	360°
hexagon	6	720°	60°	120°	360°
heptagon	7	900°	128.57°	51.43°	360°
octagon	8	1,080°	135°	45°	360°
decagon	10	1,440°	144°	36°	360°
dodecagon	12	1,800°	150°	30°	360°
$n-$gon	n	$180(n-2)°$	$\frac{180(n-2)}{n}°$	$\frac{360}{n}°$	360°

8.
9. One possible answer.

$$\begin{aligned} 180 - \frac{180(n-2)}{n} &= \frac{180n}{n} - \frac{180(n-2)}{n} \\ &= \frac{180n - 180(n-2)}{n} \\ &= \frac{180n - 180n + 360}{n} \\ &= \frac{360}{n} \end{aligned}$$

6.3 Classifying Quadrilaterals

Learning Objectives

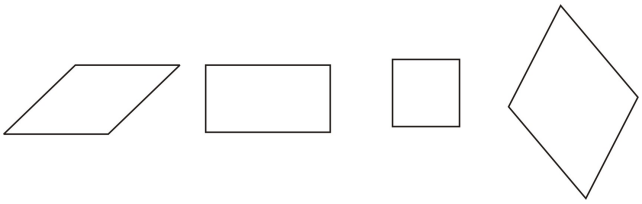
- Identify and classify a parallelogram.
- Identify and classify a rhombus.
- Identify and classify a rectangle.
- Identify and classify a square.
- Identify and classify a kite.
- Identify and classify a trapezoid.
- Identify and classify an isosceles trapezoid.
- Collect the classifications in a Venn diagram.
- Identify how to classify shapes on a coordinate grid.

Introduction

There are many different classifications of quadrilaterals. In this lesson, you will explore what defines each type of quadrilateral and also what properties each type of quadrilateral has. You have probably heard of many of these shapes before, but here we will focus on things we've learned about other polygons—the relationships among interior angles, and the relationships among the sides and diagonals. These issues will be explored in later lessons to further your understanding.

Parallelograms

A **parallelogram** is a quadrilateral with two pairs of parallel sides. Each of the shapes shown below is a parallelogram.



As you can see, parallelograms come in a variety of shapes. The **only** defining feature is that **opposite sides are parallel**. But, once we know that a figure is a parallelogram, we have two very useful theorems we can use to solve problems involving parallelograms: the Opposite Sides Theorem and the Opposite Angles Theorem.

We prove both of these theorems by adding an auxiliary line and showing that a parallelogram can be divided into two congruent triangles. Then we apply the definition of congruent triangles—the fact that if two triangles are congruent, all their corresponding parts are congruent (CPCTC).

An **auxiliary** line is a line that is added to a figure *without changing the given information.* You can always add an auxiliary line to a figure by connecting two points because of the Line Postulate. In many of the proofs in this chapter we use auxiliary lines.

Opposite Sides of Parallelogram Theorem: The opposite sides of a parallelogram are congruent.

Proof.

- Given Parallelogram $ABCD$
- Prove $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

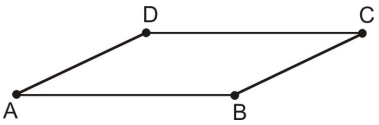
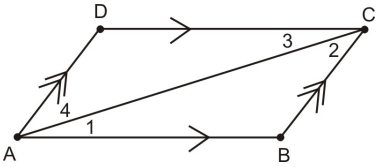


Table 6.5:

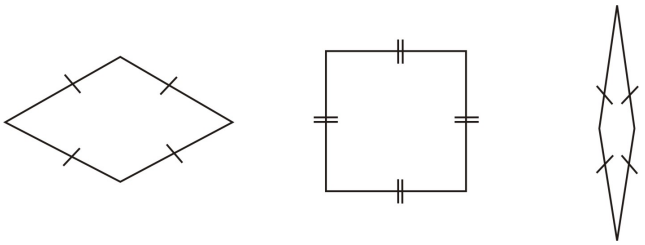
Statement	Reason
1. $ABCD$ is a parallelogram.	1. Given
2. Draw Auxiliary segment $\overline{AC}$ and label the angles as follows. 	2. Line Postulate
3. $\overline{AB} \| \overline{DC}$	3. Definition of parallelogram
4. $\angle 1 \cong \angle 3$	4. Alternate Interior Angles Theorem
5. $\overline{AD} \| \overline{BC}$	5. Definition of parallelogram
6. $\angle 2 \cong \angle 4$	6. Alternate Interior Angles Theorem
7. $\overline{AC} \cong \overline{AC}$	7. Reflexive Property
8. $\triangle ADC \cong \triangle CBA$	8. ASA Triangle Congruence Postulate
9. $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$	9. Definition of congruent triangles (all corresponding sides and angles are congruent) ♦

Opposite Angles in Parallelogram Theorem: The opposite angles of a parallelogram are congruent.

Proof. This proof is nearly the same as the one above and you will do it as an exercise.

Rhombi

A **rhombus** (plural is rhombi or rhombuses) is a quadrilateral that has four congruent sides. While it is possible for a rhombus to have four congruent angles, it's only one example. Many rhombi do NOT have four congruent angles.



Theorem: A rhombus is a parallelogram

Proof.

- Given: Rhombus $JKLM$
- Prove: $\overline{JK} \| \overline{LM}$ and $\overline{JM} \| \overline{KL}$

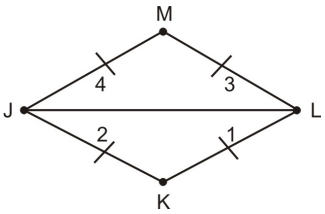


Table 6.6:

Statement	Reason
1. $JKLM$ is a Rhombus.	1. Given
2. $\overline{JK} \cong \overline{KL} \cong \overline{LM} \cong \overline{ML}$	2. Definition of a rhombus
3. Add auxiliary segment $\overline{JL}$.	3. Line Postulate
4. $\overline{JL} \cong \overline{JL}$	4. Reflexive Property
5. $\triangle JKL \cong \triangle LMJ$	5. SSS
6. $\angle 1 \cong \angle 4$	6. Definition of Congruent Triangles

Table 6.6: (continued)

Statement	Reason
7. $\overline{JM} \parallel \overline{KL}$	7. Converse of AIA Theorem
8. $\angle 2 \cong \angle 3$	8. Definition of Congruent Triangles
9. $\overline{JK} \parallel \overline{LM}$	9. Converse of AIA Theorem ♦

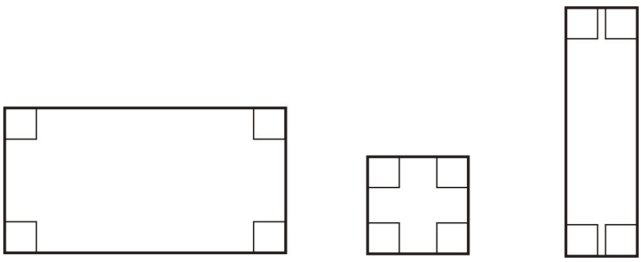
That may seem like a lot of work just to prove that a rhombus is a parallelogram. But, now that you know that a rhombus *is a type of parallelogram*, then you also know that the rhombus **inherits** all of the properties of a parallelogram. This means if you know something is true about parallelograms, it must also be true about a rhombus.

Rectangle

A **rectangle** is a quadrilateral with four congruent angles. Since you know that any quadrilateral will have interior angles that sum to 360° (using the expression $180(n-2)$), you can find the measure of each interior angle.

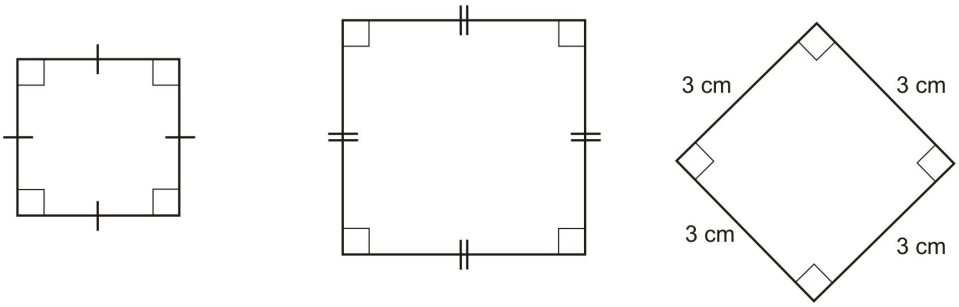
$$360 \div 4 = 90$$

Rectangles will have four right angles, or four angles that are each equal to 90°.



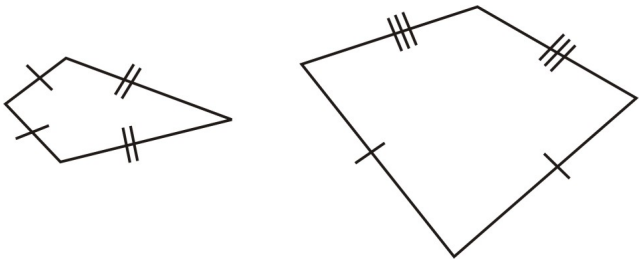
Square

A **square** is both a rhombus and a rectangle. A square has four congruent sides as well as four congruent angles. Each of the shapes shown below is a square.



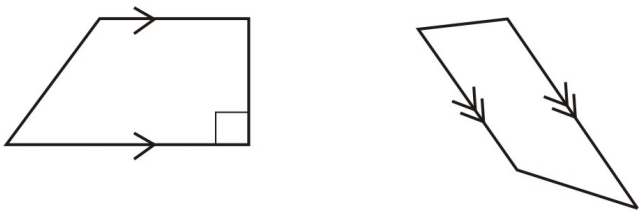
Kite

A **kite** is a different type of quadrilateral. It does not have parallel sides or right angles. Instead, a kite is defined as a quadrilateral that has two distinct pairs of adjacent congruent sides. Unlike parallelograms or other quadrilaterals, the congruent sides are adjacent (next to each other), not opposite.



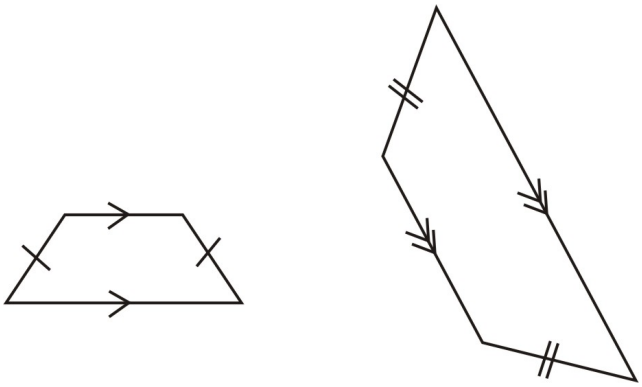
Trapezoid

A **trapezoid** is a quadrilateral that has exactly one pair of parallel sides. Unlike the parallelogram that has two pairs, the trapezoid only has one. It may or may not contain right angles, so the angles are not a distinguishing characteristic. Remember that parallelograms cannot be classified as trapezoids. A trapezoid is classified as having *exactly* one pair of parallel sides.



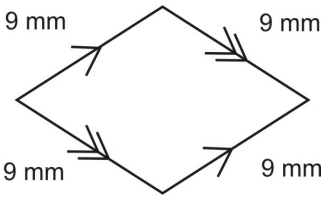
Isosceles Trapezoid

An **isosceles trapezoid** is a special type of trapezoid. Like an isosceles triangle, it has two sides that are congruent. As a trapezoid can only have one pair of parallel sides, the parallel sides cannot be congruent (because this would create *two* sets of parallel sides). Instead, the non-parallel sides of a trapezoid must be congruent.



Example 1

Which is the most specific classification for the figure shown below?



A. parallelogram

B. rhombus

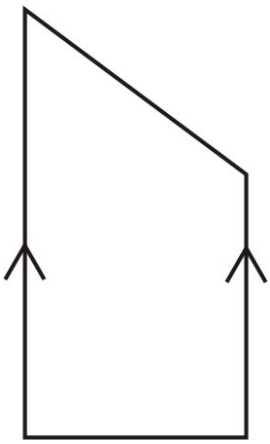
C. rectangle

D. square

The shape above has two sets of parallel sides, so it is a parallelogram. It also has four congruent sides, making it a rhombus. The angles are not right angles (and we can't assume we know the angle measures since they are unmarked), so it cannot be a rectangle or a square. While the shape is a parallelogram, the most specific classification is rhombus. The answer is choice B.

Example 2

Which is the most specific classification for the figure shown below? You may assume the diagram is drawn to scale.



A. parallelogram

B. kite

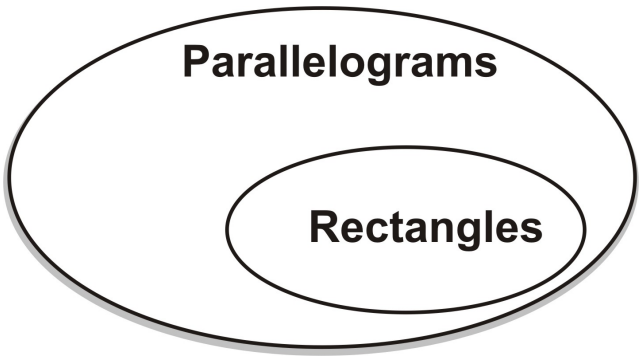
C. trapezoid

D. isosceles trapezoid

The shape above has exactly one pair of parallel sides, so you can rule out parallelogram and kite as possible classifications. The shape is definitely a trapezoid because of the one pair of parallel sides. For a shape to be an isosceles trapezoid, the other sides must be congruent. That is not the case in this diagram, so the most specific classification is trapezoid. The answer is choice C.

Using a Venn Diagram for Classification

You have just explored many different rules and classifications for quadrilaterals. There are different ways to collect and understand this information, but one of the best methods is to use a **Venn Diagram.** Venn Diagrams are a way to classify objects according to their properties. Think of a rectangle. A rectangle is a type of parallelogram (you can prove this using the Converse of the Interior Angles on the Same Side of the Transversal Theorem), but not all parallelograms are rectangles. Here's a simple Venn Diagram of that relationship:



Notice that *all rectangles are parallelograms,* but *not all parallelograms are rectangles.* If an item falls into more than one category, it is placed in the overlapping section between the appropriate classifications. If it does not meet any criteria for the category, it is placed outside of the circles.

To begin organizing the information for a Venn diagram, you can analyze the quadrilaterals we have discussed thus far by three characteristics: parallel sides, congruent sides, and congruent angles. Below is a table that shows how each quadrilateral fits these characteristics.

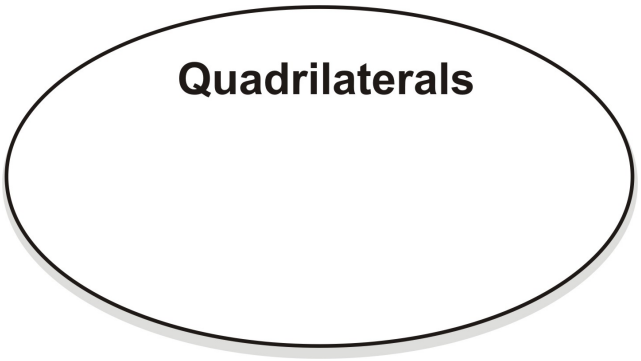
Table 6.7:

Shape	Number of pairs of parallel sides	Number of pairs of congruent sides	Four congruent angles
Parallelogram	2	2	No
Rhombus	2	2	No
Rectangle	2	2	Yes
Square	2	2	Yes
Kite	0	2	No
Trapezoid	1	0	No
Isosceles trapezoid	1	1	No

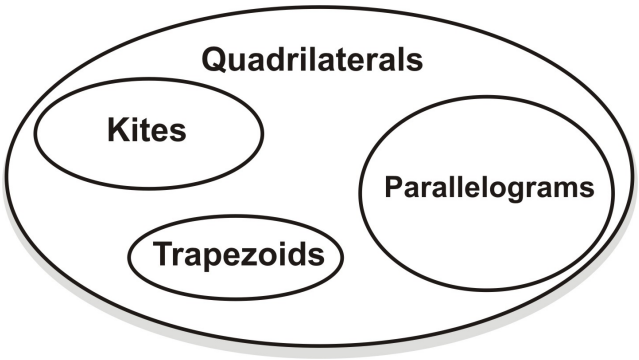
Example 3

Organize the classification information in the table above in a Venn Diagram.

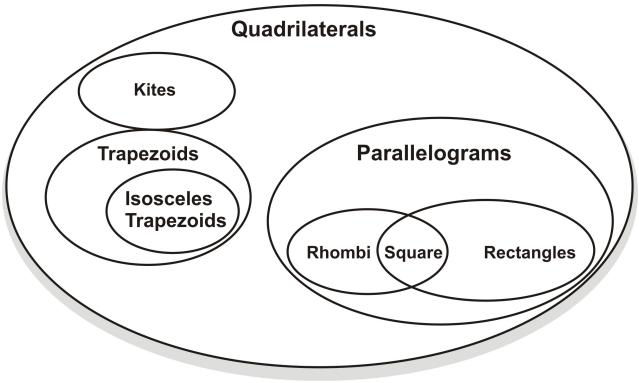
To begin a Venn Diagram, you must first draw a large ellipse representing the biggest category. In this case, that will be quadrilaterals.



Now, one class of quadrilaterals are parallelograms—all quadrilaterals with opposite sides parallel. But, not all quadrilaterals are parallelograms: kites have no pairs of parallel sides, and trapezoids only have one pair of parallel sides. In the diagram we can show this as follows:



Okay, we are almost there, but there are several types of parallelograms. Squares, rectangles, and rhombi are all types of parallelograms. Also, under the category of trapezoids we need to add isosceles trapezoids. The completed Venn diagram is like this:



You can use this Venn Diagram to quickly answer questions. For instance, is every square a rectangle? (Yes.) Is every rhombus a square? (No, but some are.)

Strategies for Shapes on a Coordinate Grid

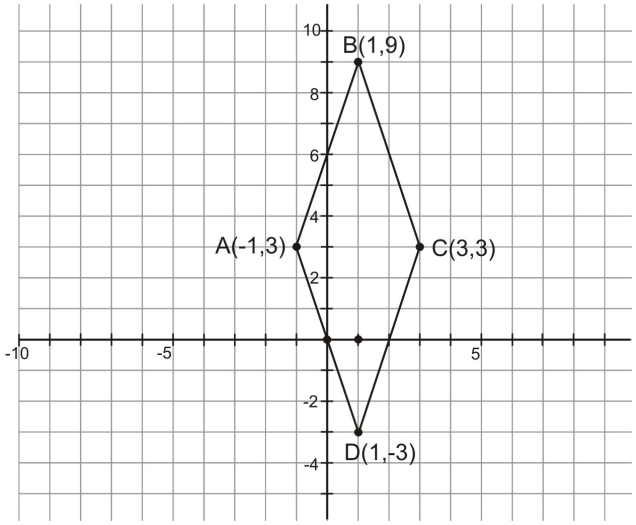
You have already practiced some of the tricks for analyzing shapes on a coordinate grid. You actually have all of the tools you need to classify any quadrilateral placed on a grid. To find out whether sides are congruent, you can use the distance formula.

Distance Formula: Distance between points (x_1, y_1) and $(x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

To find out whether lines are parallel, you can find the slope by computing slope $= \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$. If the slopes are the same, the lines are parallel. Similarly, if you want to find out if angles are right angles, you can test the slopes of their lines. Perpendicular lines will have slopes that are opposite reciprocals of each other.

Example 4

Classify the shape on the coordinate grid below.



First identify whether the sides are congruent. You can use the distance formula four times to find the distance between the vertices.

For segment $\overline{AB}$, find the distance between $(-1, 3)$ and $(1, 9)$.

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-1))^2 + (9 - 3)^2} \\ &= \sqrt{(1 + 1)^2 + (9 - 3)^2} \\ &= \sqrt{(2)^2 + (6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \end{aligned}$$

For segment $\overline{BC}$, find the distance between $(1, 9)$ and $(3, 3)$.

$$\begin{aligned} BC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 1)^2 + (3 - 9)^2} \\ &= \sqrt{(2)^2 + (-6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \end{aligned}$$

For segment $\overline{CD}$, find the distance between $(3, 3)$ and $(1, -3)$.

$$\begin{aligned} CD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-3)^2 + ((-3)-3)^2} \\ &= \sqrt{(-2)^2 + (-6)^2} \\ &= \sqrt{4+36} \\ &= \sqrt{40} \end{aligned}$$

For segment $\overline{AD}$, find the distance between $(-1, 3)$ and $(1, -3)$.

$$\begin{aligned} AD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-(-1))^2 + ((-3)-3)^2} \\ &= \sqrt{(2)^2 + (-6)^2} \\ &= \sqrt{4+36} \\ &= \sqrt{40} \end{aligned}$$

So, the length of each segment is equal to $\sqrt{40}$, and the sides are all equal. At this point, you know that the figure is either a rhombus or a square. To distinguish, you'll have to identify whether the angles are right angles. If one of the angles is a right angle, they all must be, so the shape will be a square. If it isn't a right angle, then none of them are, and it is a rhombus.

You can check whether two segments form a right angle by finding the slopes of two intersecting segments. If the slopes are opposite reciprocals, then the lines are perpendicular and form right angles.

The slope of segment $\overline{AB}$ can be calculated by finding the "rise over the run".

$$\begin{aligned} \text{slope } \overline{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{9-3}{1-(-1)} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

Now find the slope of an adjoining segment, like $\overline{BC}$.

$$\begin{aligned}\text{slope } \overline{BC} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 - 9}{3 - 1} \\ &= \frac{-6}{2} \\ &= -3\end{aligned}$$

The two slopes are -3 and 3. These are opposite numbers, but they are not reciprocals. Remember that the opposite reciprocal of -3 would be $\frac{1}{3}$, so segments $\overline{AB}$ and $\overline{BC}$ are *not* perpendicular. Since the sides of $ABCD$ do not intersect a right angle, you can rule out square. Therefore $ABCD$ is a rhombus.

Lesson Summary

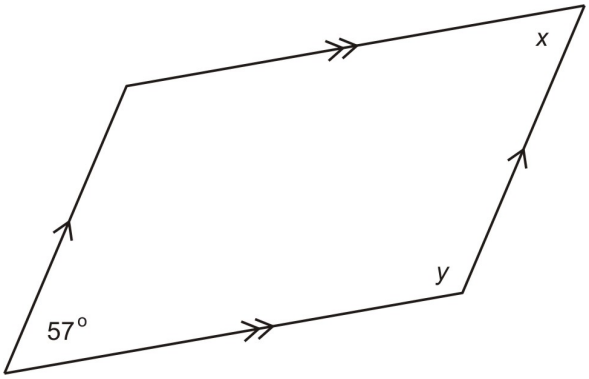
In this lesson, we explored quadrilateral classifications. Specifically, we have learned:

- How to identify and classify a parallelogram.
- How to identify and classify a rhombus.
- How to identify and classify a rectangle.
- How to identify and classify a square.
- How to identify and classify a kite.
- How to identify and classify a trapezoid.
- How to identify and classify an isosceles trapezoid.
- How to collect the classifications in a Venn diagram.
- How to identify and classify shapes using a coordinate grid.

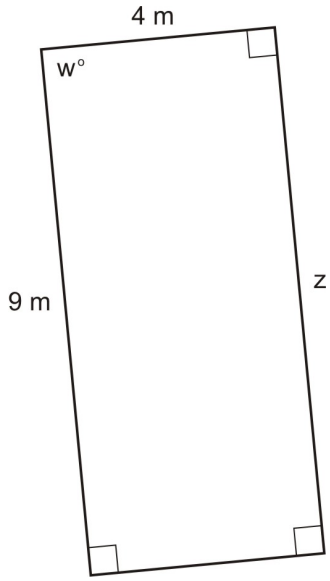
It is important to be able to classify different types of quadrilaterals in many different situations. The more you understand the differences and similarities between the shapes, the more success you'll have applying them to more complicated problems.

Review Questions

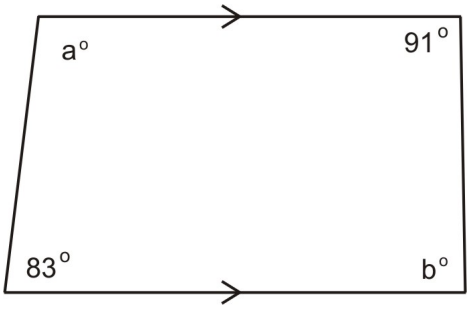
1. $x =$ ____, $y =$ ____



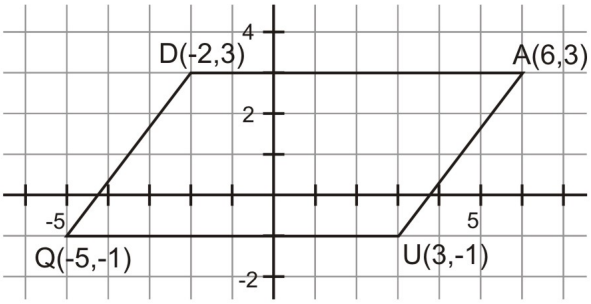
2. $w =$ ____, $z =$ ____



3. $a =$ ____, $b =$ ____



Use the diagram below for exercises 4-7:



4. Find the slope of $\overline{QU}$ and $\overline{DA}$, and find the slope of $\overline{QD}$ and $\overline{UA}$.
5. Based on 4, what can you conclude now about quadrilateral $QUAD$?
6. Find QD using the distance formula. What can you conclude about UA?
7. If $m\angle Q = 53°$, find $m\angle U$ and $m\angle A$.
8. Prove the Opposite Angles Theorem: The opposite angles of a parallelogram are congruent.
9. Draw a Venn diagram representing the relationship between Widgets, Wookies, and Wooblies (these are made-up terms) based on the following four statements:

   (a) All Wookies are Wooblies
   (b) All Widgets are Wooblies
   (c) All Wookies are Widgets
   (d) Some Widgets are not Wookies

10. Sketch a kite. Describe the symmetry of the kite and write a sentence about what you know based on the symmetry of a kite.

Review Answers

1. $x = 57°, y = 123°$
2. $w = 90°, z = 9$ m
3. $a = 97°, b = 89°$
4. The slope of $\overline{QU}$ and the slope of $\overline{DA}$ both $= 0$ since the lines are horizontal. For $\overline{QD}$, slope $\overline{QD} = \frac{3-(-1)}{-2-(-5)} = \frac{3+1}{-2+5} = \frac{4}{3}$. Finally for $\overline{UA}$, slope $\overline{UA} = \frac{3-(-1)}{6-3} = \frac{3+1}{3} = \frac{4}{3}$
5. Since the slopes of the opposite sides are equal, the opposite sides are parallel. Therefore, $QUAD$ is a parallelogram

6. Using the distance formula,

$$\begin{aligned} QD &= \sqrt{((-2)-(-5))^2+(3-(-1))^2} \\ &= \sqrt{(-2+5)^2+(3+1)^2} \\ &= \sqrt{(3)^2+(4)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Since $QUAD$ is a parallelogram, we know that $UA = QD = 5$

7. $m\angle U = 127^\circ$ and $m\angle A = 53^\circ$
8. First, we convert the theorem into "given" information and what we need to prove: Given: Parallelogram $ABCD$. Prove: $\angle A \cong \angle C$ and $\angle D \cong \angle B$

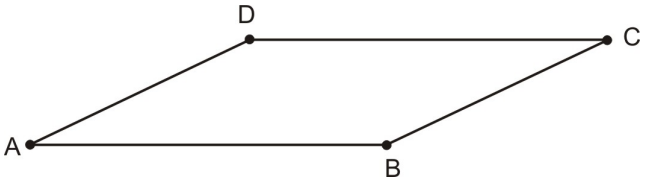


Table 6.8:

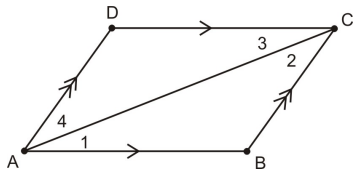
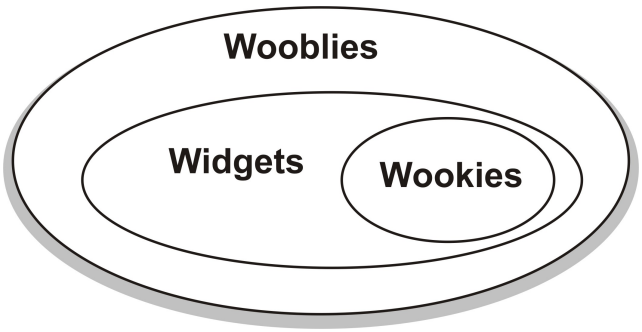
Statement	**Reason**
1. $ABCD$ is a parallelogram	1. Given
2. Draw auxiliary segment $\overline{AC}$ and label the angles as follows 	2. Line Postulate
3. $\overline{AB} \parallel \overline{DC}$	3. Definition of parallelogram
4. $\angle 1 \cong \angle 3$	4. Alternate Interior Angles Theorem
5. $\overline{AD} \parallel \overline{BC}$	5. Definition of parallelogram
6. $\angle 2 \cong \angle 4$	6. Alternate Interior Angles Theorem

Table 6.8: (continued)

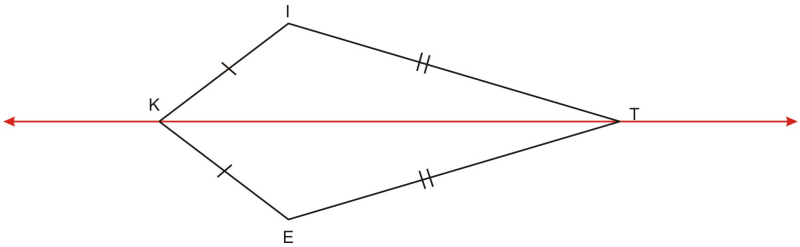
Statement	**Reason**
7. $\overline{AC} \cong \overline{AC}$	7. Reflexive property
8. $\triangle ADC \cong \triangle CBA$	8. ASA Triangle Congruence Postulate
9. $\angle D \cong \angle B$	9. Definition of congruent triangles (all corresponding sides and angles of congruent triangles are congruent)
10. $m\angle DAB = m\angle 1 + m\angle 4$	10. Angle addition postulate
11. $m\angle DCB = m\angle 2 + m\angle 3$	11. Angle addition postulate
12. $\angle DAB \cong \angle DCB$	12. Substitution

Now we have shown that opposite angles of a parallelogram are congruent

9.



10. See below. The red line is a line of reflection. Given this symmetry, we can conclude that $\angle I \cong \angle E$.



6.4 Using Parallelograms

Learning Objectives

- Describe the relationships between opposite sides in a parallelogram.
- Describe the relationship between opposite angles in a parallelogram.
- Describe the relationship between consecutive angles in a parallelogram.
- Describe the relationship between the two diagonals in a parallelogram.

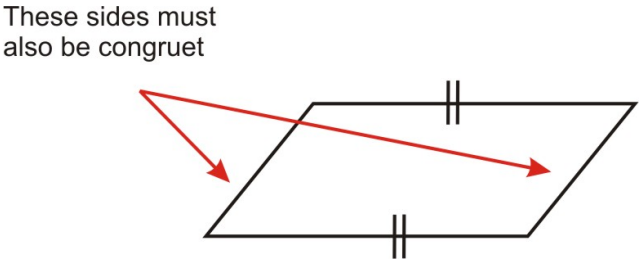
Introduction

Now that you have studied the different types of quadrilaterals and their defining characteristics, you can examine each one of them in greater depth. The first shape you'll look at more closely is the parallelogram. It is defined as a quadrilateral with two pairs of parallel sides, but there are many more characteristics that make a parallelogram unique.

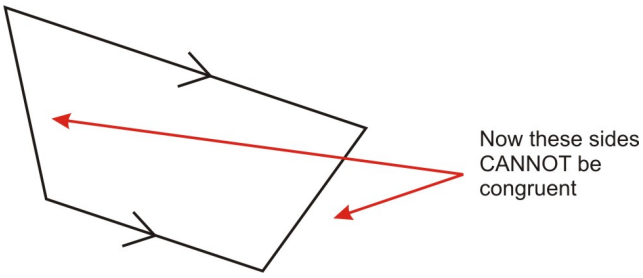
Opposite Sides in a Parallelogram

By now, you recognize that there are many types of parallelograms. They can look like squares, rectangles, or diamonds. Either way, opposite sides are always parallel. One of the most important things to know, however, is that opposite *sides* in a parallelogram are also congruent.

To test this theory, you can use pieces of string on your desk. Place two pieces of string that are the same length down so that they are parallel. You'll notice that the only way to connect the remaining vertices will be two parallel, congruent segments. There will be only one possible fit given two lengths.



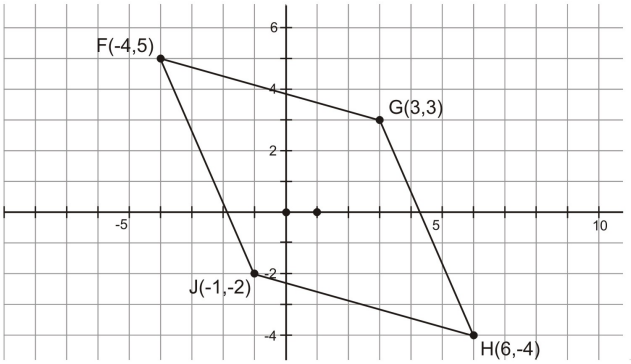
Try this again with two pieces of string that are different lengths. Again, lay them down so that they are parallel on your desk. What you should notice is that if the two segments are different lengths, the missing segments (if they connect the vertices) will not be parallel. Therefore, it will not create a parallelogram. In fact, there is no way to construct a parallelogram if opposite sides aren't congruent.



So, even though parallelograms are *defined* by their parallel opposite sides, one of their *properties* is that opposite sides be congruent.

Example 1

Parallelogram $FGHJ$ is shown on the following coordinate grid. Use the distance formula to show that opposite sides in the parallelogram are congruent.



You can use the distance formula to find the length of each segment. You are trying to prove that FG is the same as HJ, and that GH is the same as FJ. (Recall that FG means the same as $m\overline{FG}$, or the length of $\overline{FG}$.)

Start with FG. The coordinates of F are $(-4, 5)$ and the coordinates of G are $(3, 3)$.

$$\begin{aligned} FG &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-4))^2 + (3 - 5)^2} \\ &= \sqrt{(3 + 4)^2 + (3 - 5)^2} \\ &= \sqrt{(7)^2 + (-2)^2} \\ &= \sqrt{49 + 4} \\ &= \sqrt{53} \end{aligned}$$

So $FG = \sqrt{53}$.

Next find GH. The coordinates of G are $(3, 3)$ and the coordinates of H are $(6, -4)$.

$$\begin{aligned} GH &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6-3)^2 + (-4-3)^2} \\ &= \sqrt{(3)^2 + (-7)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58} \end{aligned}$$

So $GH = \sqrt{58}$.

Next find HJ. The coordinates of H are $(6, -4)$ and the coordinates of J are $(-1, -2)$.

$$\begin{aligned} HJ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-1-6)^2 + (-2-(-4))^2} \\ &= \sqrt{(-7)^2 + (2)^2} \\ &= \sqrt{49 + 4} \\ &= \sqrt{53} \end{aligned}$$

So $HJ = \sqrt{53}$.

Finally, find the length of FJ. The coordinates of F are $(-4, 5)$ and the coordinates of J are $(-1, -2)$.

$$\begin{aligned} FJ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-1-(-4))^2 + (-2-5)^2} \\ &= \sqrt{(3)^2 + (-7)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58} \end{aligned}$$

So $FJ = \sqrt{58}$.

Thus, in parallelogram $FGHJ$, $FG = HJ$ and $GH = FJ$. The opposite sides are congruent.

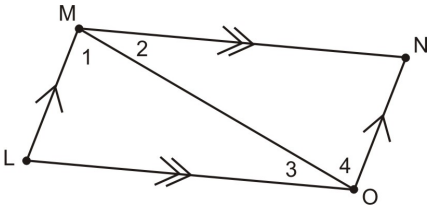
This example shows that in this parallelogram, the opposite sides are congruent. In the last section we **proved** this fact is true for all parallelograms using congruent triangles. Here we have shown an example of this property in the coordinate plane.

Opposite Angles in a Parallelogram

Not only are opposite sides in a parallelogram congruent. Opposite angles are also congruent. You can prove this by drawing in a diagonal and showing ASA congruence between the two triangles created. Remember that when you have congruent triangles, all corresponding parts will be congruent.

Example 2

Fill in the blanks in the two-column proof below.



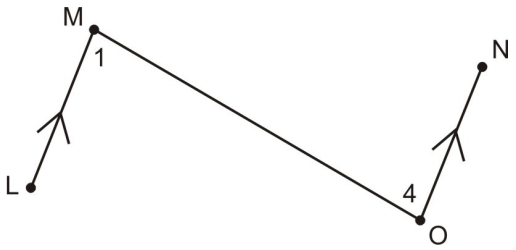
- Given: $LMNO$ is a parallelogram
- Prove: $\angle OLM \cong \angle MNO$

Table 6.9:

Statement	Reason
1. $LMNO$ is a parallelogram	1. Given
2. $\overline{LM} \| \overline{ON}$	2. Definition of a parallelogram
3. $\angle$ ____ $\cong \angle$ ____	3. Alternate Interior Angles Theorem
4. ____ $\|$ ____	4. Definition of a parallelogram
5. $\angle 2 \cong \angle 3$	5. _____
6. $\overline{MO} \cong \overline{MO}$	6. Reflexive Property
7. $\triangle$ ____ $\cong \triangle$ ____	7. ASA Triangle Congruence Postulate
8. $\angle OLM \cong \angle MNO$	8. Corresponding parts of congruent triangles are congruent

The missing statement in step 3 should be related to the information in step 2. $\overline{LM}$ and $\overline{ON}$

are parallel, and $\overline{MO}$ is a transversal. Look at the following figure (with the other segments removed) to see the angles formed by these segments:



Therefore the missing step is $\angle 1 \cong \angle 4$.

Work backwards to fill in step 4. Since step 5 is about $\angle 2 \cong \angle 3$, the sides we need parallel are $\overline{LO}$ and $\overline{MN}$. So step 4 is $\overline{LO} \| \overline{MN}$.

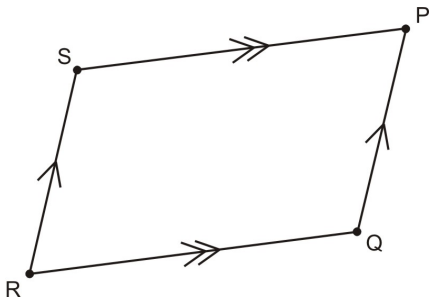
The missing reason on step 5 will be the same as the missing reason in step 3: alternate interior angles.

Finally, to fill in the triangle congruence statement, BE CAREFUL to make sure you match up corresponding angles. The correct form is $\triangle LMO \cong \triangle NOM$. (Students commonly get this reversed, so don't feel bad if you take a few times to get it correct!)

As you can imagine, the same process could be repeated with diagonal $\overline{LN}$ to show that $\angle LON \cong \angle LMN$. Opposite angles in a parallelogram are congruent. Or, even better, we can use the fact that $\angle 1 \cong \angle 4$ and $\angle 2 \cong \angle 3$ together with the Angle Addition Postulate to show $\angle LON \cong \angle LMN$. We leave the details of these operations for you to fill in.

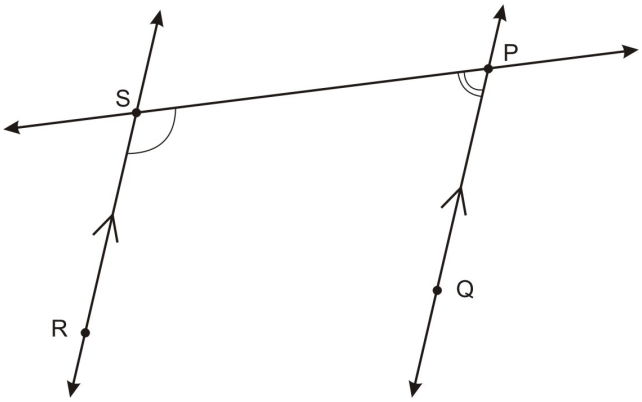
Consecutive Angles in a Parallelogram

So at this point, you understand the relationships between opposite sides and opposite angles in parallelograms. Think about the relationship between consecutive angles in a parallelogram. You have studied this scenario before, but you can apply what you have learned to parallelograms. Examine the parallelogram below.



Imagine that you are trying to find the relationship between $\angle SPQ$ and $\angle PSR$. To help you understand the relationship, extend all of the segments involved with these angles and

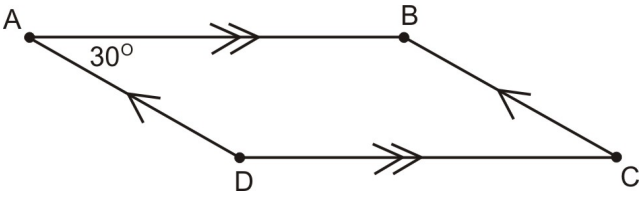
remove $\overline{RQ}$.



What you should notice is that PQ and SR are two parallel lines cut by transversal PS. So, you can find the relationships between the angles as you learned in Chapter 1. Earlier in this course, you learned that in this scenario, two consecutive interior angles are supplementary; they sum to 180°. The same is true within the parallelogram. Any two consecutive angles inside a parallelogram are supplementary.

Example 3

Fill in the remaining values for the angles in parallelogram ABCD below.



You already know that $m\angle DAB = 30°$ since it is given in the diagram. Since opposite angles are congruent, you can conclude that $m\angle BCD = 30°$.

Now that you know that consecutive angles are supplementary, you can find the measures of the remaining angles by subtracting 30° from 180°.

$$\begin{aligned} m\angle BAD + m\angle ADC &= 180° \\ 30° + m\angle ADC &= 180° \\ 30° + m\angle ADC - 30° &= 180° - 30° \\ m\angle ADC &= 150° \end{aligned}$$

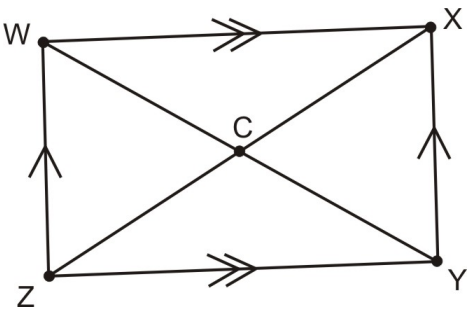
So, $m\angle ADC = 150°$. Since opposite angles are congruent, $\angle ABC$ will also measure 150°.

Diagonals in a Parallelogram

There is one more relationship to examine within parallelograms. When you draw the two diagonals inside parallelograms, they bisect each other. This can be very useful information for examining larger shapes that may include parallelograms. The easiest way to demonstrate this property is through congruent triangles, similarly to how we proved opposite angles congruent earlier in the lesson.

Example 4

Use a two-column proof for the theorem below.



- Given: $WXYZ$ is a parallelogram
- Prove: $\overline{WC} \cong \overline{CY}$ and $\overline{XC} \cong \overline{ZC}$

Table 6.10:

Statement	Reason
1. $WXYZ$ is a parallelogram	1. Given.
2. $\overline{WX} \cong \overline{YZ}$	2. Opposite sides in a parallelogram are congruent.
3. $\angle WCX \cong \angle ZCY$	3. Vertical angles are congruent.
4. $\angle XWC \cong \angle CYZ$	4. Alternate interior angles are congruent.
5. $\triangle WXC \cong \triangle YZC$	5. AAS congruence theorem: If two angles and one side in a triangle are congruent, the triangles are congruent.
6. $\overline{WC} \cong \overline{CY}$ and $\overline{XC} \cong \overline{ZC}$	6. Corresponding parts of congruent triangles are congruent. ♦

Lesson Summary

In this lesson, we explored parallelograms. Specifically, we have learned:

- How to describe and prove the distance relationships between opposite sides in a parallelogram.
- How to describe and prove the relationship between opposite angles in a parallelogram.
- How to describe and prove the relationship between consecutive angles in a parallelogram.
- How to describe and prove the relationship between the two diagonals in a parallelogram.

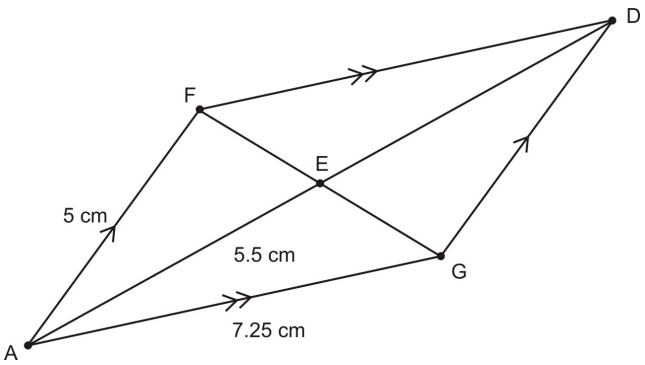
It is helpful to be able to understand the unique properties of parallelograms. You will be able to use this information in many different ways.

Points to Consider

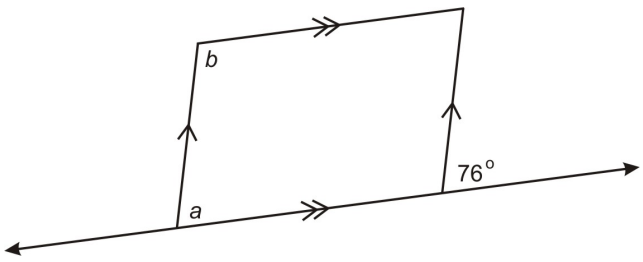
Now that you have learned the many relationships in parallelograms, it is time to learn how you can prove that shapes are parallelograms.

Review Questions

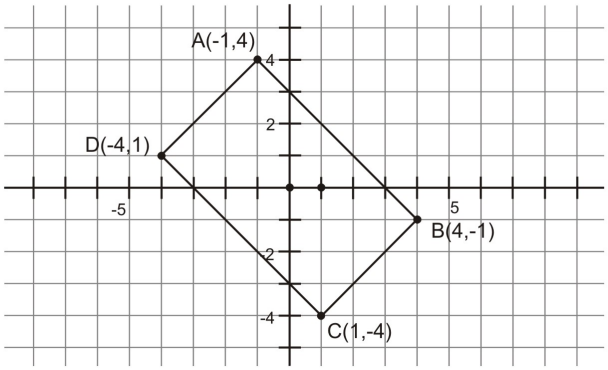
1. $DG =$ ____, $DF =$ ____, $AD =$ ____



2. $a =$ ____, $b =$ ____

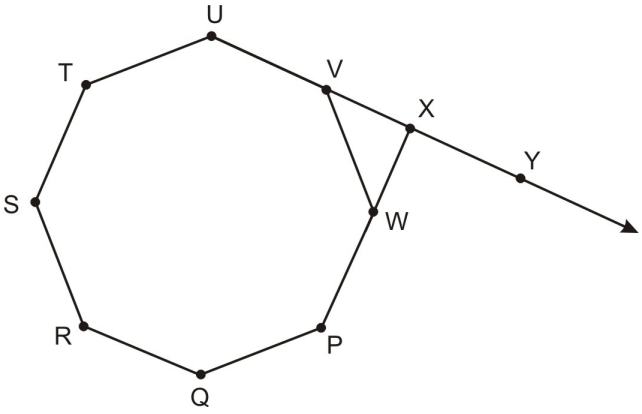


Use the following figure for exercises 3-6.



3. Find the slopes of $\overline{AD}$ and $\overline{CB}$.
4. Find the slopes of $\overline{DC}$ and $\overline{AB}$.
5. What kind of quadrilateral is $ABCD$? Give an answer that is as detailed as possible.
6. If you add diagonals to $ABCD$, where will they intersect?

Use the figure below for questions 7-11. Polygon $PQRSTUVW$ is a regular polygon. Find each indicated measurement.



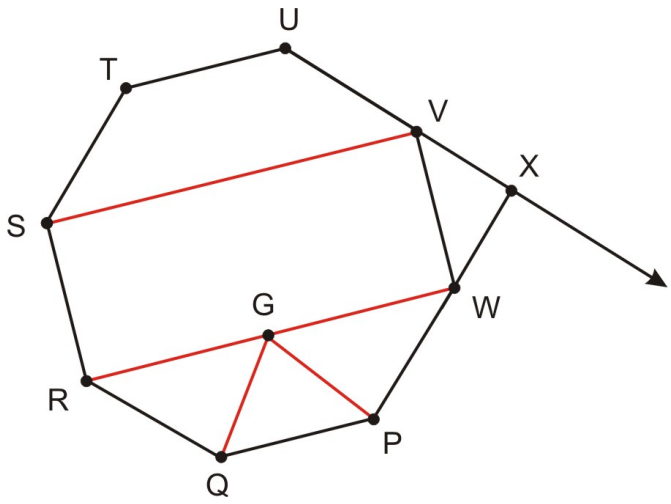
7. $m\angle RST =$
8. $m\angle VWX =$
9. $m\angle WXY =$
10. What kind of triangle is $\triangle WVX$?
11. Copy polygon $PQRSTUVW$ and add auxiliary lines to make each of the following:
    (a) a parallelogram

(b) a trapezoid
(c) an isosceles triangle

Review Answers

1. $DG = 5$ cm, $DF = 7.25$ cm , $AD = 11$ cm
2. $a = 76°$, $b = 104°$

3. Slopes of $\overline{AD}$ and $\overline{CB}$ both $= 1$
4. Both $= -1$
5. This figure is a **parallelogram** since opposite sides have equal slopes (i.e., opposite sides are parallel). Additionally, it is a **rectangle** because each angle is a 90° angle. We know this because the slopes of adjacent sides are opposite reciprocals
6. The diagonals would intersect at $(0, 0)$. One way to see this is to use the symmetry of the figure—each corner is a 90° rotation around the origin from adjacent corners
7. $m\angle RST = 135°$
8. $m\angle VWX = 45°$
9. $m\angle WXY = 90°$
10. $\triangle WVX$ is an isosceles right triangle
11. There are many possible answers. Here is one: Auxiliary lines are in red:



(a) $SRWV$ is a parallelogram (in fact it is a rectangle).
(b) $STUV$ is a trapezoid.
(c) QPG is an isosceles triangle

6.5 Proving Quadrilaterals are Parallelograms

Learning Objectives

- Prove a quadrilateral is a parallelogram given congruent opposite sides.
- Prove a quadrilateral is a parallelogram given congruent opposite angles.
- Prove a quadrilateral is a parallelogram given that the diagonals bisect each other.
- Prove a quadrilateral is a parallelogram if one pair of sides is both congruent and parallel.

Introduction

You'll remember from earlier in this course that you have studied converse statements. A converse statement reverses the order of the hypothesis and conclusion in an if-then statement, and is only *sometimes* true. For example, consider the statement: "If you study hard, then you will get good grades." Hopefully this is true! However, the converse is "If you get good grades, then you study hard." This may be true, but is it not *necessarily* true—maybe there are many other reasons why you get good grades—i.e., the class is really easy!

An example of a statement that is true and whose converse is also true is as follows: If I face east and then turn a quarter-turn to the right, I am facing south. Similarly, if I turn a quarter-turn to the right and I am facing south, then I was facing east to begin with.

Also all geometric definitions have true converses. For example, if a polygon is a quadrilateral then it has four sides and if a polygon has four sides then it is a quadrilateral.

Converse statements are important in geometry. It is crucial to know which theorems have true converses. In the case of parallelograms, almost all of the theorems you have studied this far have true converses. This lesson explores which characteristics of quadrilaterals ensure that they are parallelograms.

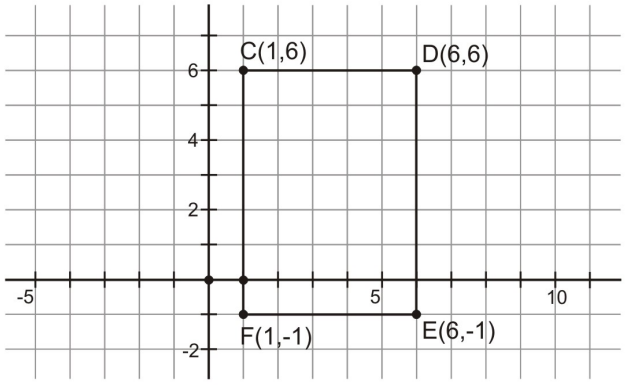
Proving a Quadrilateral is a Parallelogram Given Congruent Sides

In the last lesson, you learned that a parallelogram has congruent opposite sides. We proved this earlier and then looked at one example of this using the distance formula on a coordinate grid to verify that opposite sides of a parallelogram had identical lengths.

Here, we will show on the coordinate grid that the converse of this statement is also true: If a quadrilateral has two pairs of opposite sides that are congruent, then it is a parallelogram.

Example 1

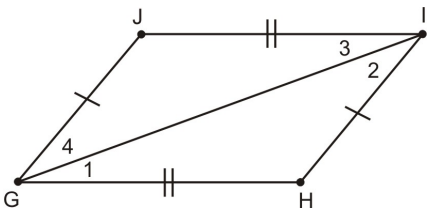
Show that the figure on the grid below is a parallelogram.



We can see that the lengths of opposite sides in this quadrilateral are congruent. For example, to find the length of $\overline{EF}$ we can find the difference in the $x-$coordinates $(6-1=5)$ because $\overline{EF}$ is horizontal (it's generally very easy to find the length of horizontal and vertical segments). $EF = CD = 5$ and $CF = DE = 7$. So, we have established that opposite sides of this quadrilateral are congruent.

But is it a parallelogram? Yes. One way to argue that $CDEF$ is a parallelogram is to note that $m\angle CFE = m\angle FED = 90°$. We can think of $\overline{FE}$ as a transversal that crosses $\overline{CF}$ and $\overline{DE}$. Now, interior angles on the same side of the transversal are supplementary, so we can apply the postulate *if interior angles on the same side of the transversal are supplementary then the lines crossed by the transversal are parallel.*

Note: This example does not **prove** that if opposite sides of a quadrilateral are congruent then the quadrilateral is a parallelogram. To do that you need to use any quadrilateral with congruent opposite sides, and then you use congruent triangles to help you. We will let you do that as an exercise, but here's the basic picture. What triangle congruence postulate can you use to show $\triangle GHI \cong \triangle IJG$?

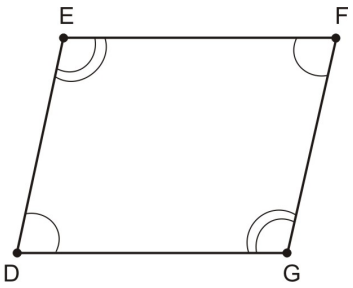


Proving a Quadrilateral is a Parallelogram Given Congruent Opposite Angles

Much like the converse statements you studied about opposite side lengths, if you can prove that opposite angles in a quadrilateral are congruent, the figure is a parallelogram.

Example 2

Complete the two-column proof below.



- Given: Quadrilateral $DEFG$ with $\angle D \cong \angle F$ and $\angle E \cong \angle G$
- Prove: $DEFG$ is a parallelogram

Table 6.11:

Statement	Reason
1. $DEFG$ is a quadrilateral with $\angle D \cong \angle F$ and $\angle E \cong \angle G$	1. Given
2. $m\angle D + m\angle E + \angle F + m\angle G = 360°$	2. Sum of the angles in a quadrilateral is 360°
3. $m\angle D + m\angle E + \angle D + m\angle E = 360°$	3. Substitution ($\angle D \cong \angle F$ and $\angle E \cong \angle G$)
4. $2(m\angle D) + 2(m\angle E) = 360°$	4. Combine like terms
5. $2(m\angle D + m\angle E) = 360°$	5. Factoring
6. $m\angle D + m\angle E = 180°$	6. Division property of equality (divided both sides by 2)
7. $\overline{DG} \| \overline{EF}$	7. If interior angles on the same side of a transversal are supplementary then the lines crossed by the transversal are parallel
8. $m\angle D + m\angle G = 180°$	8. Substitution on line 6 ($\angle E \cong \angle G$)
9. $\overline{DE} \| \overline{FG}$	9. Same reason as step 7

Table 6.11: (continued)

Statement	Reason
10. $DEFG$ is a parallelogram	10. Definition of a parallelogram ♦

Proving a Quadrilateral is a Parallelogram Given Bisecting Diagonals

In the last lesson, you learned that in a parallelogram, the diagonals bisect each other. This can also be turned around into a converse statement. If you have a quadrilateral in which the diagonals bisect each other, then the figure is a parallelogram. See if you can follow the proof below which shows how this is explained.

Example 3

Complete the two-column proof below.

- Given: $\overline{QV} \cong \overline{VS}$, and $\overline{TV} \cong \overline{VR}$
- Prove: $QRST$ is a parallelogram

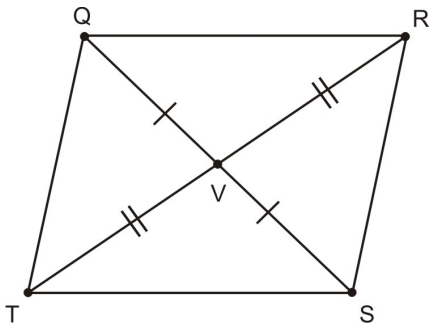


Table 6.12:

Statement	Reason
1. $\overline{QV} \cong \overline{VS}$	1. Given
2. $\overline{TV} \cong \overline{VR}$	2. Given
3. $\angle QVT \cong \angle RVS$	3. Vertical angles are congruent

Table 6.12: (continued)

Statement	Reason
4. $\triangle QVT \cong \triangle SVR$	4. $SAS \cong SAS$ If two sides and the angle between them are congruent, the two triangles are congruent
5. $\overline{QT} \cong \overline{RS}$	5. Corresponding parts of congruent triangles are congruent
6. $\angle TVS \cong \angle RVQ$	6. Vertical angles are congruent
7. $\triangle TVS \cong \triangle RVQ$	7. $SAS \cong SAS$ If two sides and the angle between them are congruent, then the two triangles are congruent
8. $\overline{TS} \cong \overline{RQ}$	8. Corresponding parts of congruent triangle are congruent
9. $QRST$ is a parallelogram	9. If two pairs of opposite sides of a quadrilateral are congruent, the figure is a parallelogram ♦

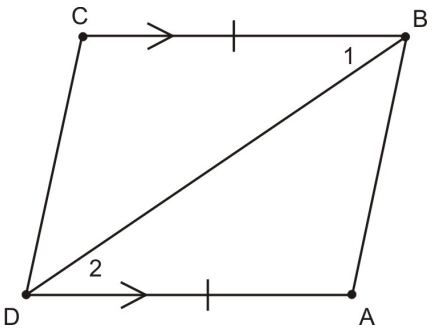
So, given only the information that the diagonals bisect each other, you can prove that the shape is a parallelogram.

Proving a Quadrilateral is a Parallelogram Given One Pair of Congruent and Parallel Sides

The last way you can prove a shape is a parallelogram involves only one pair of sides.

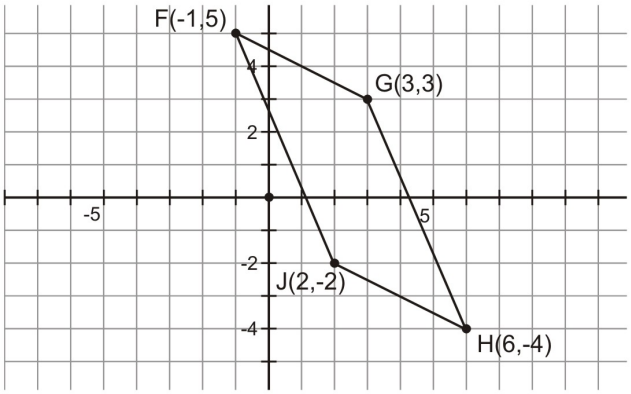
The proof is very similar to the previous proofs you have done in this section so we will leave it as an exercise for you to fill in. To set up the proof (which often IS the most difficult step), draw the following:

- Given: Quadrilateral $ABCD$ with $\overline{DA} \| \overline{CB}$ and $\overline{DA} \cong \overline{CB}$
- Prove: $ABCD$ is a parallelogram



Example 4

Examine the quadrilateral on the coordinate grid below. Can you show that it is a parallelogram?



To show that this shape is a parallelogram, you could find all of the lengths and compare opposite sides. However, you can also study one pair of sides. If they are both congruent and parallel, then the shape is a parallelogram.

Begin by showing two sides are congruent. You can use the distance formula to do this.

Find the length of $\overline{FG}$. Use $(-1, 5)$ for F and $(3, 3)$ for G.

$$\begin{aligned} FG &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-1))^2 + (3 - 5)^2} \\ &= \sqrt{(4)^2 + (-2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \end{aligned}$$

Next, find the length of the opposite side, $\overline{JH}$. Use $(2, -2)$ for J and $(6, -4)$ for H.

$$\begin{aligned} JH &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6-2)^2 + ((-4) - (-2))^2} \\ &= \sqrt{(4)^2 + (-2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \end{aligned}$$

So, $FG = JH = \sqrt{20}$; they have equal lengths. Now you need to show that $\overline{FG}$ and $\overline{JH}$ are parallel. You can do this by finding their slopes. Recall that if two lines have the same slope, they are parallel.

$$\begin{aligned} \text{Slope of } \overline{FG} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-5}{3-(-1)} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2} \end{aligned}$$

So, the slope of $\overline{FG} = -\frac{1}{2}$. Now, check the slope of $\overline{JH}$.

$$\begin{aligned} \text{Slope of } \overline{JH} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(-4) - (-2)}{6-2} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2} \end{aligned}$$

So, the slope of $\overline{JH} = -\frac{1}{2}$. Since the slopes of $\overline{FG}$ and $\overline{JH}$ are the same, the two segments are parallel. Now that have shown that the opposite segments are both parallel and congruent, you can identify that the shape is a parallelogram.

Lesson Summary

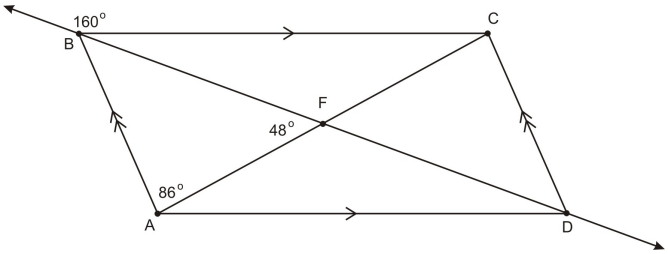
In this lesson, we explored parallelograms. Specifically, we have learned:

- How to prove a quadrilateral is a parallelogram given congruent opposite sides.
- How to prove a quadrilateral is a parallelogram given congruent opposite angles.
- How to prove a quadrilateral is a parallelogram given that the diagonals bisect each other.
- How to prove a quadrilateral is a parallelogram if one pair of sides is both congruent and parallel.

It is helpful to be able to prove that certain quadrilaterals are parallelograms. You will be able to use this information in many different ways.

Review Questions

Use the following diagram for exercises 1-3.



1. Find each angle:

   (a) $m\angle FBC =$
   (b) $m\angle FBA =$
   (c) $m\angle ADC =$
   (d) $m\angle BCD =$

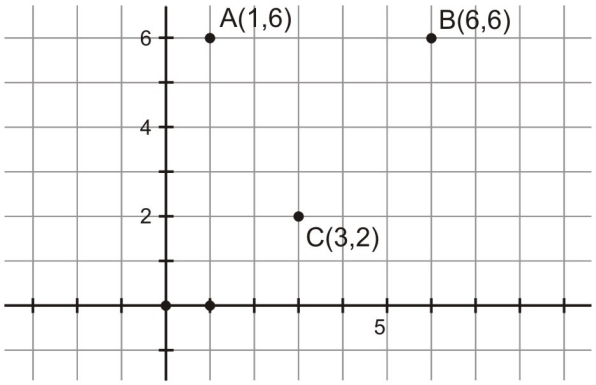
2. If $AB = 4.5$ m and $BC = 9.5$ m, find each length:

   (a) $AD =$
   (b) $DC =$

3. If $AC = 8.1$ m and $BF = 6$ m,, find each length:

   (a) $AF =$
   (b) $BD =$

Use the following figure for exercises 4-7.



4. Suppose that A $(1, 6)$, B $(6, 6)$, and C $(3, 2)$ are three of four vertices (corners) of a parallelogram. Give two possible locations for the fourth vertex, D, if you know that the $y-$coordinate of D is 2.
5. Depending on where you choose to put point D in 4, the name of the parallelogram you draw will change. Sketch a picture to show why.
6. If you know the parallelogram is named $ABDC$, what is the slope of the side parallel to AC?
7. Again, assuming the parallelogram is named $ABDC$, what is the length of $\overline{BD}$?
8. Prove: If opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: $ABCD$ with $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

Prove: $\overline{AB} \| \overline{DC}$ and $\overline{AD} \| \overline{BC}$ (i.e., $ABCD$ is a parallelogram).

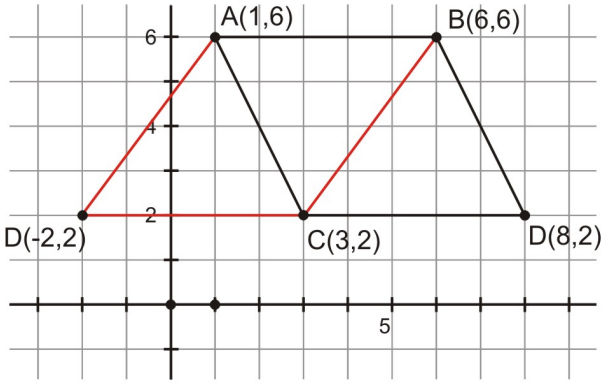
9. Prove: If a quadrilateral has one pair of congruent parallel sides, then it is a parallelogram.
10. Note in 9 that the parallel sides must also be the congruent sides for that theorem to work. Sketch a counterexample to show that if a quadrilateral has one pair of parallel sides and one pair of congruent sides (which are not the parallel sides) then the resulting figure it is not necessarily a parallelogram. What kind of quadrilaterals can you make with this arrangement?

Review Answers

1. (a) $m\angle FBC = 20°$
   (b) $m\angle FBA = 46°$
   (c) $m\angle ADC = 66°$
   (d) $m\angle BCD = 114°$ (note: you need to find almost all angle measures in the diagram to answer this question)

2. (a) $AD = 9.5$ m, $DC = 4.5$ m
3. (a) $AF = 4.35$ m,
   (b) $BD = 12$ m

4. D can be at either $(-2, 2)$ or $(8, 2)$
5. If D is at $(-2, 2)$ the parallelogram would be named $ABCD$ (in red in the following illustration). If D is at $(8, 2)$ then the parallelogram will take the name $ABDC$.



6. BD would have a slope of -2.
7. $BD = \sqrt{20}$
8. Given: $ABCD$ with $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$Prove: $\overline{AB} \| \overline{DC}$ and $\overline{AD} \| \overline{BC}$ (i.e., $ABCD$ is a parallelogram)

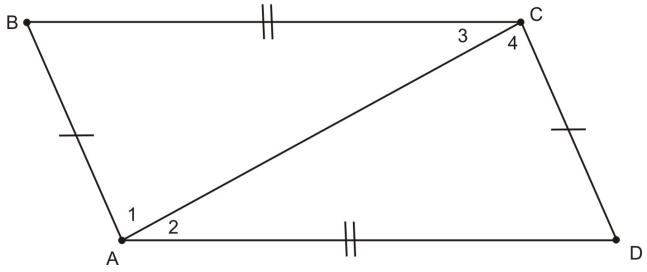


Table 6.13:

Statement	Reason
1. $\overline{AB} \cong \overline{DC}$	1. Given
2. $\overline{AD} \cong \overline{BC}$	2. Given
3. Add auxiliary line $\overline{AC}$	3. Line Postulate
4. $\overline{AC} \cong \overline{AC}$	4. Reflexive Property

Table 6.13: (continued)

Statement	Reason
5. $\triangle ACD \cong \triangle CAB$	5. SSS Congruence Postulate
6. $\angle 2 \cong \angle 3$	6. Definition of congruent triangles
7. $\overline{AD} \| \overline{BC}$	7. Converse of Alternate Interior Angles Postulate
8. $\angle 4 \cong \angle 1$	8. Definition of congruent triangles
9. $\overline{AB} \| \overline{DC}$	9. Converse of Alternate Interior Angles Postulate

9. First, translate the theorem into given and prove statements: Given: $ABCD$ with $\overline{AB} \| \overline{CD}$ and $\overline{AB} \cong \overline{CD}$
Prove: $\overline{BC} \| \overline{AD}$

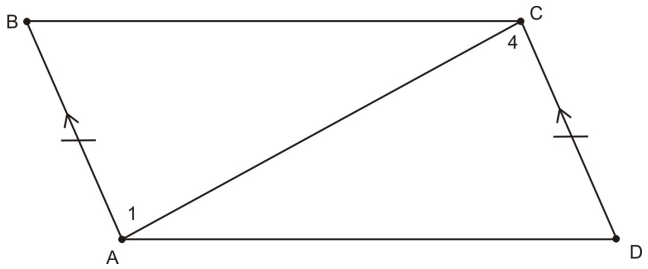


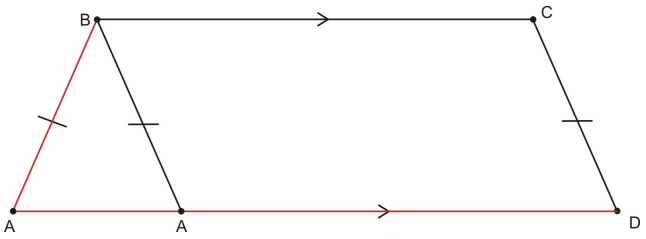
Table 6.14:

Statement	Reason
1. $\overline{AB} \cong \overline{CD}$	1. Given
2. $\overline{AB} \| \overline{CD}$	2. Given
3. $\angle 4 \cong \angle 1$	3. Alternate Interior Angles Theorem
4. Add auxiliary line $\overline{AC}$	4. Line Postulate
5. $\overline{AC} \cong \overline{AC}$	5. Reflexive Property
6. $\triangle ABC \cong \triangle CDA$	6. SAS Triangle Congruence Postulate

Table 6.14: (continued)

Statement	Reason
7. $\angle BCA \cong \angle DAC$	7. Definition of congruent triangles
8. $\overline{BC} \| \overline{AD}$	8. Converse of Alternate Interior Angles Theorem

10. If the congruent sides are not the parallel sides, then you can make either a parallelogram (in black) or an isosceles trapezoid (in red):



6.6 Rhombi, Rectangles, and Squares

Learning Objectives

- Identify the relationship between the diagonals in a rectangle.
- Identify the relationship between diagonals in a rhombus.
- Identify the relationship between diagonals and opposite angles in a rhombus.
- Identify and explain biconditional statements.

Introduction

Now that you have a much better understanding of parallelograms, you can begin to look more carefully into certain types of parallelograms. This lesson explores two very important types of parallelograms—rectangles and rhombi. Remember that all of the rules that apply to parallelograms still apply to rectangles and rhombi. In this lesson, you'll learn about rules specific to these shapes that are not true for all parallelograms.

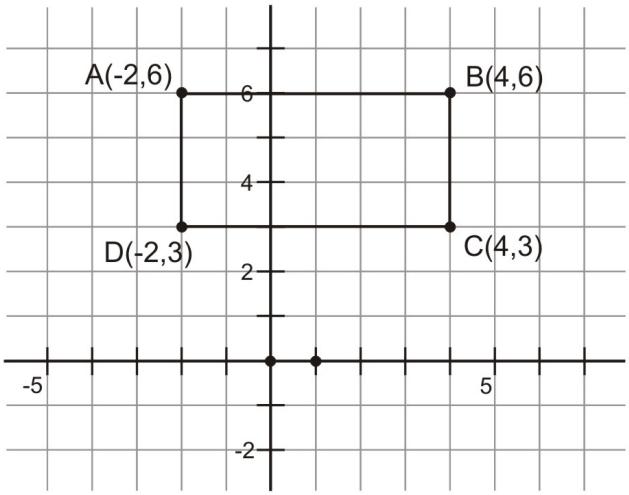
Diagonals in a Rectangle

Recall from previous lessons that the diagonals in a parallelogram bisect each other. You can prove this with congruence of triangles within the parallelogram. In a rectangle, there is

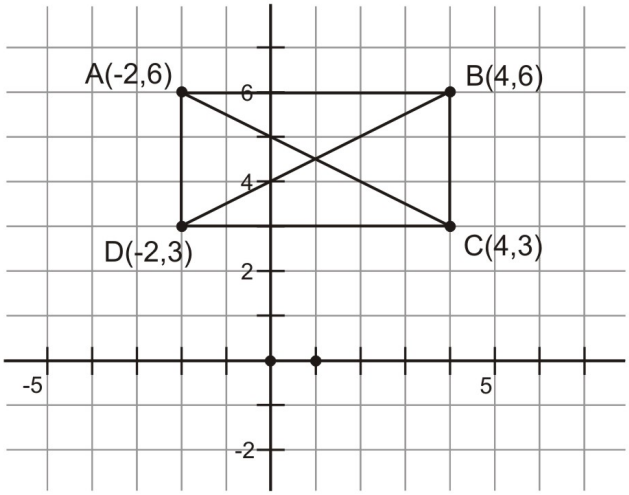
an even more special relationship between the diagonals. The two diagonals in a rectangle will always be congruent. We can show this using the distance formula on a coordinate grid.

Example 1

Use the distance formula to demonstrate that the two diagonals in the rectangle below are congruent.



To solve this problem, you need to find the lengths of both diagonals in the rectangle. First, draw line segments that connect the vertices of the rectangle. So, draw a segment from $(-2, 3)$ to $(4, 6)$ and from $(-2, 6)$ to $(4, 3)$.



You can use the distance formula to find the length of the diagonals. Diagonal $\overline{BD}$ goes from $B(-2, 3)$ to $D(4, 6)$.

$$\begin{aligned} BD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2 - 4)^2 + (3 - 6)^2} \\ &= \sqrt{(-6)^2 + (-3)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \end{aligned}$$

Next, find the length of diagonal $\overline{AC}$. That diagonal goes from $A(-2, 6)$ to $C(4, 3)$.

$$\begin{aligned} AC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-2))^2 + (3 - 6)^2} \\ &= \sqrt{(6)^2 + (-3)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \end{aligned}$$

So, $BD = AC = \sqrt{45}$. In this example, the diagonals are congruent. Are the diagonals of rectangles always congruent? The answer is yes.

Theorem: The diagonals of a rectangle are congruent

The proof of this theorem relies on the definition of a rectangle (a quadrilateral in which all angles are congruent) as well as the property that rectangles are parallelograms.

- Given: Rectangle $RECT$
- Prove: $\overline{RC} \cong \overline{ET}$

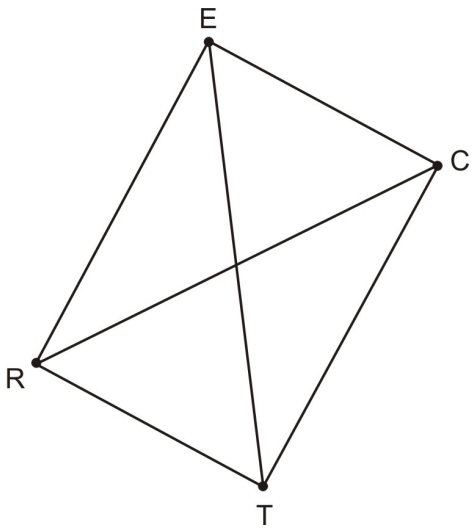


Table 6.15:

Statement	Reason
1. $RECT$ is a rectangle	1. Given
2. $\angle RTC \cong \angle TCE$	2. Definition of a rectangle
3. $\overline{RT} \cong \overline{EC}$	3. Opposite sides of a parallelogram are $\cong$
4. $\overline{TC} \cong \overline{TC}$	4. Reflexive Property of $\cong$
5. $\triangle RTC \cong \triangle ECT$	5. SAS Congruence Postulate
6. $\overline{RC} \cong \overline{ET}$	6. Definition of congruent triangles (corresponding parts of congruent triangles are congruent)

Perpendicular Diagonals in Rhombi

Remember that rhombi are quadrilaterals that have four congruent sides. They don't necessarily have right angles (like squares), but they are also parallelograms. Also, all squares are parallelograms.

The diagonals of a rhombus not only bisect each other (because they are parallelograms), they do so at a right angle. In other words, the diagonals are perpendicular. This can be very helpful when you need to measure angles inside rhombi or squares.

Theorem: The diagonals of a rhombus are perpendicular bisectors of each other

The proof of this theorem uses the fact that the diagonals of a parallelogram bisect each other and that if two angles are congruent and supplementary, then they are right angles.

- Given: Rhombus $RMBS$ with diagonals $\overline{RB}$ and $\overline{MS}$ intersecting at point A
- Prove: $\overline{RB} \perp \overline{MS}$

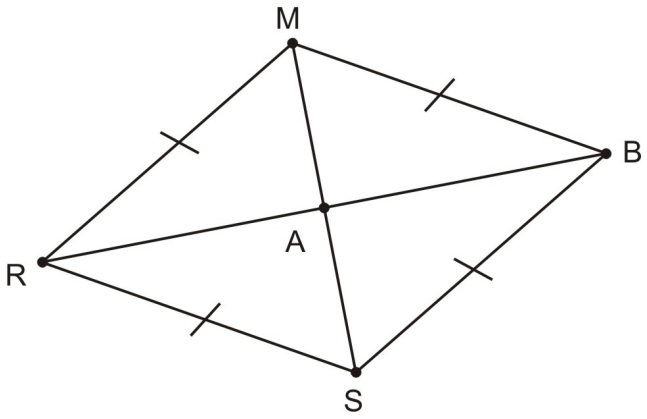


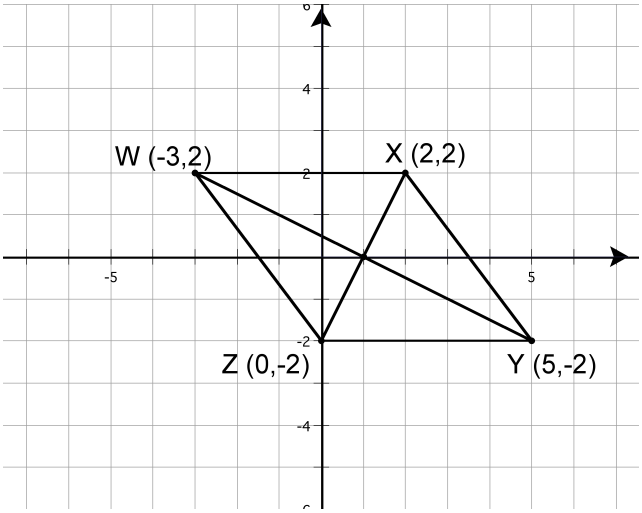
Table 6.16:

Statement	Reason
1. $RMBS$ is a rhombus	1. Given
2. $RMBS$ is a parallelogram	2. Theorem: All rhombi are parallelograms
3. $\overline{RM} \cong \overline{MB}$	3. Definition of a rhombus
4. $\overline{AM} \cong \overline{AM}$	4. Reflexive Property of $\cong$
5. $\overline{RA} \cong \overline{AB}$	5. Diagonals of a parallelogram bisect each other
6. $\triangle RAM \cong \triangle BAM$	6. SSS Triangle Congruence Postulate
7. $\angle RAM \cong \angle BAM$	7. Definition of congruent triangles (corresponding parts of congruent triangles are congruent)
8. $\angle RAM$ and $\angle BAM$ are supplementary	8. Linear Pair Postulate
9. $\angle RAM$ and $\angle BAM$ are right angles	9. Congruent supplementary angles are right angles
10. $\overline{RB} \perp \overline{MS}$	10. Definition of perpendicular lines

Remember that you can also show that lines or segments are perpendicular by comparing their slopes. Perpendicular lines have slopes that are opposite reciprocals of each other.

Example 2

Analyze the slope of the diagonals in the rhombus below. Use slope to demonstrate that they are perpendicular.



Notice that the diagonals in this diagram have already been drawn in for you. To find the slope, find the change in y over the change in x. This is also referred to as *rise over run.*

Begin by finding the slope of the diagonal $\overline{WY}$, which goes from $W(-3, 2)$ to$Y(5, -2)$.

$$\begin{aligned}\text{slope of } \overline{WY} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(-2) - 2}{5 - (-3)} \\ &= \frac{-4}{8} \\ &= -\frac{1}{2}\end{aligned}$$

Now find the slope of the diagonal $\overline{ZX}$ from $Z(0, -2)$ to $X(2, 2)$.

$$\begin{aligned}\text{slope of } \overline{ZX} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - (-2)}{2 - 0} \\ &= \frac{4}{2} \\ &= \frac{2}{1} \\ &= 2\end{aligned}$$

The slope of $\overline{WY} = -\frac{1}{2}$ and the slope of $\overline{ZX} = 2$. These two slopes are opposite reciprocals of each other, so the two segments are perpendicular.

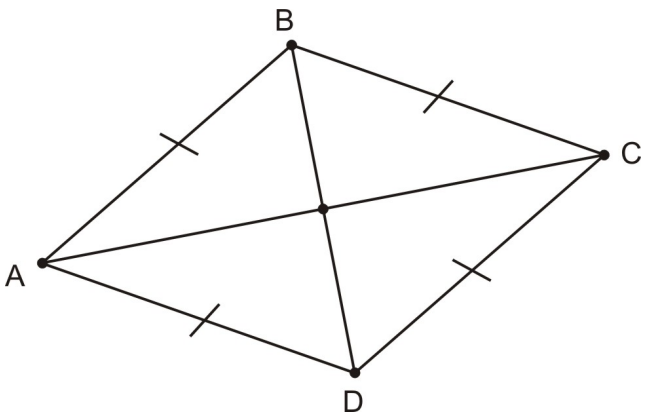
Diagonals as Angle Bisectors

Since a rhombus is a parallelogram, opposite angles are congruent. One property unique to rhombi is that in any rhombus, the diagonals will bisect the interior angles. Here we will prove this theorem using a different method than the proof we showed above.

Theorem: The diagonals of a rhombus bisect the interior angles

Example 3

Complete the two-column proof below.



- Given: $ABCD$ is a rhombus
- Prove: $\angle BDA \cong \angle BDC$

Table 6.17:

Statement	Reason
1. $ABCD$ is a rhombus	1. Given
2. $\overline{DC} \cong \overline{BC}$	2. All sides in a rhombus are congruent
3. $\triangle BCD$ is isosceles	3. Any triangle with two congruent sides is isosceles
4. $\angle BDC \cong \angle DBC$	4. The base angles in an isosceles triangle are congruent
5. $\angle BDA \cong \angle DBC$	5. Alternate interior angles are congruent
6. $\angle BDA \cong \angle BDC$	6. Transitive Property

Segment BD bisects $\angle ADC$. You could write a similar proof for every angle in the rhombus. Diagonals in rhombi bisect the interior angles.

Biconditional Statements

Recall that a **conditional statement** is a statement in the form "If ... then ... ." For example, *if a quadrilateral is a parallelogram, then opposite sides are congruent.*

You have learned a number of theorems as conditional statements. Many times you have also investigated the converses of these theorems. Sometimes the converse of a statement is true, and sometimes the converse are not. For example, you could say that *if you live in Los Angeles, you live in California.* However, the converse of this statement is not true. *If you live in California, you don't* ***necessarily*** *live in Los Angeles.*

A **biconditional statement** is a conditional statement that also has a true converse. For example, a true biconditional statement is, "If a quadrilateral is a square then it has exactly four congruent sides and four congruent angles." This statement is true, as is its converse: "If a quadrilateral has exactly four congruent sides and four congruent angles, then that quadrilateral is a square." When a conditional statement can be written as a biconditional, then we use the term "if and only if." In the previous example, we could say: "A quadrilateral is a square if and only if it has four congruent sides and four congruent angles."

Example 4

Which of the following is a true biconditional statement?

A. A polygon is a square if and only if it has four right angles.

B. A polygon is a rhombus if and only if its diagonals are perpendicular bisectors.

C. A polygon is a parallelogram if and only if its diagonals bisect the interior angles.

D. A polygon is a rectangle if and only if its diagonals bisect each other.

Examine each of the statements to see if it is true. Begin with choice A. It is true that if a polygon is a square, it has four right angles. However, the converse statement is not necessarily true. A rectangle also has four right angles, and a rectangle is not necessarily a square. Providing an example that shows something is not true is called a **counterexample.**

The second statement seems correct. It is true that rhombi have diagonals that are perpendicular bisectors. The same is also true in converse—if a figure has perpendicular bisectors as diagonals, it is a rhombus. Check the other statements to make sure that they are not biconditionally true.

The third statement isn't necessarily true. While rhombi have diagonals that bisect the interior angles, it is not true of all parallelograms. Choice C is not biconditionally true.

The fourth statement is also not necessarily true. The diagonals in a rectangle do bisect each other, but parallelograms that are not rectangles also have bisecting diagonals. Choice D is not correct.

So, after analyzing each statement carefully, only B is true. Choice B is the correct answer.

Lesson Summary

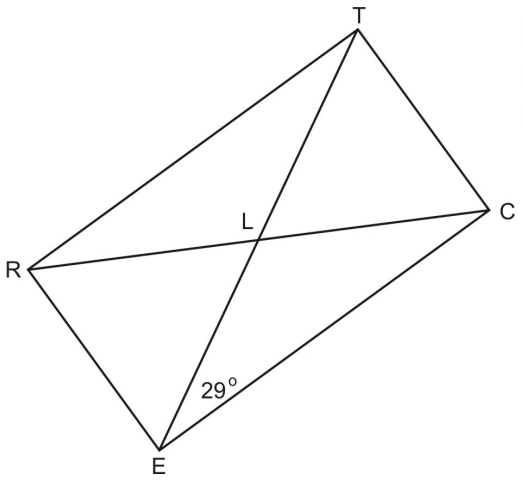
In this lesson, we explored rhombi, rectangles, and squares. Specifically, we have learned:

- How to identify and prove the relationship between the diagonals in a rectangle.
- How to identify and prove the relationship between diagonals in a rhombus.
- How to identify and prove the relationship between diagonals and opposite angles in a rhombus.
- How to identify and explain biconditional statements.

It is helpful to be able to identify specific properties in quadrilaterals. You will be able to use this information in many different ways.

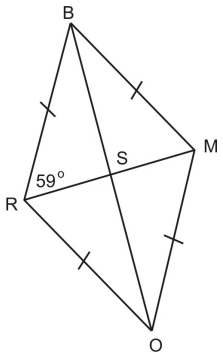
Review Questions

Use Rectangle RECT for exercises 1-3.



1. (a) $TC =$
   (b) $EC =$
2. (a) $ET =$
   (b) $RL =$
3. (a) $m\angle REC =$
   (b) $m\angle LTC =$

Use rhombus $ROMB$ for exercises 4-7.



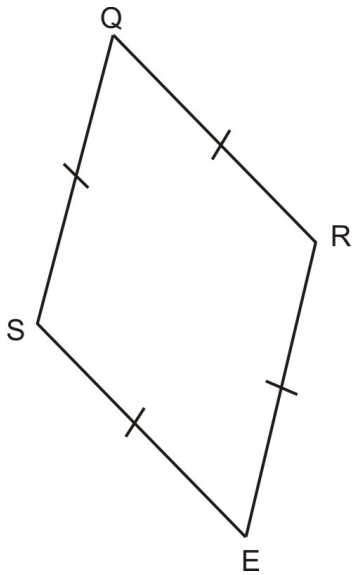
4. If $RO = 54$ in. and $RM = 52$ in., then
   (a) $RB =$ _____
   (b) $RS =$ _____
5. (a) $m\angle RMO =$
   (b) $m\angle RBM =$
6. What is the perimeter of $ROMB$?
7. $\overline{BO}$ is the _____ of $\overline{RM}$

For exercises 8 and 9, rewrite each given statement as a biconditional statement. Then state whether it is true. If the statement is false, draw a counterexample.

8. If a quadrilateral is a square, then it is a rhombus.
9. If a quadrilateral has for right angles, then it is a rectangle.
10. Give an example of an if-then statement whose converse is true. Then write that statement as a biconditional.

Review Answers

1. (a) $TC = 3.5$ cm
   (b) $EC = 6.4$ cm
2. (a) $ET = 7.2$ cm
   (b) $RL = 3.6$ cm
3. (a) $m\angle REC = 90^\circ$
   (b) $m\angle LTC = 62^\circ$
4. (a) $RB = 54$ in.
   (b) $RS = 26$ in.
5. (a) $m\angle RMO = 59^\circ, m\angle RBM = 62^\circ$
6. The perimeter is 216 in.
7. Perpendicular bisector
8. A quadrilateral is a square if and only if it is a rhombus. This is FALSE because some rhombi are not squares. Quadrilateral $SQRE$ below is a counterexample—it is a rhombus, but not a square



9. A quadrilateral has four right angles if and only if it is a rectangle. This is TRUE by the definition of rectangle.
10. *Answers will vary, but any geometric definition can be written as a biconditional.*

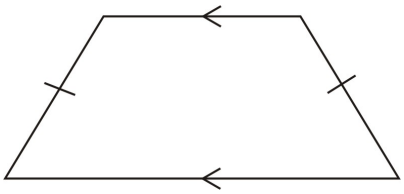
6.7 Trapezoids

Learning Objectives

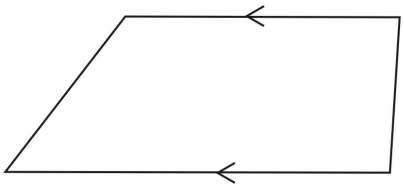
- Understand and prove that the base angles of isosceles trapezoids are congruent.
- Understand and prove that if base angles in a trapezoid are congruent, it is an isosceles trapezoid.
- Understand and prove that the diagonals in an isosceles trapezoid are congruent.
- Understand and prove that if the diagonals in a trapezoid are congruent, the trapezoid is isosceles.
- Identify the median of a trapezoid and use its properties.

Introduction

Trapezoids are particularly unique figures among quadrilaterals. They have exactly one pair of parallel sides so unlike rhombi, squares, and rectangles, they are **not** parallelograms. There are special relationships in trapezoids, particularly in isosceles trapezoids. Remember that isosceles trapezoids have non-parallel sides that are of the same lengths. They also have symmetry along a line that passes perpendicularly through both bases.



Isosceles Trapezoid



Non-isosceles Trapezoid

Base Angles in Isosceles Trapezoids

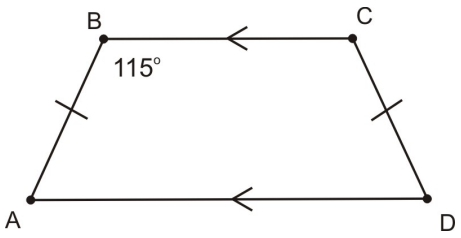
Previously, you learned about the Base Angles Theorem. The theorem states that in an isosceles *triangle*, the two base angles (opposite the congruent sides) are congruent. The same property holds true for isosceles *trapezoids*. The two angles along the same base in an isosceles triangle will also be congruent. Thus, this creates two pairs of congruent angles—one

pair along each base.

Theorem: The base angles of an isosceles trapezoid are congruent

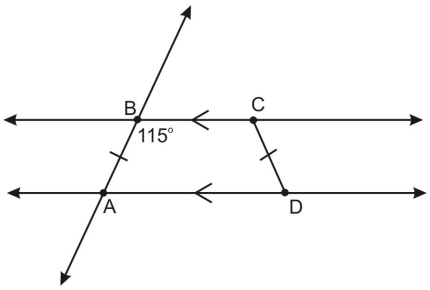
Example 1

Examine trapezoid ABCD below.



What is the measure of angle ADC?

This problem requires two steps to solve. You already know that base angles in an isosceles triangle will be congruent, but you need to find the relationship between adjacent angles as well. Imagine extending the parallel segments $\overline{BC}$ and $\overline{AD}$ on the trapezoid and the transversal $\overline{AB}$. You'll notice that the angle labeled 115° is a consecutive interior angle with $\angle BAD$.



Consecutive interior angles along two parallel lines will be supplementary. You can find $m\angle BAD$ by subtracting 115° from 180°.

$$m\angle BAD + 115^\circ = 180^\circ$$
$$m\angle BAD = 65^\circ$$

So, $\angle BAD$ measures 65°. Since $\angle BCD$ is adjacent to the same base as $\angle ADC$ in an isosceles trapezoid, the two angles must be congruent. Therefore, $m\angle ADC = 65^\circ$.

Here is a proof of this property.

- Given: Isosceles trapezoid $TRAP$ with $\overline{TR} \| \overline{PA}$ and $\overline{TP} \cong \overline{RA}$
- Prove: $\angle PTR \cong \angle ART$

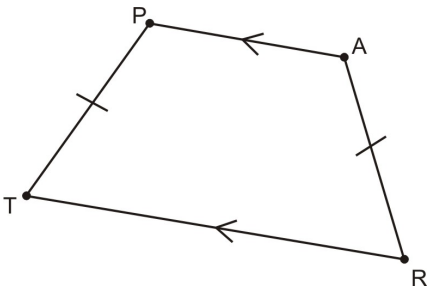


Table 6.18:

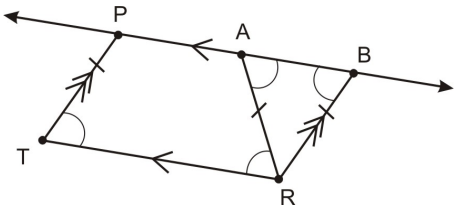
Statement	Reason
1. $TRAP$ is an isosceles trapezoid with $\overline{TP} \cong \overline{RA}$	1. Given
2. Extend $\overline{AP}$	2. Line Postulate
3. Construct $\overline{RB}$ as shown in the figure below such that $\overline{RB} \| \overline{TP}$  $TRAP$ with added auxiliary lines and markings	3. Parallel Postulate
4. $TRBR$ is a parallelogram	4. Definition of a parallelogram
5. $\angle PBR \cong \angle PTR$	5. Opposite angles in a parallelogram are $\cong$
6. $\overline{BR} \cong \overline{TP}$	6. Opposite sides of a parallelogram are congruent
7. $\triangle ABR$ is isosceles	7. Definition of isosceles triangle
8. $\angle RAB \cong \angle ABR$	8. Base angles in an isosceles triangle are $\cong$
9. $\angle ART \cong \angle RAB$	9. Alternate Interior Angles Theorem

Table 6.18: (continued)

Statement	Reason
10. $\angle ART \cong \angle ABR$	10. Transitive Property of $\cong$
11. $\angle PTR \cong \angle ART$	11. Transitive Property of $\cong$ ♦

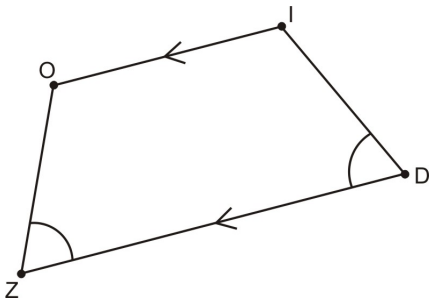
Identify Isosceles Trapezoids with Base Angles

In the last lesson, you learned about biconditional statements and converse statements. You just learned that *if a trapezoid is an isosceles trapezoid then base angles are congruent.* The converse of this statement is also true. If a trapezoid has two congruent angles along the same base, then it is an isosceles trapezoid. You can use this fact to identify lengths in different trapezoids.

First, we prove that this converse is true.

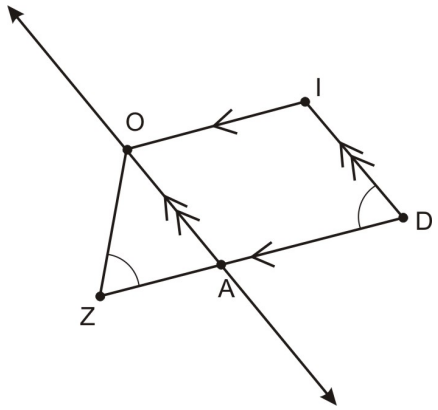
Theorem: If two angles along one base of a trapezoid are congruent, then the trapezoid is an isosceles trapezoid

- Given: Trapezoid $ZOID$ with $\overline{ZD} \| \overline{OI}$ and $\angle OZD \cong \angle ZDI$
- Prove: $\overline{ZO} \cong \overline{ID}$



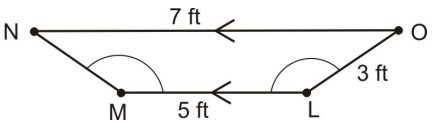
This proof is very similar to the previous proof, and it also relies on isosceles triangle properties.

Table 6.19:

Statement	Reason
1. Trapezoid $ZOID$ has $\overline{ZO} \parallel \overline{OI}$ and $\angle OZD \cong \angle ZDI$	1. Given
2. Construct $\overline{OA} \parallel \overline{ID}$	2. Parallel Postulate
3. $\angle ZAO \cong \angle ADI$	3. Corresponding Angles Postulate
4. $AOID$ is a parallelogram	4. Definition of a parallelogram
5. $\overline{AO} \cong \overline{ID}$	5. Opposite sides of a parallelogram are $\cong$
 Trapezoid $ZOID$ with auxiliary lines	
6. $\angle OZA \cong \angle OAZ$	6. Transitive Property
7. $\triangle OZA$ is isosceles	7. Definition of isosceles triangle
8. $\overline{OZ} \cong \overline{OA}$	8. Converse of the Base Angles Theorem
9. $\overline{OZ} \cong \overline{ID}$	9. Transitive Property ♦

Example 2

What is the length of MN in the trapezoid below?



Notice that in trapezoid $LMNO$, two base angles are marked as congruent. So, the trapezoid is isosceles. That means that the two non-parallel sides have the same length. Since you are looking for the length of $\overline{MN}$, it will be congruent to $\overline{LO}$. So, $MN = 3$ feet .

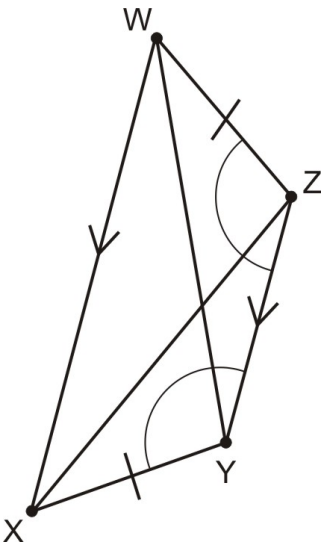
Diagonals in Isosceles Trapezoids

The angles in isosceles trapezoids are important to study. The diagonals, however, are also important. The diagonals in an isosceles trapezoid will not necessarily be perpendicular as in rhombi and squares. They are, however, congruent. Any time you find a trapezoid that is isosceles, the two diagonals will be congruent.

Theorem: The diagonals of an isosceles trapezoid are congruent

Example 3

Review the two-column proof below.



- Given: $WXYZ$ is a trapezoid and $\overline{WZ} \cong \overline{XY}$
- Prove: $\overline{WY} \cong \overline{XZ}$

Table 6.20:

Statement	Reason
1. $\overline{WZ} \cong \overline{XY}$	1. Given
2. $\angle WZY \cong \angle XYZ$	2. Base angles in an isosceles trapezoid are congruent
3. $\overline{ZY} \cong \overline{ZY}$	3. Reflexive Property.
4. $\triangle WZY \cong \triangle XYZ$	4. $SAS \cong SAS$
5. $\overline{WY} \cong \overline{XZ}$	5. Corresponding parts of congruent triangles are congruent

So, the two diagonals in the isosceles trapezoid are congruent. This will be true in any isosceles trapezoids.

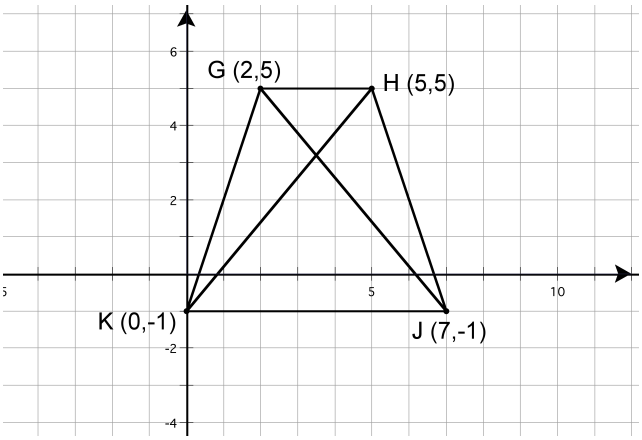
Identifying Isosceles Trapezoids with Diagonals

The converse statement of the theorem stating that diagonals in an isosceles triangle are congruent is also true. If a trapezoid has congruent diagonals, it is an isosceles trapezoid. You can either use measurements shown on a diagram or use the distance formula to find the lengths. If you can prove that the diagonals are congruent, then you can identify the trapezoid as isosceles.

Theorem: If a trapezoid has congruent diagonals, then it is an isosceles trapezoid

Example 4

Is the trapezoid on the following grid isosceles?



It is true that you could find the lengths of the two sides to identify whether or not this trapezoid is isosceles. However, for the sake of this lesson, compare the lengths of the diagonals.

Begin by finding the length of $\overline{GJ}$. The coordinates of G are $(2, 5)$ and the coordinates of J are $(7, -1)$.

$$\begin{aligned} GJ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(7-2)^2 + (-1-5)^2} \\ &= \sqrt{(5)^2 + (-6)^2} \\ &= \sqrt{25 + 36} \\ &= \sqrt{61} \end{aligned}$$

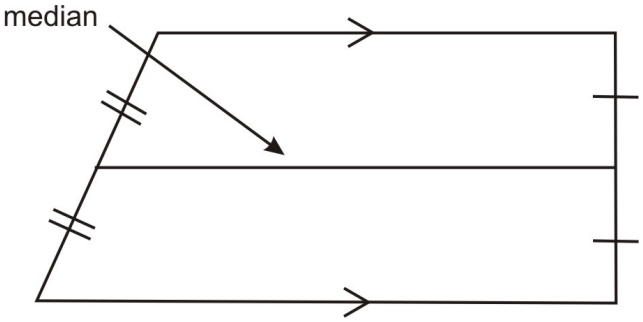
Now find the length of $\overline{HK}$. The coordinates of H are $(5, 5)$ and the coordinates of K are $(0, -1)$.

$$\begin{aligned} HK &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0-5)^2 + ((-1)-5)^2} \\ &= \sqrt{(-5)^2 + (-6)^2} \\ &= \sqrt{25 + 36} \\ &= \sqrt{61} \end{aligned}$$

Thus, we have shown that the diagonals are congruent. $GJ = HK = \sqrt{61}$. Therefore, trapezoid $GHJK$ is isosceles.

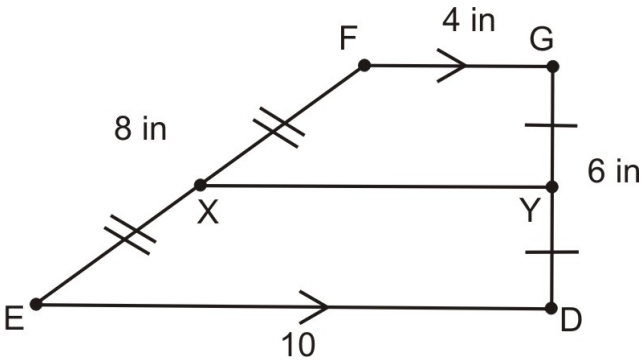
Trapezoid Medians

Trapezoids can also have segments drawn in called **medians.** The median of a trapezoid is a segment that connects the midpoints of the non-parallel sides in a trapezoid. The median is located half way between the bases of a trapezoid.



Example 5

In trapezoid DEFG below, segment XY is a median. What is the length of $\overline{EX}$?



The median of a trapezoid is a segment that is equidistant between both bases. So, the length of $\overline{EX}$ will be equal to half the length of $\overline{EF}$. Since you know that $EF = 8$ inches, you can divide that value by 2. Therefore, XE is 4 inches .

Theorem: The length of the median of a trapezoid is equal to half of the sum of the lengths of the bases

This theorem can be illustrated in the example above,

$$XY = \frac{FG + ED}{2}$$
$$XY = \frac{4 + 10}{2}$$
$$XY = 7$$

Therefore, the measure of segment XY is 7 inches . We leave the proof of this theorem as an exercise, but it is similar to the proof that the length of the triangle midsegment is half the length of the base of the triangle.

Lesson Summary

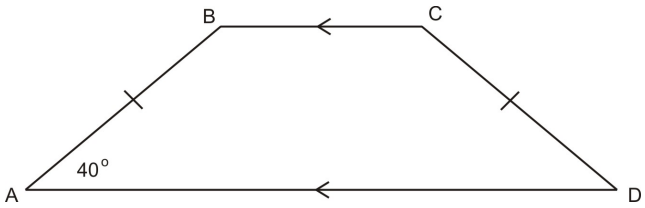
In this lesson, we explored trapezoids. Specifically, we have learned to:

- Understand and prove that the base angles of isosceles trapezoids are congruent.
- Understand that if base angles in a trapezoid are congruent, it is an isosceles trapezoid.
- Understand that the diagonals in an isosceles trapezoid are congruent.
- Understand that if the diagonals in a trapezoid are congruent, the trapezoid is isosceles.
- Identify the properties of the median of a trapezoid.

It is helpful to be able to identify specific properties in trapezoids. You will be able to use this information in many different ways.

Review Questions

Use the following figure for exercises 1-2.

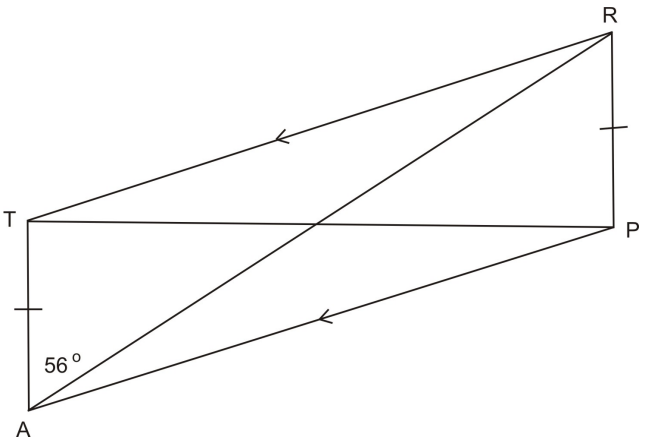


1. $m\angle ADC =$
2. $m\angle BCD =$

Use the following figure for exercises 3-5.

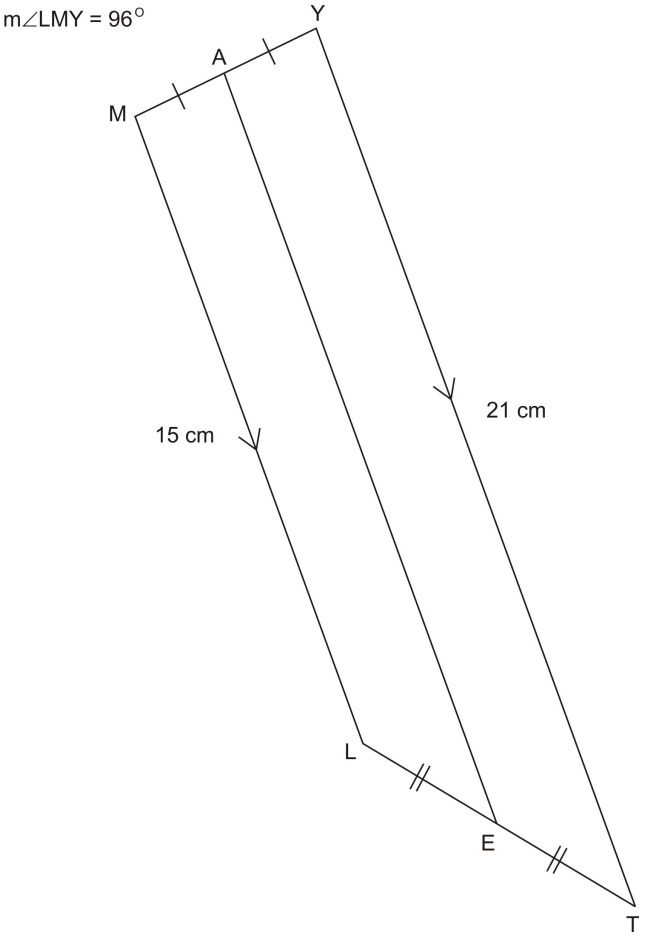
$m\angle APR = 73°$

$TP = 11.5$ cm



3. $m\angle RAP =$
4. $AR =$ _____
5. $m\angle ATR =$

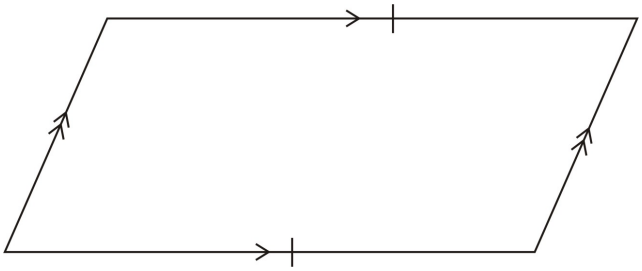
Use the following diagram for exercises 6-7.



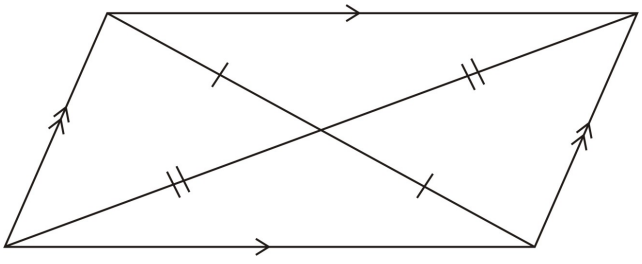
6. $m\angle MAE =$
7. $EA =$
8. Can the parallel sides of a trapezoid be congruent? Why or why not? Use a sketch to illustrate your answer.
9. Can the diagonals of a trapezoid bisect each other? Why or why not? Use a sketch to illustrate your answer.
10. Prove that the length of the median of a trapezoid is equal to half of the sum of the lengths of the bases.

Review Answers

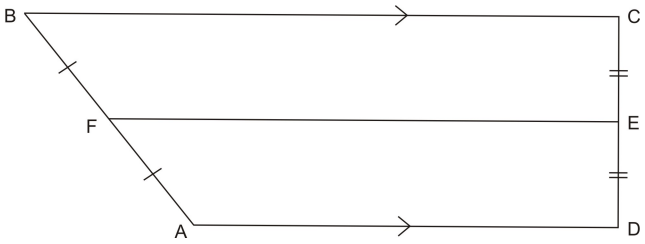
1. 40°
2. 140°
3. 17°
4. 11.5 cm
5. 107°
6. 84°
7. 18 cm
8. No, if the parallel (and by definition opposite) sides of a quadrilateral are congruent then the quadrilateral MUST be a parallelogram. When you sketch it, the two other sides must also be parallel and congruent to each other (proven in a previous section).



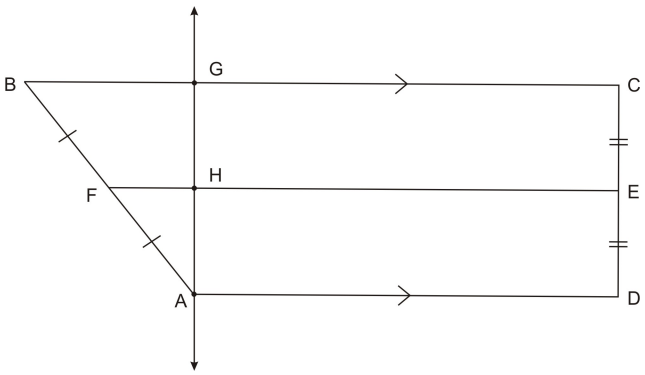
9. No, if the diagonals of a trapezoid bisect each other, then you have a parallelogram. We also proved this in a previous section.



10. We will use a paragraph proof. Start with trapezoid $ABCD$ and midsegment $\overline{FE}$.



Now, using the parallel postulate, construct a line through point A that is parallel to $\overline{CD}$. Label the new intersections as follows:



Now quadrilateral $AGCD$ is a parallelogram by construction. Thus, the theorem about opposite sides of a parallelogram tells us $AD = GC = HE$. The triangle midsegment theorem tells us that
$FH = \frac{1}{2}BG$ or $BG = 2FH$
So,

$$\begin{aligned}
\frac{BC + AD}{2} &= \frac{BG + GC + AD}{2} && \text{by the segment addition postulate} \\
&= \frac{2FH + 2HE}{2} && \text{by substitution} \\
&= FH + HE && \text{by factoring out and canceling the 2} \\
&= FE && \text{by the segment addition postulate. Which is exactly what we wan}
\end{aligned}$$

6.8 Kites

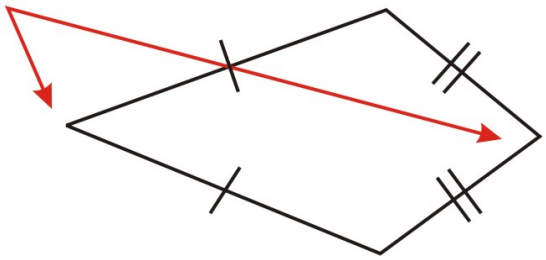
Learning Objectives

- Identify the relationship between diagonals in kites.
- Identify the relationship between opposite angles in kites.

Introduction

Among all of the quadrilaterals you have studied thus far, kites are probably the most unusual. Kites have no parallel sides, but they do have congruent sides. Kites are defined by two pairs of congruent sides that are adjacent to each other, instead of opposite each other.

Vertex angles



A **vertex angle** is between two congruent sides and a **non-vertex angle** is between sides of different lengths.

Kites have a few special properties that can be proven and analyzed just as the other quadrilaterals you have studied. This lesson explores those properties.

Diagonals in Kites

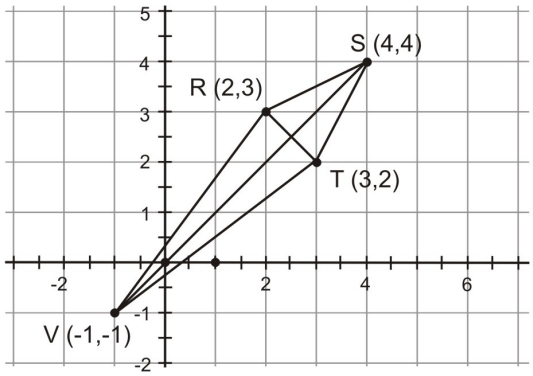
The relationship of diagonals in kites is important to understand. The diagonals are not congruent, but they are always perpendicular. In other words, the diagonals of a kite will always intersect at right angles.

Theorem: The diagonals of a kite are perpendicular

This can be examined on a coordinate grid by finding the slope of the diagonals. Perpendicular lines and segments will have slopes that are opposite reciprocals of each other.

Example 1

Examine the kite RSTV on the following coordinate grid. Show that the diagonals are perpendicular.



To find out whether the diagonals in this diagram are perpendicular, find the slope of each segment and compare them. The slopes should be opposite reciprocals of each other.

Begin by finding the slope of $\overline{RT}$. Remember that the slope is the change in the $y-$coordinate over the change in the $x-$coordinate.

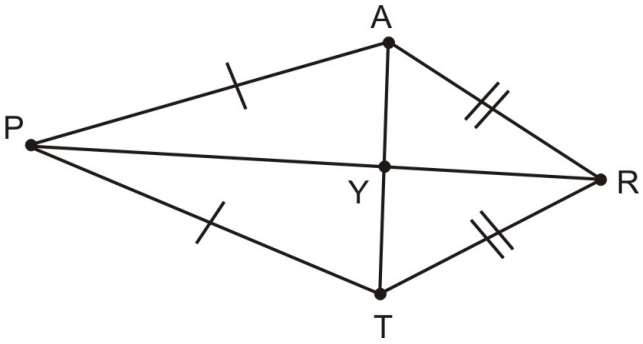
$$\begin{aligned}\text{slope of } \overline{RT} &= \frac{(y_2 - y_1)}{(x_2 - x_1)} \\ &= \frac{(2-3)}{(3-2)} \\ &= \frac{-1}{1} \\ &= -1\end{aligned}$$

The slope of $\overline{RT}$ is -1. You can also find the slope of $\overline{VS}$ using the same method.

$$\begin{aligned}\text{slope of } \overline{VS} &= \frac{(y_2 - y_1)}{(x_2 - x_1)} \\ &= \frac{(4-(-1))}{(4-(-1))} \\ &= \frac{5}{5} \\ &= 1\end{aligned}$$

The slope of $\overline{VS}$ is 1. If you think of both of these numbers as fractions, $-\frac{1}{1}$ and $\frac{1}{1}$, you can tell that they are opposite reciprocals of each other. Therefore, the two line segments are perpendicular.

Proving this property in general requires using congruent triangles (surprise!). We will do this proof in two parts. First, we will prove that one diagonal (connecting the vertex angles) bisects the vertex angles in the kite.



Part 1:

- Given: Kite $PART$ with $\overline{PA} \cong \overline{PT}$ and $\overline{AR} \cong \overline{RT}$
- Prove: $\overline{PR}$ bisects $\angle APT$ and $\angle ART$

Table 6.21:

Statement	Reason
1. $\overline{PA} \cong \overline{PT}$ and $\overline{AR} \cong \overline{RT}$	1. Given
2. $\overline{PR} \cong \overline{PR}$	2. Reflexive Property
3. $\triangle PAR \cong \triangle PTR$	3. SSS Congruence Postulate
4. $\angle APR \cong \angle TPR$	4. Corresponding parts of congruent triangles are congruent
5. $\angle ARP \cong \angle TRP$	5. Corresponding parts of congruent triangles are congruent
6. $\overline{PR}$ bisects $\angle APT$ and $\angle ART$	6. Definition of angle bisector ♦

Now we will prove that the diagonals are perpendicular.

Part 2:

- Given: Kite $PART$ with $\overline{PA} \cong \overline{PT}$ and $\overline{AR} \cong \overline{RT}$
- Prove: $\overline{PR} \perp \overline{AT}$

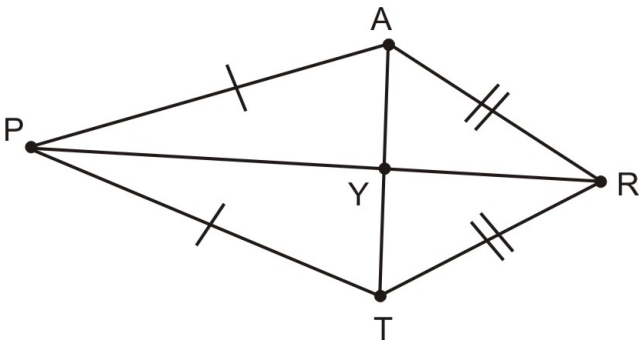
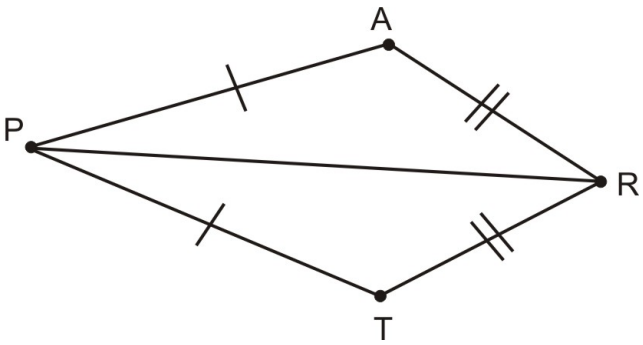


Table 6.22:

Statement	Reason
1. Kite $PART$ with $\overline{PA} \cong \overline{PT}$ and $\overline{AR} \cong \overline{RT}$	1. Given
2. $\overline{PY} \cong \overline{PY}$	2. Reflexive Property of $\cong$
3. $\angle APR \cong \angle TPR$	3. By part 1 above: The diagonal between vertex angles bisects the angles
4. $\triangle PAY \cong \triangle PTY$	4. SAS Congruence Postulate
5. $\angle AYP \cong \angle TYP$	5. Corresponding parts of congruent triangles are congruent
6. $\angle AYP$ and $\angle TYP$ are supplementary	6. Linear Pair Postulate
7. $\angle AYP$ and $\angle TYP$ are right angles	7. Congruent supplementary angles are right angles
8. $\overline{PR} \perp \overline{AT}$	8. Definition of perpendicular ♦

Opposite Angles in Kites

In addition to the bisecting property, one other property of kites is that the non-vertex angles are congruent.



So, in the kite $PART$ above, $\angle PAR \cong \angle PTR$.

Example 2

Complete the two-column proof below.

- Given: $\overline{PA} \cong \overline{PT}$ and $\overline{AR} \cong \overline{RT}$
- Prove: $\angle PAR \cong \angle PTR$

Table 6.23:

Statement	Reason
1. $\overline{PA} \cong \overline{PT}$	1. Given
2. $\overline{AR} \cong \overline{RT}$	2. Given
3. _____	3. Reflexive Property
4. _____	4. $SSS \cong SSS$ If two triangles have three pairs of congruent sides, the triangles are congruent.
5. $\angle PAR \cong \angle PTR$	5. _____- _____

We will let you fill in the blanks on your own, but a hint is that this proof is nearly identical to the first proof in this section.

So, you have successfully proved that the angles between the congruent sides in a kite are congruent.

Lesson Summary

In this lesson, we explored kites. Specifically, we have learned to:

- Identify the relationship between diagonals in kites.
- Identify the relationship between opposite angles in kites.

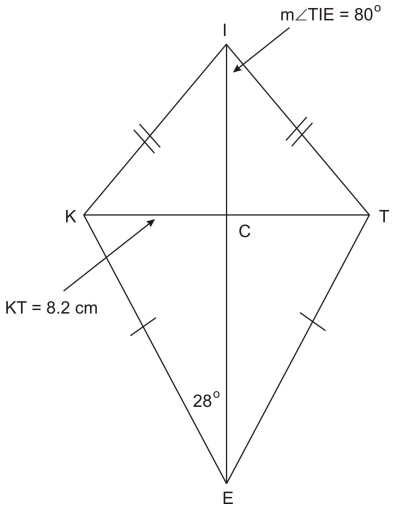
It is helpful to be able to identify specific properties in kites. You will be able to use this information in many different ways.

Points to Consider

Now that you have learned about different types of quadrilaterals, it is important to learn more about the relationships between shapes. The next chapter deals with similarity between shapes.

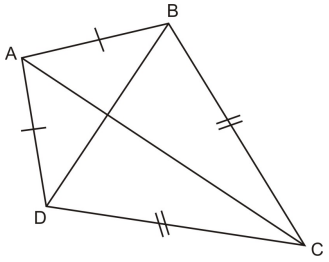
Review Questions

For exercises 1-5, use kite $KITE$ below with the given measurements.



1. $m\angle KIT =$ ____
2. $m\angle TEI =$ ____
3. $m\angle EKI =$ ____
4. $m\angle KCE =$ ____
5. $KC =$ ____

For exercises 6-10, fill in the blanks in each sentence about Kite ABCD below:



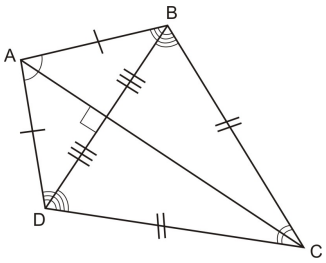
6. The vertex angles of kite $ABCD$ are _____ and _____.
7. _____ is the perpendicular bisector of _____.
8. Diagonal _____ bisects $\angle$ _____ and $\angle$ _____.
9. $\angle$ ____ $\cong$ $\angle$____, $\angle$____ $\cong$ $\angle$ _____, and $\angle$____ $\cong$ $\angle$____.
10. The line of symmetry in the kite is along segment _____.
11. Can the diagonals of a kite be congruent to each other? Why or why not?

Review Answers

1. 160°
2. 28°
3. 72°
4. 90°
5. 4.1 cm
6. The vertex angles of kite $ABCD$ are $\angle DAB$ and $\angle BCD$.
7. $\overline{AC}$ is the perpendicular bisector of $\overline{DB}$.
8. Diagonal $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$.
9. *There are many possible answers:*

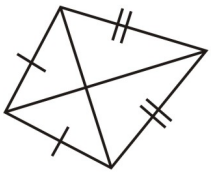
   $\angle ADC \cong ABC, \angle BAC \cong \angle DAC, \angle BCA \cong \angle DCA, \angle ABD \cong \angle ADB, \angle CDB \cong \angle CBD$

10. $\overline{AC}$ is a line of reflection. *Below is kite ABCD fully annotated with geometric markings.*

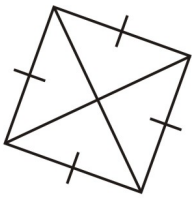


11. No, if the diagonals were congruent then the "kite" would be a square. Since the two pairs of congruent sides cannot be congruent to each other (i.e., they must be

distinct), the diagonals will have different lengths.



Diagonals not $\cong$, and this is a kite



Diagonals are $\cong$, but shape is not a kite

Chapter 7

Similarity

7.1 Ratios and Proportions

Learning Objectives

- Write and simplify ratios.
- Formulate proportions.
- Use ratios and proportions in problem solving.

Introduction

Words can have different meanings, or even shades of meanings. Often the exact meaning depends on the *context* in which a word is used. In this chapter you'll use the word *similar.*

What does similar mean in *ordinary* language? Is a rose similar to a tulip? They're certainly both flowers. Is an elephant similar to a donkey? They're both mammals (and symbols of national political parties in the United States!). Maybe you'd rather say that a sofa is similar to a chair? In loose terms, by similar we usually mean that things are like each other in some way or ways, but maybe not the same.

Similar has a very precise meaning in geometry, as we'll see in upcoming lessons. To understand *similar* we first need to review some basic skills in ratios and proportions.

Using Ratios

A **ratio** is a type of *fraction.* Usually a ratio is a fraction that *compares* two parts. "The ratio of x to y" can be written in several ways.

- $\frac{x}{y}$
- $x : y$
- x to y

Example 1

Look at the data below, giving sales at Bagel Bonanza one day.

Bagel Bonanza Monday Sales

Table 7.1:

Type of bagel	Number sold
Plain	80
Cinnamon Raisin	30
Sesame	25
Garlic	20
Whole grain	45
Everything	50

a) What is the ratio of the number of cinnamon raisin bagels sold to the number of plain bagels sold?

Ratio of cinnamon raisin to plain $= \frac{30}{80}, 30 : 80$, or 30 to 80.

Note: Depending on the problem, ratios are often written in *simplest form*. In this case the ratio can be reduced or simplified because $\frac{30}{80} = \frac{3}{8}$.

b) What is the ratio, in simplest form, of the number of whole grain bagels sold to the number of "everything" bagels sold?

Ratio of whole grain to everything $= \frac{45}{50} = \frac{9}{10}, 9 : 10$, or 9 to 10.

c) What is the ratio, in simplest form, of everything bagels sold to the number of whole grain bagels sold?

Answer: This ratio is just the reciprocal of the ratio in b. If the ratio of whole grain to everything is, $\frac{45}{50} = \frac{9}{10}, 9 : 10$, or 9 to 10, then the ratio of everything to whole grain is, $\frac{10}{9}, 10 : 9$, or 10 to 9.

d. What is the ratio, in simplest form, of the number of sesame bagels sold to the total number of all bagels sold?

First find the total number of bagels sold: $80 + 30 + 25 + 20 + 45 + 50 = 250$.

Ratio of sesame to total sold $= \frac{25}{250} = \frac{1}{10}, 1 : 10$, or 1 to 10.

Note that this also means that $\frac{1}{10}$, or 10%, of all the bagels sold were sesame.

In some situations you need to write a ratio of more than two numbers. For example, the ratio, in simplest form, of the number of cinnamon raisin bagels to the number of sesame bagels to the number of garlic bagels is $6 : 5 : 4$ (or $30 : 25 : 20$ before simplifying).

Example 2

A talent show features only dancers and singers.

- The ratio of dancers to singers is $3 : 2$.
- There are 30 performers in all.

How many singers are there?

There is a whole number n so that the total number of each group can be represented as

$$\text{dancers} = 3n, \ \text{singers} = 2n.$$

Since there are 30 dancers and singers in all,

$$\begin{aligned} 3n + 2n &= 30 \\ 5n &= 30 \\ n &= 6 \end{aligned}$$

The number of dancers is $3n = 3 \cdot 6 = 18$. The number of singers is $2n = 2 \cdot 6 = 12$. It's easy to check these answers. The numbers of dancers and singers have to add up to 30, and they have to be in a $3-$ to -2 ratio.

Check: $18 + 12 = 30$. The ratio of dancers to singers is $\frac{18}{12} = \frac{3}{2}$, or 3 to 2.

Proportions

A **proportion** is an equation. The two sides of the equation are ratios that are equal to each other. Proportions are often found in situations involving direct variation. A scale drawing would make a good example.

Example 3

Leo uses a scale drawing of his barn. He recorded actual measurements and the lengths on the scale drawing that represent those actual measurements.

Table 7.2: **Barn dimensions**

	Actual length	Length on scale drawing
Door opening	16 feet	4 inches
Interior wall	25 feet	6.25 inches
Water line	10 feet	?

a) Since he is using a scale drawing, the ratio of actual length to length on the scale drawing should be the same all the time. We can write two ratios that should be equal. This is the proportion below.

$$\frac{16}{4} = \frac{25}{6.25}$$

Is the proportion true?

We could write the fractions with a common denominator. One common denominator is 4×6.25.

$$\frac{16}{4} =? \frac{25}{6.25} \Rightarrow \frac{16 \cdot 6.25}{4 \cdot 6.25} =? \frac{25 \cdot 4}{6.25 \cdot 4} \Rightarrow \frac{100}{25} = \frac{100}{25}.$$

The proportion is true.

b) Depending on how you think, you might have written a different proportion. You could say that the ratio of the actual lengths must be the same as the ratio of the lengths on the scale drawing.

$$\frac{16}{25} =? \frac{4}{6.25} \Rightarrow \frac{16 \cdot 6.25}{25 \cdot 6.25} =? \frac{4 \cdot 25}{6.25 \cdot 25} \Rightarrow \frac{100}{25 \cdot 6.25} = \frac{100}{6.25 \cdot 25}.$$

This proportion is also true. One nice thing about working with proportions is that there are several proportions that correctly represent the same data.

c) What length should Leo use on the scale drawing for the water line?

Let x represent the scale length. Write a proportion.

$$\left[\frac{\text{actual}}{\text{scale}}\right] \Rightarrow \frac{16}{4} = \frac{10}{x} \Rightarrow \frac{16x}{4x} = \frac{10 \cdot 4}{x(4)} \Rightarrow \frac{16x}{4x} = \frac{40}{4x}$$

If two fractions are equal, and they have the same denominator, then the numerators must be equal.

$$16x = 40 \Rightarrow x = \frac{40}{16} = 2.5$$

The scale length for the water line is 2.5 inches.

Note that the scale for this drawing can be expressed as 1 inch to 4 feet, or $\frac{1}{4}$ inch to 1 foot.

Proportions and Cross Products

Look at example 3b above.

$$\frac{16}{25} = \frac{4}{6.25} \Rightarrow \frac{16 \cdot 6.25}{25 \cdot 6.25} = \frac{4 \cdot 25}{6.25 \cdot 25}$$

$\frac{16}{25} = \frac{4}{6.25}$ is true if and only if $16 \cdot 6.25 = 4 \cdot 25$.

In the proportion, $\frac{16}{25} = \frac{4}{6.25}$, 25 and 4 are called the **means** (they're in the middle); 16 and 6.25 are called the **extremes** (they're on the ends). You can see that for the proportion to be true, the product of the means ($25 \cdot 4$) must equal the product of the extremes ($16 \cdot 6.25$). Both products equal 100.

It is easy to generalize this means-and-extremes rule for any true proportion.

Means and Extremes Theorem or The Cross Multiplication Theorem

Cross Multiplication Theorem: Let a, b, c, and d be real numbers, with $b \neq 0$ and $d \neq 0$.

If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

The proof of the cross multiplication theorem is example 4. The proof of the converse is in the Lesson Exercises.

Example 4

Prove The Cross Multiplication Theorem: For real numbers a, b, c, and d with $b \neq 0$ and $d \neq 0$, If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

We will start by summarizing the given information and what we want to prove. Then we will use a two-column proof.

- Given: a, b, c, and d are real numbers, with $b \neq 0$ and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$
- Prove: $ad = bc$

Table 7.3:

Statement	Reason
1. a, b, c, and d are real numbers, with $b \neq 0$ and $d \neq 0$	1. Given
2. $\frac{a}{b} = \frac{c}{d}$	2. Given
3. $\frac{a}{b} \cdot \frac{d}{d} = \frac{c}{d} \cdot \frac{b}{b}$	3. $\frac{d}{d} = \frac{b}{b} = 1$, identity property of multiplication
4. $\frac{a}{b} \cdot \frac{d}{d} = \frac{b}{b} \cdot \frac{c}{d}$	4. Commutative property of multiplication
5. $a \cdot d = b \cdot c$ or $ad = bc$	5. If equal fractions have the same denominator, then the numerators must be equal ♦

This theorem allows you to use the method of cross multiplication with proportions.

Lesson Summary

Ratios are a useful way to compare things. Equal ratios are proportions. With the Means-and-Extremes Theorem we have a simple but powerful method for solving any proportion.

Points to Consider

Proportions are very "forgiving"—there are many different ways to write proportions that are equivalent to each other. There are hints of some of these in the Lesson Exercises. In the next lesson, we'll prove that these proportions are equivalent.

You know about figures that are *congruent*. But many figures that are *alike* are not congruent. They may have the same shape, even though they are not the same size. These are

similar figures; ratios and proportions are integral to defining and understanding similar figures.

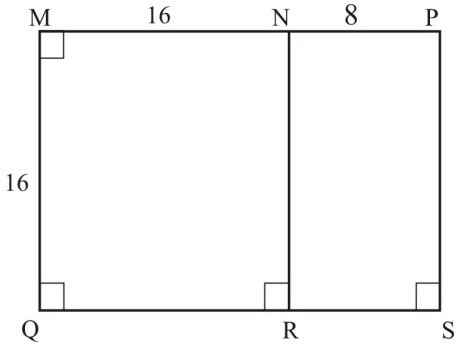
Review Questions

The votes for president in a club election were:

Suarez, 24 Milhone, 32 Cho, 20

1. Write each of the following ratios in simplest form.
   (a) votes for Milhone to votes for Suarez
   (b) votes for Cho to votes for Milhone
   (c) votes for Suarez to votes for Milhone to votes for Cho
   (d) votes for Suarez or Cho to total votes

   Use the diagram below for exercise 2.



2. Write each of the following ratios in simplest form.
   (a) $MN : MQ$
   (b) $MN : NP$
   (c) $NP : MN$
   (d) $MN : MP$
   (e) area of $MNRQ$: area of $NPSR$
   (f) area of $NPSR$: area of $MNRQ$
   (g) area of $MNRQ$: area of $MPSQ$
3. The measures of the angles of a triangle are in the ratio $3 : 3 : 4$. What are the measures of the angles?
4. The length and width of a rectangle are in a $3 : 5$ ratio. The area of the rectangle is 540 square inches. What are the length and width?
5. Prove the converse of Theorem 7-1: For real numbers a, b, c, and d, with, $b \neq 0$ and $d \neq 0, ad = bc \Rightarrow a/b = c/d$.

Given: a, b, c, and d are real numbers, with $b \neq 0$ and $d \neq 0$ and $ad = bc$

Prove: $\frac{a}{b} = \frac{c}{d}$

6. Which of the following statements are true for all real numbers a, b, c, and $d, b \neq 0$ and $d \neq 0$?
   (a) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{d} = \frac{c}{b}$.
   (b) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$.
   (c) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$.
   (d) If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{c} = \frac{a}{d}$.
7. Solve each proportion for w.
   (a) $\frac{6}{w} = \frac{4}{5}$
   (b) $\frac{w}{3} = \frac{12}{w}$
   (c) $\frac{3}{4} = \frac{w}{w+2}$
8. Shawna drove 245 miles and used 8.2 gallons of gas. At that rate, she would use x gallons of gas to drive 416 miles. Write a proportion that could be used to find the value of x.
9. Solve the proportion you wrote in exercise 8. How much gas would Shawna expect to use to drive 416 miles?
10. Rashid, Leon, and Maria are partners in a company. They divide the profits in a $3 : 2 : 4$ ratio, with Rashid getting the largest share and Leon getting the smallest share. In 2006 the company had a total profit of $\$1,800,000$. How much profit did each person receive?

Review Answers

1. (a) $4 : 3$
   (b) $5 : 8$
   (c) $6 : 8 : 5$
   (d) $11 : 19$
2. (a) $1 : 1$
   (b) $2 : 1$
   (c) $1 : 2$
   (d) $2 : 3$
   (e) $2 : 1$
   (f) $1 : 2$
   (g) $2 : 3$
3. $54°, 54°, 72°$
4. 30 inches and 18 inches

Table 7.4:

Statement	Reason
A. a, b, c, and d, with $b \neq 0$ and $d \neq 0$, and $ad = bc$.	A. Given
B. $ad \times \frac{1}{bd} = bc \times \frac{1}{bd}$	B. Multiplication Property of Equality
C. $\frac{ad}{bd} = \frac{bc}{bd}$	C. Arithmetic
D. $\frac{a}{b} \times \frac{d}{d} = \frac{b}{b} \times \frac{c}{d}$	D. Arithmetic
E. $\frac{a}{b} = \frac{c}{d}$	E. $\frac{d}{d} = \frac{b}{b} = 1$, identity property of equality

5.

6. (a) No
   (b) Yes
   (c) Yes
   (d) No
7. (a) $w = 7.5$
   (b) $w = 6$ or $w = -6$
   (c) $w = 6$
8. $\frac{245}{8.2} = \frac{416}{x}$ or equivalent
9. $x \approx 13.9$. At that rate she would use about 13.9 gallons of gas.
10. Rashid gets $\$800,000$, Leon gets $\$400,000$, and Maria gets $\$600,000$.

7.2 Properties of Proportions

Learning Objectives

- Prove theorems about proportions.
- Recognize true proportions.
- Use proportions theorems in problem solving.

Introduction

The Cross Multiplication Theorem is the basic, defining property of proportions. Whenever you are in doubt about whether a proportion is true, you can always check it by cross multiplication. Additionally, there are also a number of "sub-theorems" about proportions that are useful to apply for solving problems. In each case the sub-theorem is easy to prove using cross multiplication.

Properties of Proportions

Technically speaking, the theorems in this lesson are not called sub-theorems. The formal term is *corollary.* The word corollary is rather loosely defined in mathematics. Basically, a corollary is a theorem that follows quickly, easily, and directly from another theorem—in this case from the Cross multiplication Theorem.

The corollaries in this section are not absolutely essential—you could always go back to using cross multiplication. But there may be times when the corollaries make things quicker or easier, so it's good to have them if and when they are needed.

Cross Multiplication Corollaries

Below are three corollaries that are immediate results of the Cross Multiplication Theorem and the fundamental laws of algebra.

Corollaries 1, 2, and 3 of The Cross Multiplication Theorem

If $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$, then ....

1. $\frac{a}{c} = \frac{b}{d}$.
2. $\frac{d}{b} = \frac{c}{a}$.
3. $\frac{b}{a} = \frac{d}{c}$.

In words.

1. A true proportion is also true if you "swap" the "means."
2. A true proportion is also true if you "swap" the "extremes."
3. A true proportion is also true if you "flip" it upside down.

Example 1

Look at the diagram below.

Suppose we're given that $\frac{10}{6} = \frac{15}{9} = \frac{x}{y}$.

We know $\frac{10}{6} = \frac{15}{9}$, *since* $10 \cdot 9 = 6 \cdot 15 = 90$

Here are some other proportions that must also be true by corollaries 1-3.

$$\frac{15}{x} = \frac{9}{y}$$
$$\frac{y}{6} = \frac{x}{10}$$
$$\frac{15}{9} = \frac{x}{y}$$
$$\frac{15}{10} = \frac{9}{6}$$
$$\frac{x}{15} = \frac{y}{9}$$

Two Additional Corollaries to the Cross Multiplication Theorem

Here we have two more corollaries to the Cross Multiplication Theorem. The "if" part of these theorems is the same as above. So the given in each proof remains the same too.

Corollary 4:

If $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$, then

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

Proof.

Table 7.5:

Statement	Reason
1. $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$	1. Given
2. $ad = bc$	2. Cross Multiplication Theorem
3. $(a+b) \times d = ad + bd$	3. Distributive Property
4. $b(c+d) = bc + bd$	4. Distributive Property
5. $b(c+d) = ad + bd$	5. Substitution
6. $(a+b) \times d = b(c+d)$	6. Substitution
7. $\frac{a+b}{b} = \frac{c+d}{d}$	7. Cross Multiplication Theorem ♦

This second theorem is nearly the same as the previous,

Corollary 5:

If $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$, then

$$\frac{a-b}{b} = \frac{c-d}{d}.$$

The proof of this corollary is in the Lesson Exercises.

Example 2

Suppose we're given that $\frac{10}{6} = \frac{15}{9} = \frac{x}{y}$ *again, as in example 1.*

Here are some other proportions that must also be true, and the theorems that guarantee them.

$$\frac{16}{6} = \frac{24}{9} \qquad \text{Corollary 4}$$

$$\frac{4}{6} = \frac{6}{9} \qquad \text{Corollary 5}$$

$$\frac{24}{9} = \frac{x+y}{y} \qquad \text{Corollary 4} \blacklozenge$$

Lesson Summary

Proportions were probably not new to you in this lesson; you may have studied them in previous courses. What probably is new is the larger structure of theorems and corollaries that serve as tools for working with proportions.

The most basic fact about proportions is the Cross Multiplication Theorem:

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$$

assuming a, b, c, and $d \neq 0$. The corollaries in this lesson are really just variations on the Cross Multiplication Theorem. They may be useful in problems, but we could always revert back to Cross Multiplication if we had to.

Some people find proportions nice to work with, because there are so many different—and correct—ways to write a given proportion, as you saw in the corollaries. It sometimes seems that you would really have to work at it to write a proportion that is **not** equivalent to the proportion you are given!

Points to Consider

As we move ahead we will meet important concepts that require the use of ratios and proportions. Proportions are mandatory for understanding the geometric meaning of *similar*. Later when we work with transformations and scale factors, ratios will also be useful.

Finally, one proof of the Pythagorean Theorem relies on proportions.

Review Questions

Given that $\frac{10}{6} = \frac{15}{d} = \frac{x}{y}, x \neq 0, y \neq 0$. For each of the following, write "true" if the proportion must be true. Otherwise write "false."

1. $\frac{10}{y} = \frac{x}{6}$
2. $\frac{10}{15} = \frac{6}{9}$
3. $\frac{10}{y} = \frac{6}{x}$
4. $\frac{y}{6} = \frac{x}{10}$
5. $\frac{9}{15} = \frac{6}{10}$
6. $\frac{6}{x} = \frac{10}{y}$
7. $\frac{25}{15} = \frac{x}{y}$
8. $\frac{10}{16} = \frac{x}{x+y}$
9. $\frac{33}{9} = \frac{x+2y}{y}$
10. $\frac{4}{6} = \frac{y-x}{y}$
11. Prove: If $\frac{a}{b} = \frac{c}{d}, b \neq 0, d \neq 0$, then $\frac{a}{a+b} = \frac{c}{c+d}$.
12. Prove Corollary 5 to the Cross Multiplication Theorem.

Review Answers

1. False
2. True
3. False
4. True
5. True
6. False
7. True
8. True
9. True
10. False

Table 7.6:

Statement	Reason
A. $\frac{a}{b} = \frac{c}{d}, b \neq 0, d \neq 0$	A. Given
B. $ad = bc$	B. Cross Multiplication Theorem
C. $a(c + d) = ac + ad$	C. Distributive Property
D. $a(c + d) = ac + bc$	D. Substitution
E. $a(c + d) = (a + b)c$	E. Distributive Property
F. $\frac{a}{a+b} = \frac{c}{c+d}$	F. Cross Multiplication Theorem

11.

Table 7.7:

Statement	Reason
A. $\frac{a}{b} = \frac{c}{d}, b \neq 0, d \neq 0$	A. Given
B. $ad = bc$	B. Cross Multiplication Theorem
C. $(a - b)d = ad - bd$	C. Distributive Property
D. $(a - b)d = bc - bd$	D. Substitution
E. $(a - b)d = (c - d)b$	E. Distributive Property
F. $\frac{a-b}{b} = \frac{c-d}{d}$	F. Cross Multiplication Theorem

12.

7.3 Similar Polygons

Learning Objectives

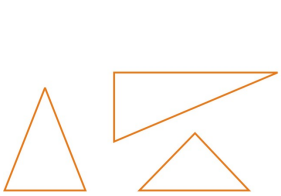
- Recognize similar polygons.
- Identify corresponding angles and sides of similar polygons from a statement of similarity.
- Calculate and apply scale factors.

Introduction

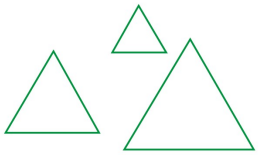
Similar figures, rectangles, triangles, etc., have the same shape. Same shape, however, is not a precise enough term for geometry. In this lesson, we'll learn a precise definition for similar, and apply it to measures of the sides and angles of similar polygons.

Similar Polygons

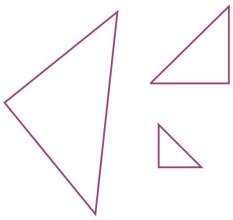
Look at the triangles below.



Triangles - Not Similar



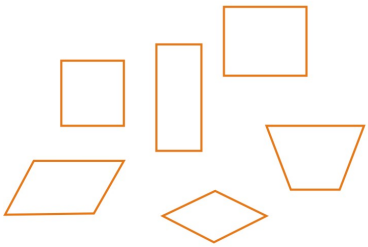
Triangles - Similar



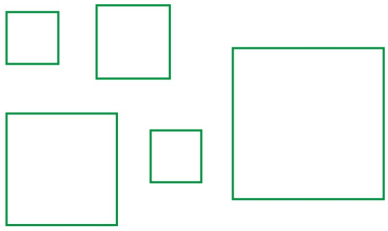
Triangles - Similar

- The triangles on the left are *not* similar because they are not the same shape.
- The triangles in the middle *are* similar. They are all the same shape, no matter what their sizes.
- The triangles on the right *are* similar. They are all the same shape, no matter how they are turned or what their sizes.

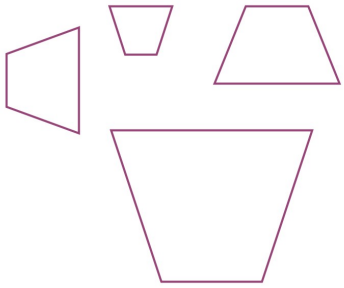
Look at the quadrilaterals below.



Quadrilaterals - None are similar to each other



Quadrilaterals - All similar.

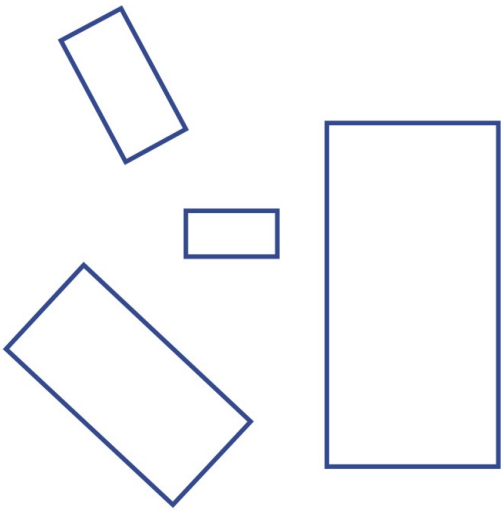


Quadrilaterals - All similar.

- The quadrilaterals in the upper left are *not* similar because they are not the same shape.

- The quadrilaterals in the upper right *are* similar. They are all the same shape, no matter what their sizes.
- The quadrilaterals in the lower left *are* similar. They are all the same shape, no matter how they are turned or what their sizes.

Now let's get serious about what it means for figures to be similar. The rectangles below *are* all similar to each other.



Rectangles - All Similar

These rectangles are similar, but it's not just because they're rectangles. Being rectangles guarantees that these figures all have congruent angles. But that's not enough. You've seen lots of rectangles before, some are long and narrow, others are more blocky and closer to square in shape.

The rectangles above are all the same shape. To convince yourself of this you could measure the length and width of each rectangle. Each rectangle has a length that is exactly twice its width. So the ratio of length-to-width is $2 : 1$ for each rectangle. Now we can make a more formal statement of what similar means in geometry.

Two polygons are **similar** if and only if:

- they have the same number of sides
- for each angle in either polygon there is a corresponding angle in the other polygon that is congruent
- the lengths of all corresponding sides in the polygons are proportional

Reminder: Just as we did with congruent figures, we name similar polygons according to corresponding parts. The symbol $\sim$ is used to represent "is similar to." Some people call this "the congruent sign without the equals part."

Example 1

Suppose $\triangle ABC \sim \triangle JKL$ Based on this statement, which angles are congruent and which sides are proportional? Write true congruence statements and proportions.

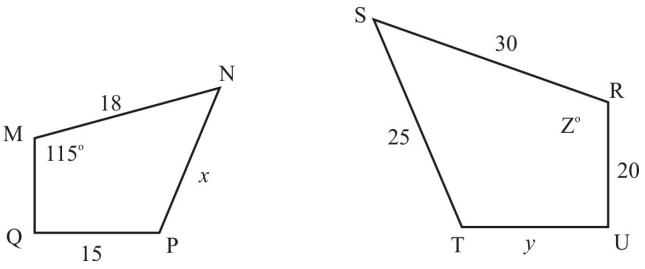
$\angle A \cong \angle J, \angle B \cong \angle K$, and $\angle C \cong \angle L$

$$\frac{AB}{JK} = \frac{BC}{KL} = \frac{AC}{JL}$$

Remember that there are many equivalent ways to write a proportion. The answer above is not the only set of true proportions you can create based on the given similarity statement. Can you think of others?

Example 2

Given: $MNPQ \sim RSTU$



What are the values of x, y, and z in the diagram below?

Set up a proportion to solve for x:

$$\begin{aligned} \frac{x}{25} &= \frac{18}{30} \\ \frac{x}{25} &= \frac{3}{5} \\ 5x &= 75 \\ x &= 15 \end{aligned}$$

Now set up a proportion to solve for y:

$$\frac{y}{15} = \frac{30}{18}$$
$$\frac{y}{15} = \frac{5}{3}$$
$$3y = 75$$
$$y = 25$$

Finally, since Z is an angle, we are looking for $m\angle R$

$$Z = m\angle R = m\angle M = 115^\circ$$

Example 3

$ABCD$ is a rectangle with length 12 and width 8.

$UVWX$ is a rectangle with length 24 and width 18.

A. *Are corresponding angles in the rectangles congruent?*

Yes. Since both are rectangles, all the angles in both are congruent right angles.

B. *Are the lengths of the sides of the rectangles proportional?*

No. The ratio of the lengths is $12 : 24 = 1 : 2$. The ratio of the widths is $8 : 18 = 4 : 9 \neq 1 : 2$. Therefore, the lengths of the sides are not proportional.

C. Are the rectangles similar?

No. Corresponding angles are congruent, but lengths of corresponding sides are not proportional.

Example 4

Prove that all squares are similar.

Our proof is a "paragraph" proof in bullet form, rather than a two-column proof:

Given two squares.

- All the angles of both squares are right angles, so all angles of both squares are congruent—and this includes corresponding angles.
- Let the length of each side of one square be k, and the length of each side of the other square be m. Then the ratio of the length of any side of the first square to the length of any side of the second square is $k : m$. So the lengths of the sides are proportional.
- The squares satisfy the definition of similar polygons: congruent angles and proportional side lengths - so they are similar

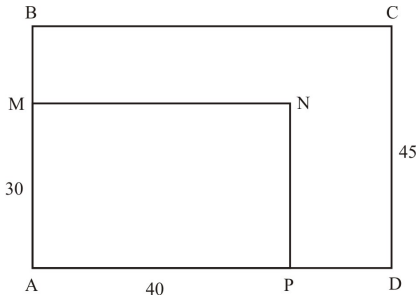
Scale Factors

If two polygons are similar, we know that the lengths of corresponding sides are proportional. If k is the length of a side in one polygon, and m is the length of the corresponding side in the other polygon, then the ratio $\frac{k}{m}$ is called the **scale factor** relating the first polygon to the second. Another way to say this is:

The length of every side of the first polygon is $\frac{k}{m}$ times the length of the corresponding side of the other polygon.

Example 5

Look at the diagram below, where $ABCD$ and $AMNP$ are similar rectangles.



A. *What is the scale factor?*

Since $ABCD \sim AMNP$, then AM and AB are corresponding sides. Since $ABCD$ is a rectangle, you know that $AB = DC = 45$.

The scale factor is the ratio of the lengths of any two corresponding sides.

So the scale factor (relating $ABCD$ to $AMNP$) is $\frac{45}{30} = \frac{3}{2} = 1.5$. We now know that the length of each side of $ABCD$ is 1.5 times the length of the corresponding side in $AMNP$.

Comment: We can turn this relationship around "backwards" and talk about the scale factor relating $AMNP$ to $ABCD$. This scale factor is just $\frac{30}{45} = \frac{2}{3}$, which is the *reciprocal* of the scale factor relating $ABCD$ to $AMNP$.

B. *What is the ratio of the perimeters of the rectangles?*

$ABCD$ is a 45 by 60 rectangle. Its perimeter is $45 + 60 + 45 + 60 = 210$.

$AMNP$ is a 30 by 40 rectangle. Its perimeter is $30 + 40 + 30 + 40 = 140$.

The ratio of the perimeters of $ABCD$ to $AMNP$ is $\frac{210}{140} = \frac{3}{2}$.

Comment: You see from this example that the ratio of the perimeters of the rectangles is the same as the scale factor. This relationship for the perimeters holds true in general for any similar polygons.

Ratio of Perimeters of Similar Polygons

Let's prove the theorem that was suggested by example 5.

Ratio of the Perimeters of Similar Polygons:

If P and Q are two similar polygons, each with n sides and the scale factor of the polygons is s, then the ratio of the perimeters of the polygons is s.

- Given: P and Q are two similar polygons, each with n sides

The scale factor of the polygons is s

- Prove: The ratio of the perimeters of the polygons is s

Table 7.8:

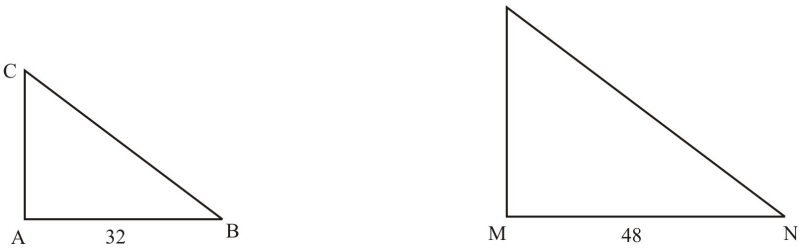
Statement	Reason
1. P and Q are similar polygons, each with n sides	1. Given
2. The scale factor of the polygons is s	2. Given
3. Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ be the lengths of corresponding sides of P and Q	3. Given (polygons have n sides each)
4. $p_1 = sq_1,\ p_2 = sq_2,\ \ldots\ , p_n = sq_n$	4. Definition of scale factor
5. Perimeter of $P = p_1 + p_2 + \ldots + p_n$	5. Definition of perimeter
6. $= sq_1 + sq_2 + \ldots + sq_n$	6. Substitution
7. $= s\,(q_1 + q_2 + \ldots + q_n)$	7. Distributive Property
8. $= s$, the perimeter of Q	8. Definition of perimeter ♦

Comment: The ratio of the perimeters of any two similar polygons is the same as the scale factor. In fact, the ratio of *any two corresponding* ***linear*** measures in similar figures is the same as the scale factor. This applies to corresponding sides, perimeters, diagonals, medians, midsegments, altitudes, etc.

As we'll see in an upcoming lesson, this is definitely *not true* for the **areas** of similar polygons. The ratio of the areas of similar polygons (that are not congruent) is *not* the same as the scale factor.

Example 6

$\triangle ABC \sim \triangle MNP$. *The perimeter of* $\triangle MNP$ *is* 150.



What is the perimeter of $\triangle ABC$?

The scale factor relating $\triangle ABC$ to $\triangle MNP$ is $\frac{32}{48} = \frac{2}{3}$. According to the Ratio of the Perimeter's Theorem, the perimeter of $\triangle ABC$ is $\frac{2}{3}$ of the perimeter of $\triangle MNP$. Thus, the perimeter of $\triangle ABC$ is $\frac{2}{3} \cdot 150 = 100$.

Lesson Summary

Similar has a very specific meaning in geometry. Polygons are similar if and only if the lengths of their sides are proportional and corresponding angles are congruent. This is *same shape* translated into geometric terms.

The ratio of the lengths of corresponding sides in similar polygons is called the scale factor. Lengths of other corresponding linear measures, such as perimeter, diagonals, etc. have the same scale factor.

Points to Consider

Scale factors show the relationship between corresponding linear measures in similar polygons. The story is not quite that simple for the relationship between the areas or volumes of similar polygons and polyhedra (three-dimensional figures). We'll study these relationships in future lessons.

Similar triangles are the basis for the study of *trigonometry*. The fact that the ratios of the lengths of corresponding sides in right triangles depends only on the measure of an angle, not on the size of the triangle, makes trigonometric functions the property of an angle, as you will study in Chapter 8.

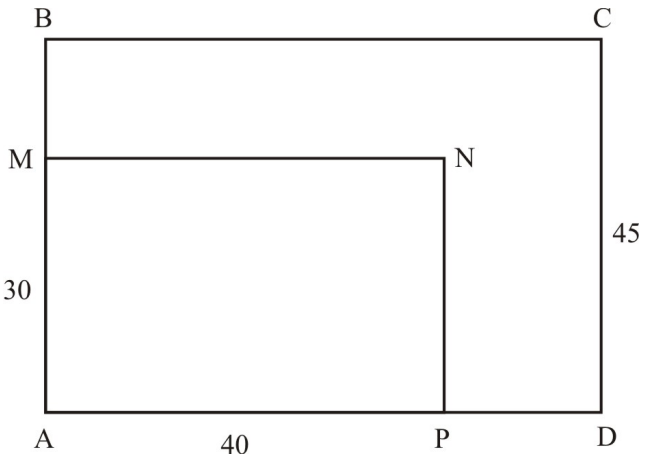
Review Questions

True or false?

1. All equilateral triangles are similar.
2. All isosceles triangles are similar.
3. All rectangles are similar.

4. All rhombuses are similar.
5. All squares are similar.
6. All congruent polygons are similar.
7. All similar polygons are congruent.

Use the following diagram for exercises 8-11.



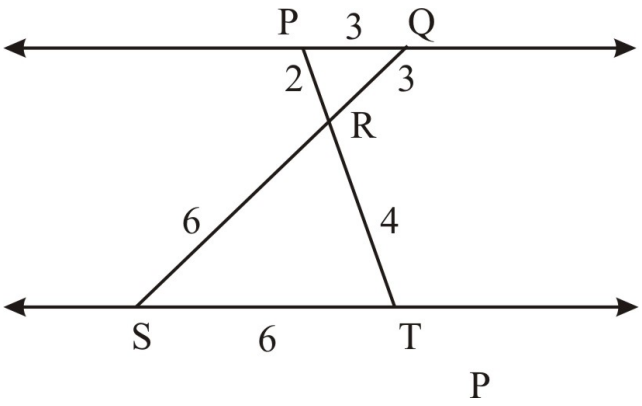
Given that rectangle $ABCD$: rectangle $AMNP$.

What is the value of each expression?

8. AB
9. BC
10. MB
11. PD
12. Given that $\triangle ABC \cong \triangle MNP$, what is the scale factor of the triangles?

Use the diagram below for exercises 13-16.

Given: $\overline{PQ} : \overline{ST}$



13. What is the perimeter of $\triangle PQR$?
14. What is the perimeter of $\triangle TSR$?
15. What is the ratio of the perimeter of $\triangle PQR$ to the perimeter of $\triangle TSR$?
16. Prove: : $PQR : TSR$. [Write a flow proof.]
17. M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{AB}$ in $\triangle ABC$.

    (a) Name a pair of parallel segments.
    (b) Name two pairs of congruent angles.
    (c) Write a statement of similarity of two triangles.
    (d) If the perimeter of the larger triangle in c is p, what is the perimeter of the smaller triangle?
    (e) If the area of $\triangle ABC$ is 100, what is the area of quadrilateral $AMNC$?

Review Answers

1. True
2. False
3. False
4. False
5. True
6. True
7. False
8. 45
9. 60
10. 15
11. 20
12. $1 : 1, 1$, or 1.0
13. 8
14. 16
15. $1 : 2, \frac{1}{2}, 0.5$ or equivalent
16. $PQ : TS = QR : SR = PR : TR = 1 : 2$, so the sides are all proportional.

$\angle PRQ \cong \angle TSR$	(vertical angles)
$\angle RPQ \cong \angle RTS, \angle RQP \cong \angle RST$	(parallel lines, alternate interior angles are congruent)
$\triangle PQR : \triangle TSR$	(definition of similar polygons: angles are congruent, lengths of sides are proportional)

17. (a) $\overline{MN}, \overline{AC}$
    (b) $\angle BMN \cong \angle BAC \angle BNM \cong \angle BCA$
    (c) $\triangle BAC : \triangle BMN$
    (d) $\frac{1}{2}p, \frac{p}{2}$ or equivalent
    (e) 75

7.4 Similarity by AA

Learning Objectives

- Determine whether triangles are similar.
- Understand AAA and AA rules for similar triangles.
- Solve problems about similar triangles.

Introduction

You have an understanding of what similar polygons are and how to recognize them. Because triangles are the most basic building block on which other polygons can be based, we now focus specifically on similar triangles. We'll find that there's a surprisingly simple rule for triangles to be similar.

Angles in Similar Triangles

Tech Note - Geometry Software

Use your geometry software to experiment with triangles. Try this:

1. Set up two triangles, $\triangle ABC$ and $\triangle MNP$.
2. Measure the angles of both triangles.
3. Move the vertices until the measures of the corresponding angles are the same in both triangles.
4. Compute the ratios of the lengths of the sides

$$\frac{AB}{MN}\frac{BC}{NP}\frac{AC}{MP}.$$

Repeat steps 1-4 with different triangles. Observe what happens in step 4 each time. Record your observations.

What did you see during your experiment? You might have noticed this: When you adjust triangles to make their angles congruent, you automatically make the sides proportional (the ratios in step 4 are the same). Once we have triangles with congruent angles and sides with proportional lengths, we know that the triangles are similar.

Conclusion: *If the angles of a triangle are congruent to the corresponding angles of another triangle, then the triangles are similar.* This is a handy rule for similar triangles—a rule based on just the angles of the triangles. We call this the AAA rule.

Caution: The AAA rule is a rule for *triangles* only. We already know that other pairs of polygons can have all corresponding angles congruent even though the polygons are **not** similar.

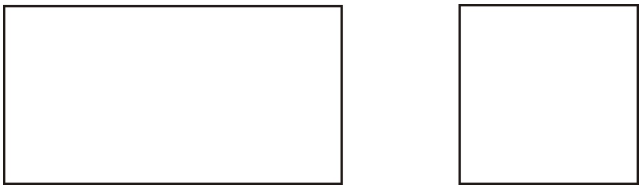
Example 1

The following is false statement: If the corresponding angles of two polygons are congruent, then the polygons are similar.

*What is a counterexample to the **false** statement above?*

Draw two polygons that are *not* similar, but which *do* have all corresponding angles congruent.

Rectangles such as the ones below make good examples.



Note: All rectangles have congruent (right) angles. However, we saw in an earlier lesson that rectangles can have different shapes—long and narrow vs. stubby and square-ish. In formal terms, these rectangles have congruent angles, but their side lengths are obviously not proportional. The rectangles are not similar. Congruent angles are not enough to ensure similarity for rectangles.

The AA Rule for Similar Triangles

Some artists and designers apply the principle that "less is more." This idea has a place in geometry as well. Some geometry scholars feel that it is more satisfying to prove something with the least possible information. Similar triangles are a good example of this principle.

The AAA rule was developed for similar triangles earlier. Let's take another look at this rule, and see if we can reduce it to "less" rather than "more."

Suppose that triangles $\triangle ABC$ and $\triangle MNP$ have two pairs of congruent angles, say

$\angle A \cong \angle M$ and $\angle B \cong \angle N$.

But we know that if triangles have *two* pairs of congruent angles, then the *third* pair of angles are also congruent (by the Triangle Sum Theorem).

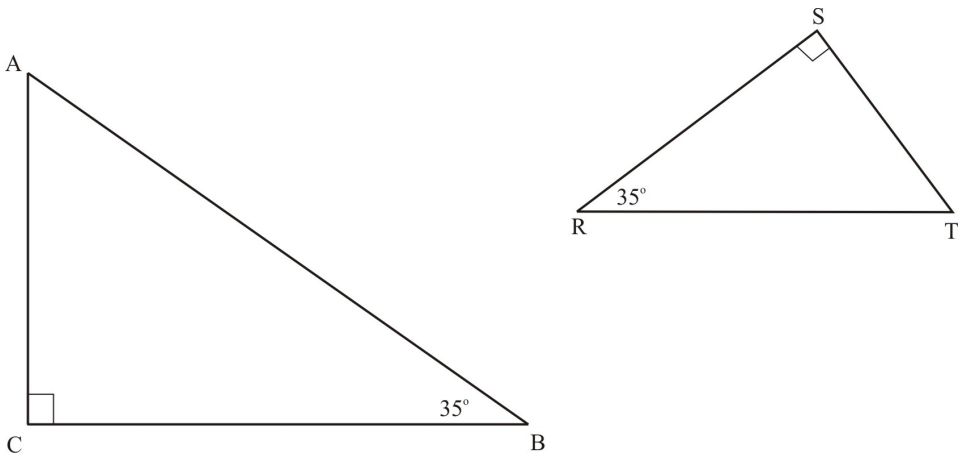
Summary: Less is more. The AAA rule for similar triangles reduces to the AA triangle similarity postulate.

The AA Triangle Similarity Postulate:

If two pairs of corresponding angles in two triangles are congruent, then the triangles are similar.

Example 2

Look at the diagram below.



A. *Are the triangles similar? Explain your answer.*

Yes. They both have congruent right angles, and they both have a 35° angle. The triangles are similar by AA.

B. *Write a similarity statement for the triangles.*

$\triangle ABC : \triangle TRS$ or equivalent

C. *Name all pairs of congruent angles.*

$\angle A \cong \angle T, \angle B \cong \angle R, \angle C \cong \angle S$

D. Write equations stating the proportional side lengths in the triangles.

$\frac{AB}{TR} = \frac{BC}{RS} = \frac{AC}{TS}$ or equivalent

Indirect Measurement

A traditional application of similar triangles is to measure lengths *indirectly*. The length to be measured would be some feature that was not easily accessible to a person. This length might be:

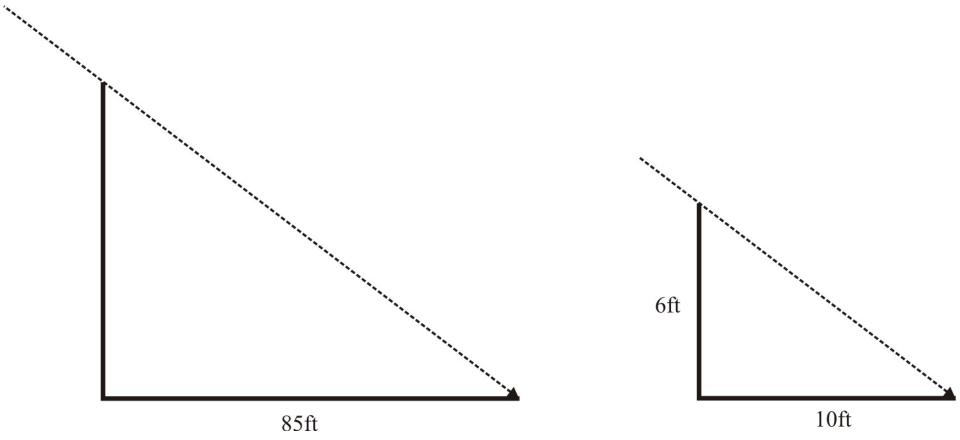
- the width of a river
- the height of a tall object
- the distance across a lake, canyon, etc.

To measure indirectly, a person would set up a pair of similar triangles. The triangles would have three known side lengths and the unknown length. Once it is clear that the triangles are similar, the unknown length can be calculated using proportions.

Example 3

Flo wants to measure the height of a windmill. She held a 6 foot *vertical pipe with its base touching the level ground, and the pipe's shadow was* 10 feet *long. At the same time, the shadow of the tower was* 85 feet *long. How tall is the tower?*

Draw a diagram.



Note: It is safe to assume that the sun's rays hit the ground at the same angle. It is also proper to assume that the tower is vertical (perpendicular to the ground).

The diagram shows two similar right triangles. They are similar because each has a right angle, and the angle where the sun's rays hit the ground is the same for both objects. We can write a proportion with only one unknown, x, the height of the tower.

$$\begin{aligned} \frac{x}{85} &= \frac{6}{10} \\ 10x &= 85 \cdot 6 \\ 10x &= 510 \\ x &= 51 \end{aligned}$$

Thus, the tower is 51 feet tall.

Note: This is method considered *indirect* measurement because it would be difficult to *directly* measure the height of tall tower. Imagine how difficult it would be to hold a tape measure up to a $51-$ foot $-$ tall tower.

Lesson Summary

The most basic way—because it requires the least input of information—to assure that triangles are similar is to show that they have two pairs of congruent angles. The AA postulate states this: If two triangles have two pairs of congruent angles, then the triangles are similar.

Once triangles are known to be similar, we can write many true proportions involving the lengths of their sides. These proportions were the basis for doing indirect measurement.

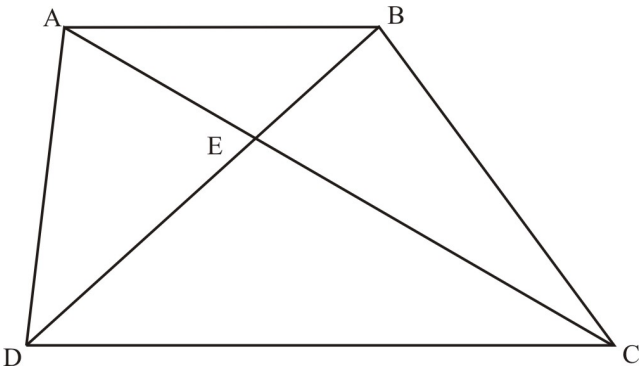
Points to Consider

Think about some right triangles for a minute. Suppose two right triangles both have an acute angle that measures 58°. Then the ratio $\frac{\text{length of long leg}}{\text{length of short leg}}$ is the same in both triangles. In fact, this ratio, called "the tangent of 58° " is the same in any right triangle with a 58° angle. As mentioned earlier, this is the reason for trigonometric functions of a given angle being constant, regardless of the specific triangle involved.

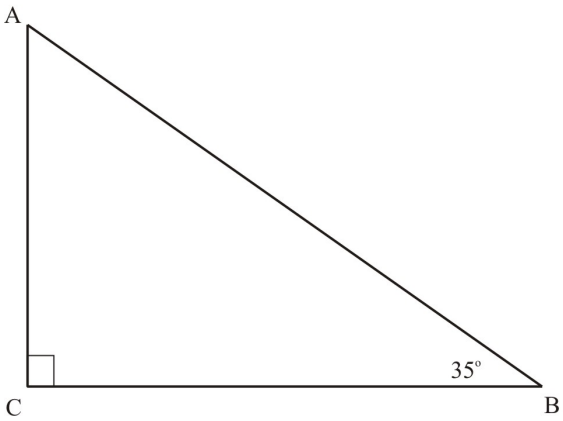
Review Questions

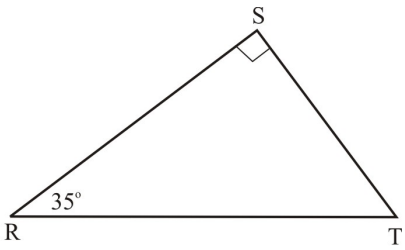
Use the diagram below for exercises 1-5.

Given that $\overline{AB} : \overline{DC}$



1. Name two similar triangles.
2. Explain how you know that the triangles you named in exercise 1 are similar.
3. Write a true proportion.
4. Name two triangles that might *not* be similar.
5. If $AB = 10, AE = 7$, and $DC = 22$, what is the length of $\overline{AC}$?
6. Given that $AB = 8$, $TR = 6$, and $BC = k$ in the diagram below





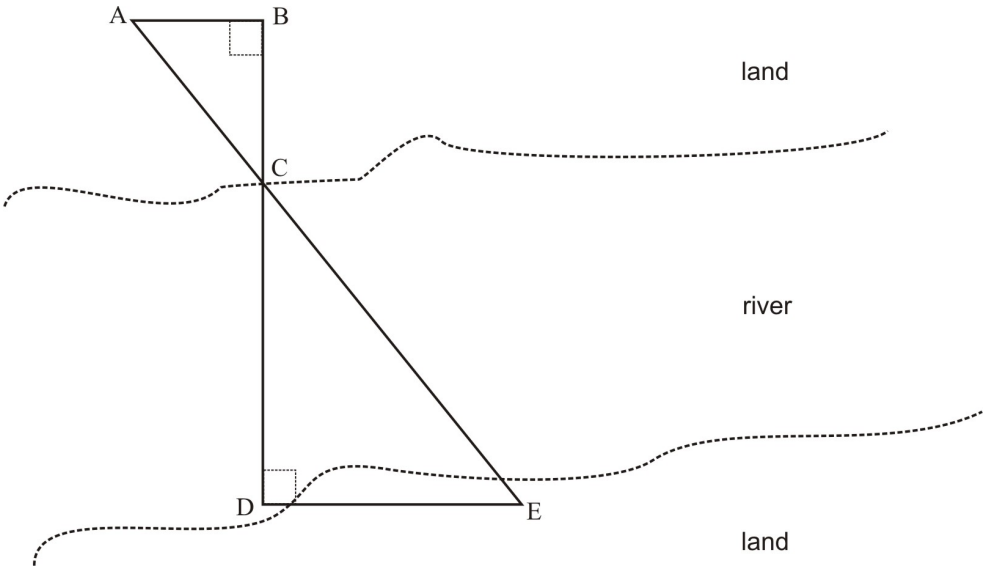
Write an expression for RS in terms of k.

7. Prove the following theorem:

If an acute angle of a right triangle is congruent to an acute angle of another right triangle, then the triangles are congruent.

Write a flow proof.

Use the following diagram for exercises 8-12.



In a geometry reality competition, the teams must estimate the width of the river shown in the diagram. Here's what they did.

- Anna, Bela, and Carlos stayed on the upper bank of the river.

- Darryl and Eva paddled across to the lower bank of the river.
- Carlos placed a marker at C.
- Darryl placed a marker directly across from Carlos at D.
- Bela walked 50 feet back from the bank in a line with the markers at C and D and placed a marker at B.
- Anna walked 30 feet on a path perpendicular to $\overline{BD}$ and placed a marker at A.
- Eva moved along the lower bank until she was lined up with A and C, and placed a marker at E.

$\overline{AB}, \overline{BC}$, and $\overline{DE}$ are on land, so they can be measured easily. DE was measured to be 80 feet.

8. Name two similar triangles.
9. Explain how you know that the triangles in exercise 8 are similar.
10. Write a proportion in which the only unknown measure is CD.
11. How wide is the river?
12. Discuss whether or not the triangles used to answer exercises 8-11 are good models for a river and its banks.

Review Answers

1. $\triangle ABE, \triangle CDE$ or equivalent
2. The triangles have two pairs of congruent alternate interior angles and one pair of congruent vertical angles. They are similar by AAA and AA.
3. Any proportion obtained from $\frac{AB}{CD} = \frac{BE}{DE} = \frac{AE}{CE}$
4. $\triangle AED, \triangle BEC$ or $\triangle ABD, \triangle BAC$ for example
5. $\frac{AB}{CD} = \frac{AE}{CE}$
   $\frac{10}{22} = \frac{7}{CE}$
   $CE = \frac{7 \times 22}{10} = \frac{154}{10} = 15.4$
   $AC = AE + CE = 7 + 15.4 = 22.4$
6. $RS = \frac{3}{4}k, RS = \frac{3k}{4}$
7. One acute angle in each triangle is congruent to an acute angle in the other triangle. Also, since they are right triangles, both triangles have a right angle, and these right angles are congruent. The triangles are congruent by AA.
8. $\triangle ABC : \triangle EDC$
9. $\angle B$ and $\angle D$ are congruent right angles. $\overline{AB} : \overline{DE}$, so $\angle A$ and $\angle E$ are congruent alternate interior angles. The triangles are similar by AA.
10. $\frac{AE}{BC} = \frac{DE}{CD}$
    $\frac{30}{50} = \frac{80}{CD}$
11. $\frac{30}{50} = \frac{80}{CD}$
    $30 \times CD = 50 \times 80 = 4000$
    $CD = \frac{4000}{30} \approx 133$The river is approximately 133 feet wide.

12. This seems to be a good model. The banks are roughly straight enough to be lines. The banks appear to be nearly parallel. If we can accept that parallel straight lines adequately represent the river banks, then the model is a good one.

7.5 Similarity by SSS and SAS

Learning Objectives

- Use SSS and SAS to determine whether triangles are similar.
- Apply SSS and SAS to solve problems about similar triangles.

Introduction

You have been using the AA postulate to work with similar triangles. AA is easy to state and to apply. In addition, there are other similarity postulates that should remind you of some of the congruence postulates. These are the SSS and SAS **similarity** postulates. These postulates will give us more tools for recognizing similar triangles and solving problems involving them.

Exploring SSS and SAS for Similar Triangles

We'll use geometry software and compass-and-straightedge constructions to explore relationships among triangles based on proportional side lengths and congruent angles.

SSS for Similar Triangles

Tech Note - Geometry Software

Use your geometry software to explore triangles with *proportional side lengths*. Try this.

1. Set up two triangles, $\triangle ABC$ and $\triangle MNP$, with each side length of $\triangle MNP$ being k times the length of the corresponding side of $\triangle ABC$.
2. Measure the **angles** of both triangles.
3. Record the results in a chart like the one below.

Repeat steps 1-3 for each value of k in the chart. Keep $\triangle ABC$ the same throughout the exploration.

Table 7.9: **Triangle Data**

	AB	BC	AC	$m\angle A$	$m\angle B$	$m\angle C$
	MN	NP	MP	$m\angle M$	$m\angle N$	$m\angle P$
$k = 2$						
$k = 5$						
$k = 0.6$						

- First, you know that all three side lengths in the two triangles are proportional. That's what it means for each side in $\triangle MNP$ to be k times the length of the corresponding side in $\triangle ABC$.

- You probably notice what happens with the angle measures in $\triangle MNP$. Each time you made a new triangle MNP for the given value of k, the measures of $\angle M, \angle N$, and $\angle P$ were approximately the same as the measures of $\angle A, \angle B$, and $\angle C$. Like before when we experimented with the AA and AAA relationships, there is something "automatic" that happens. If the *lengths of the sides of the triangles are proportional*, that "automatically" makes the angles in the two triangles congruent too. Of course, once we know that the angles are congruent, we also know that the triangles are similar by AAA or AA.

Hands-On Activity

Materials: Ruler/straightedge, compass, protractor, graph or plain paper.

Directions: Work with a partner in this activity. Each partner will use tools to draw a triangle.

Each partner can work on a sheet of graph paper or on plain paper. Make drawings as accurate as possible. *Note that it doesn't matter what unit of length you use.*

1. Partner 1: Draw a 6-8-10 triangle.
2. Partner 2: Draw a 9-12-15 triangle.
3. Partner 1: Measure the angles of your triangle.
4. Partner 2: Measure the angles of your triangle.
5. Partners 1 and 2: Compare your results.

What do you notice?

- First, you know that all three side lengths in the two triangles are proportional. $\frac{6}{9} = \frac{8}{12} = \frac{10}{15} \left(= \frac{2}{3}\right)$

- You also probably noticed that the angles in the two triangles are congruent. You might want to repeat the activity, drawing two triangles with proportional side lengths. You should find, again, that the angles in the triangles are automatically congruent.
- Once we know that the angles are congruent, then we know that the triangles are similar by AAA or AA.

SSS for Similar Triangles

Conclusion: *If the lengths of the sides of two triangles are proportional, then the triangles are similar.* This is known as SSS for similar triangles.

SAS for Similar Triangles

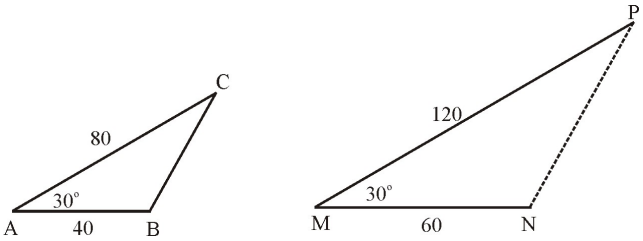
SAS for Similar Triangles

If the lengths of two corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar. This is known as SAS for similar triangles.

Example 1

Cheryl made the diagram below to investigate similar triangles more.

She drew $\triangle ABC$ first, with $AB = 40, AC = 80$, and $m\angle A = 30°$.



Then Cheryl did the following:

She drew $\overline{MN}$, and made $MN = 60$.

Then she carefully drew $\overline{MP}$, making $MP = 120$ and $m\angle M = 30°$.

At this point, Cheryl had drawn two segments ($\overline{MN}$ and $\overline{MP}$) with lengths that are proportional to the lengths of the corresponding sides of $\triangle ABC$, and she had made the included angle, $\angle M$, congruent to the included angle ($\angle A$) in $\triangle ABC$.

Then Cheryl measured angles. She found that:

- $\angle B \cong \angle N$
- $\angle C \cong \angle P$

What could Cheryl conclude? Here again we have automatic results. The other angles are automatically congruent, and the triangles are similar by AAA or AA. Cheryl's work supports the SAS for Similar Triangles Postulate.

Similar Triangles Summary

We've explored similar triangles extensively in several lessons. Let's summarize the conditions we've found that guarantee that two triangles are similar.

Two triangles are **similar** if and only if:

- the angles in the triangles are congruent.
- the lengths of corresponding sides in the polygons are proportional.

AAA: *If the angles of a triangle are congruent to the corresponding angles of another triangle, then the triangles are similar.*

AA: *It two pairs of corresponding angles in two triangles are congruent, then the triangles are similar.*

SSS for Similar Triangles: *If the lengths of the sides of two triangles are proportional, then the triangles are similar.*

SAS for Similar Triangles: *If the lengths of two corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar.*

Points to Consider

Have you ever made a model rocket? Have you seen a scale drawing? Do you know people who use blueprints? Do you enlarge pictures on your computer or shrink them? These are all examples of similar two-dimensional or three-dimensional objects.

Review Questions

Triangle 1 has sides with lengths 3 inches, 3 inches, and 4 inches.

Triangle 2 has sides with lengths 3 feet, 3 feet , and 4 feet.

1. Are Triangle 1 and Triangle 2 congruent? Explain your answer.
2. Are Triangle 1 and Triangle 2 similar? Explain your answer.
3. What is the scale factor from Triangle 1 to Triangle 2?
4. Why do we **not** study an ASA similarity postulate?

Use the chart below for exercises 4-4e.

Must $\triangle ABC$ and $\triangle MNP$ be similar?

Table 7.10:

	$m\angle A$	$m\angle B$	$m\angle C$	AB	BC	AC	$m\angle M$	$m\angle N$	$m\angle P$	MN	NP	MP
4a.				3	5	6				6	3	5
4b.	50°	40°					100°	80°				
4c.				8	4	10				10	6	12
4d.	63°			100		150		63°		20	30	
4e.			100°		24	15			110°	32	20	
4f.				30.0	20.0	32.0				22.5	15.0	24.0

5. **Hands-On Activity**

Materials: Ruler/straightedge, compass, protractor, graph or plain paper.

Directions: Work with a partner in this activity. Each partner will use tools to draw a triangle.

Each partner can work on a sheet of graph paper or on plain paper. Make drawings as accurate as possible. *Note that it doesn't matter what unit of length you use.*

Partner 1: Draw $\triangle ABC$ with $AB = 20$, $m\angle A = 40°$, and $AC = 30$

Partner 2: Draw $\triangle MNP$ with $MN = 30$, $\angle M = 40°$, and $MP = 45$.

A. Are sides $\overline{AB}$, $\overline{AC}$, $\overline{MN}$, and $\overline{MP}$ proportional? $\frac{20}{30} = \frac{30}{45} = \frac{2}{3}$

Partner 1: Measure the other angles of your triangle.

Partner 2: Measure the other angles of your triangle.

Partners 1 and 2: Compare your results.

B. Are the other angles of the two triangles (approximately) congruent?

C. Are the triangles similar? If they are, write a similarity statement and explain how you know that the triangles are similar. $\triangle ABC : \triangle MNP$.

Review Answers

1. No. One is much larger than the other.
2. Yes, SSS. The side lengths are proportional.
3. 12

4. There is no need. With the A and A parts of ASA we have triangles with two congruent angles. The triangles are similar by AA. 4a. Yes
   4b. No
   4c. No
   4d. Yes
   4e. No
   4f. Yes
5. (a) Yes
   (b) Yes
   (c) Yes. All three pairs of angles are congruent, so the triangles are similar by AAA or AA.

7.6 Proportionality Relationships

Learning Objectives

- Identify proportional segments when two sides of a triangle are cut by a segment parallel to the third side.
- Divide a segment into any given number of congruent parts.

Introduction

We'll wind up our study of similar triangles in this section. We will also extend some basic facts about similar triangles to dividing segments.

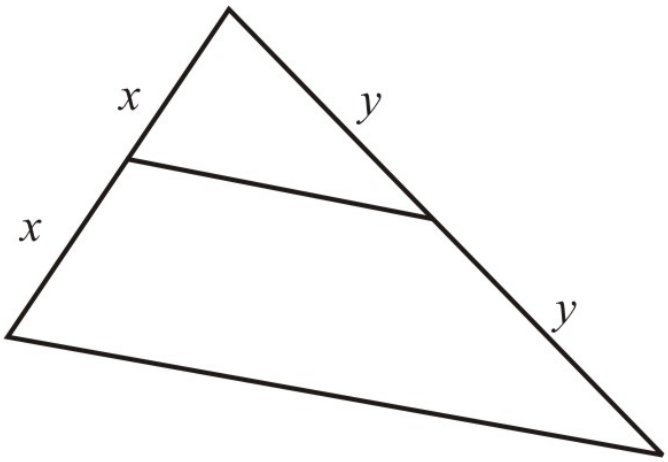
Dividing Sides of Triangles Proportionally

Think about a midsegment of a triangle. A midsegment is parallel to one side of a triangle, and that it divides the other two sides into congruent halves (because the midsegment connects the *midpoints* of those two sides). So the midsegment divides those two sides proportionally.

Example 1

Explain the meaning of "the midsegment divides the sides of a triangle proportionally."

Suppose each half of one side of a triangle is x units long, and each half of the other side is y units long.



One side is divided in the ratio $x : x$, the other side in the ratio $y : y$ Both of these ratios are equivalent to $1 : 1$ and to each other.

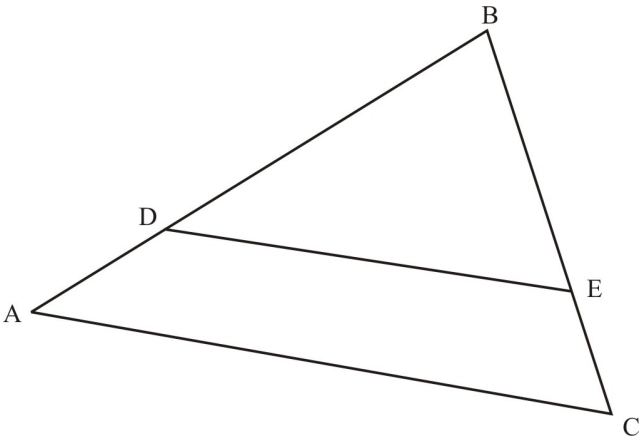
We see that a midsegment divides two sides of a triangle proportionally. But what about some other segment?

Tech Note - Geometry Software

Use your geometry software to explore triangles where a *line parallel to one side intersects the other two sides*. Try this:

1. Set up $\triangle ABC$.
2. Draw a line that is parallel to $\overline{AC}$ and that intersects both of the other sides of $\triangle ABC$.
3. Label the intersection point on $\overline{AB}$ as D; label the intersection point on $\overline{CB}$ as E.

Your triangle will look something like this.



DE parallel to AC

4. Measure lengths and calculate the following ratios.

$\frac{AD}{DB}$ = _____ and $\frac{CE}{EB}$ = _____

5. Compare your results with those of other students.

Different students can start with different triangles. They can draw different lines parallel to $\overline{AC}$. But in each case the two ratios, $\frac{AD}{DB}$ and $\frac{CE}{EB}$, are approximately the same. This is another way to say that the two sides of the triangle are divided proportionally. We can prove this result as a theorem.

Triangle Proportionality Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides into proportional segments.

Proof.

- Given: $\triangle ABC$ with $\overline{DE} : \overline{AC}$
- Prove: $\frac{AD}{DN} = \frac{CE}{EB}$

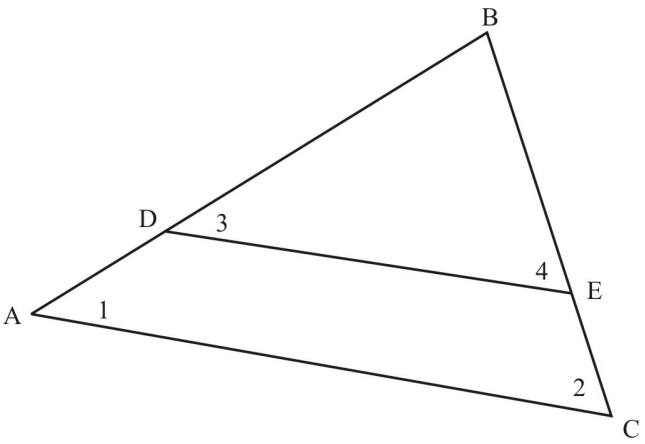


Table 7.11:

Statement	Reason
1. $\overline{DE} : \overline{AC}$	1. Given
2. $\angle 1 \cong \angle 3, \angle 2 \cong \angle 4$	2. Corresponding angles are congruent
3. $\triangle ABC : \triangle DBE$	3. AA Similarity Postulate
4. $AD + DB = AB, CE + EB = CB$	4. Segment addition postulate
5. $\frac{AD}{DB} = \frac{CE}{EB}$	5. Corresponding side lengths in similar triangles are proportional
6. $\frac{AD+DB}{DB} = \frac{CE+EB}{EB}$	6. Substitution
7. $\frac{AD+DB}{DB} = \frac{AD}{DB} + \frac{DB}{DB} = \frac{AD}{DB} + 1$ $\frac{CE+EB}{EB} = \frac{CE}{EB} + \frac{EB}{EB} = \frac{CE}{EB} + 1$	7. Algebra
8. $\frac{AD}{DB} + 1 = \frac{CE}{EB} + 1$	8. Substitution

Table 7.11: (continued)

Statement	Reason
9. $\frac{AD}{DB} = \frac{CE}{EB}$	9. Addition property of equality ♦

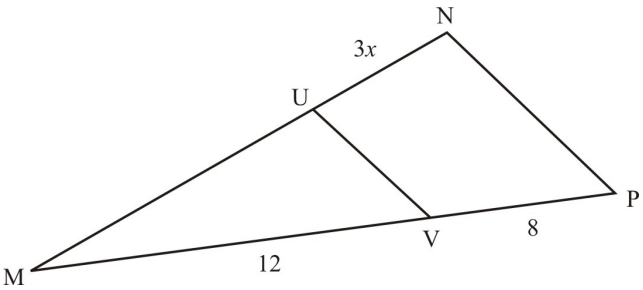
Can you see why we wrote the proportion this way, rather than as $\frac{DB}{AD+DB} = \frac{EB}{CE+EB}$, which is also a true proportion?

It's because $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$, but there is no similar way to simplify $\frac{r}{s+t}$.

Note: The converse of this theorem is also true. If a line divides two sides of a triangle into proportional segments, then the line is parallel to the third side of the triangle.

Example 2

In the diagram below, $UV : NP = 3 : 5$.



What is an expression in terms of x for the length of $\overline{MN}$?

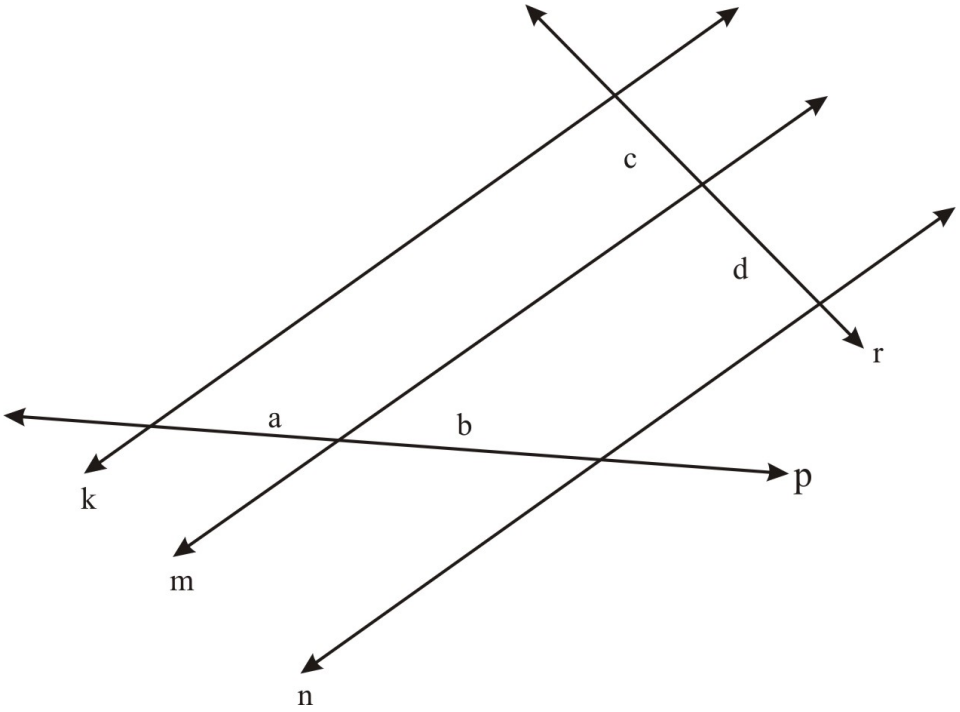
According to the Triangle Proportionality Theorem,

$$\begin{aligned} \frac{3}{5} &= \frac{MU}{MU+3x} \\ 3MU + 9x &= 5MU \\ 2MU &= 9x \\ MU &= \frac{9x}{2} = 4.5x \\ MN &= MU + UN = 4.5x + 3x \\ MN &= 7.5x \end{aligned}$$

There are some very interesting corollaries to the Triangle Proportionality Theorem. One could be called the *Lined Notebook Paper Corollary*!

Example 3

Look at the diagram below. We can make a corollary to the previous theorem.



$k,\ m,\ n,\ p,\ r$ are labels for lines

$a,\ b,\ c,\ d$ are lengths of segments

$k,\ m,\ n$ are parallel but not equally spaced

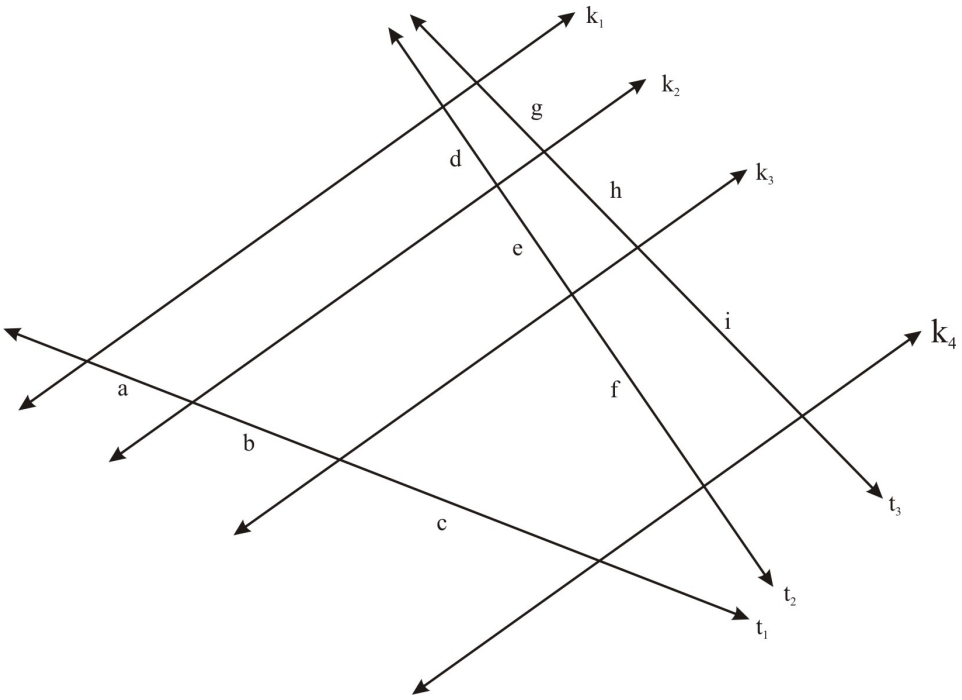
We're given that lines k, m, and n are parallel. We can see that the parallel lines cut lines p and r (transversals). A corollary to the Triangle Proportionality Theorem states that the segment lengths on one transversal are proportional to the segment lengths on the other transversal.

Conclusion: $\frac{a}{b} = \frac{c}{d}$ **and** $\frac{a}{c} = \frac{b}{d}$

Example 4

The corollary in example 3 can be broadened to ***any*** *number of parallel lines that cut* ***any*** *number of transversals. When this happens,* ***all*** *corresponding segments of the transversals are proportional*!

The diagram below shows several parallel lines, k_1, k_2, k_3 and k_4, that cut several transversals t_1, t_2, and t_3.



k lines are all parallel.

Now we have lots of proportional segments.

For example:

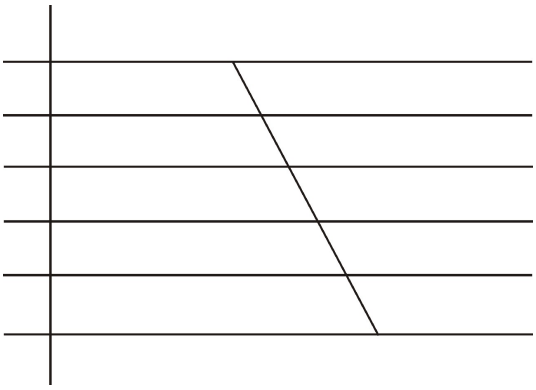
$\frac{a}{b} = \frac{d}{e}, \frac{a}{c} = \frac{g}{i}, \frac{b}{h} = \frac{a}{g}, \frac{c}{f} = \frac{b}{e}$, and many more.

This corollary extends to more parallel lines cutting more transversals.

Lined Notebook Paper Corollary

Think about a sheet of lined notebook paper. A sheet has numerous equally spaced horizontal parallel segments; these are the lines a person can write on. And there is a vertical segment running down the left side of the sheet. This is the segment setting the margin, so you don't write all the way to the edge of the paper.

Now suppose we draw a slanted segment on the sheet of lined paper.



Because the vertical margin segment is divided into congruent parts, then the slanted segment is also divided into congruent segments. This is the Lined Notebook Paper Corollary.

What we've done here is to divide the slanted segment into five congruent parts. By placing the slanted segment differently we could divide it into any given number of congruent parts.

History Note

In ancient times, mathematicians were interested in bisecting and trisecting angles and segments. Bisection was no problem. They were able to use basic geometry to bisect angles and segments.

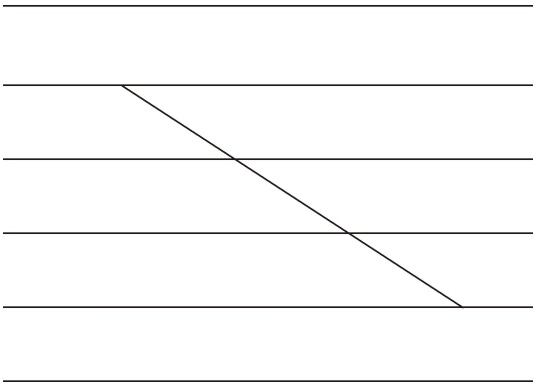
But what about trisection dividing an angle or segment into exactly three congruent parts? This was a real challenge! In fact, ancient Greek geometers proved that an **angle** cannot be trisected using only compass and straightedge.

With the Lined Notebook Paper Corollary, though, we have an easy way to trisect a given **segment**.

Example 5

Trisect the segment below.

Draw equally spaced horizontal lines like lined notebook paper. Then place the segment onto the horizontal lines so that its endpoints are on two horizontal lines that are three spaces apart.



- slanted segment is same length as segment above picture
- endpoints are on the horizontal segments shown
- slanted segment is divided into three congruent parts

The horizontal lines now trisect the segment. We could use the same method to divide a segment into any required number of congruent smaller segments.

Lesson Summary

In this lesson we began with the basic facts about similar triangles the definition and the SSS and SAS properties. Then we built on those to create numerous proportional relationships. First we examined proportional sides in triangles, then we extended that concept to dividing segments into proportional parts. We finalized those ideas with a notebook paper property that gave us a way to divide a segment into any given number of equal parts.

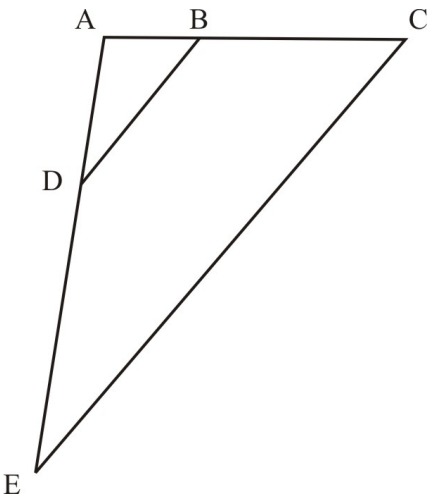
Points to Consider

Earlier in this book you studied congruence transformations. These are transformations in which the image is congruent to the original figure. You found that translations (slides), rotations (turns), and reflections (flips) are all congruence transformations. In the next lesson we'll study *similarity* transformations transformations in which the image is *similar* to the original figure. We'll focus on *dilations*. These are figures that we zoom in on, or zoom out on. The idea is very similar to blowing up or shrinking a photo before printing it.

Review Questions

Use the diagram below for exercises 1-5.

Given that $\overline{DB} : \overline{EC}$

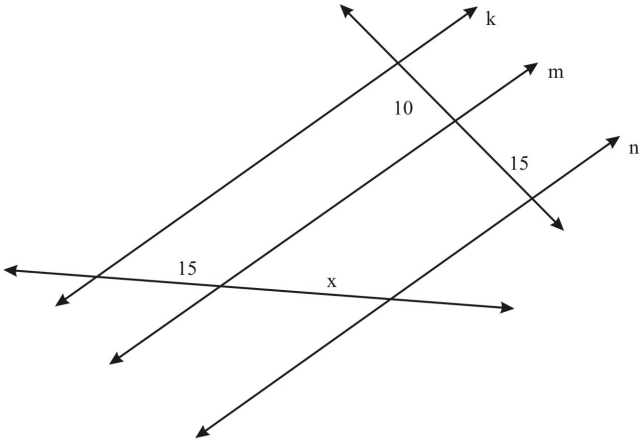


1. Name similar triangles.

Complete the proportion.

2. $\frac{AB}{BC} = \frac{?}{DE}$
3. $\frac{AB}{AD} = \frac{?}{DE}$
4. $\frac{AB}{AC} = \frac{AD}{?}$
5. $\frac{AC}{AE} = \frac{BC}{?}$

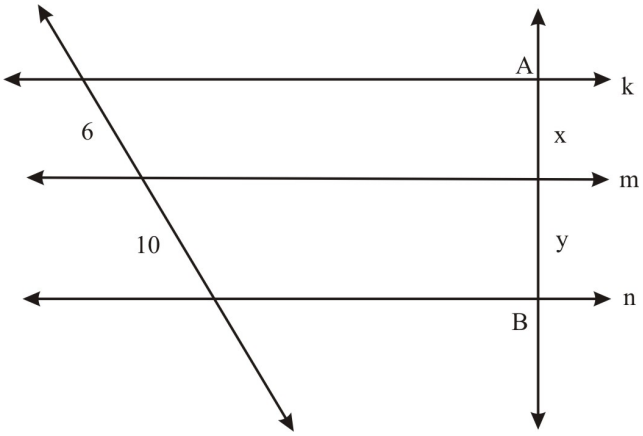
Lines k, m, and n are parallel.



k, m, and n parallel; x = 22.5; lengths proportional to indicated

6. What is the value of x?

Lines k, m, and n are parallel, and $AB = 30$.



7. What is the value of x?
8. What is the value of y?
9. Explain how to divide a segment into seven congruent segments using the Lined Notebook Paper Corollary.

Review Answers

1. $\triangle ABD : \triangle ACE$ or equivalent
2. AD
3. BC
4. AE
5. DE
6. 22.5
7. 11.25
8. 18.75
9. Place the original segment so that one endpoint is on the top horizontal line. Slant the segment so that the other endpoint is on the seventh horizontal line below the top line. These eight horizontal lines divide the original segment into seven congruent smaller segments.

7.7 Similarity Transformations

Learning Objectives

- Draw a dilation of a given figure.
- Plot the image of a point when given the center of dilation and scale factor.

- Recognize the significance of the scale factor of a dilation.

Introduction

Earlier you studied one group of transformations that "preserve" length. This means that the image of a segment is a congruent segment. These **congruence transformations** are translations, reflections, and rotations.

In this lesson, you'll study one more kind of transformation, the *dilation*. Dilations do not preserve length, meaning the image of a segment can be a segment that is not congruent to the original. You'll see that the image of a figure in a dilation is a similar, not necessarily congruent, figure.

Dilations

A dilation is like a "blow-up" of a photo to change its size. A dilation may make a figure larger, or smaller, but the same shape as the original. In other words, as you'll see, a dilation gives us a figure similar to the original.

A **dilation** is a transformation that has a **center** and a **scale factor**. The center is a point and the scale factor governs how much the figure stretches or shrinks.

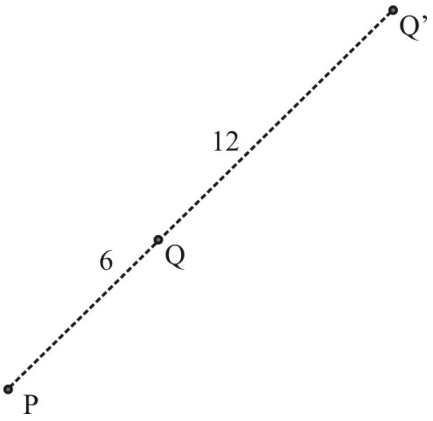
Think about watching a round balloon being inflated, and focusing on the point exactly in the middle of the balloon. The balloon stretches outwards from this point uniformly. So for example, if a circle is drawn around the point, this circle will grow as the balloon stretches away from the points.

Dilation with center at point P and scale factor k, $k > 0$

Given a point Q that is d units from point P. The image of Q for this dilation is the point Q' that is collinear with P and Q and kd units from P, the center of dilation.

Example 1

The center of dilation is P, and the scale factor is 3.

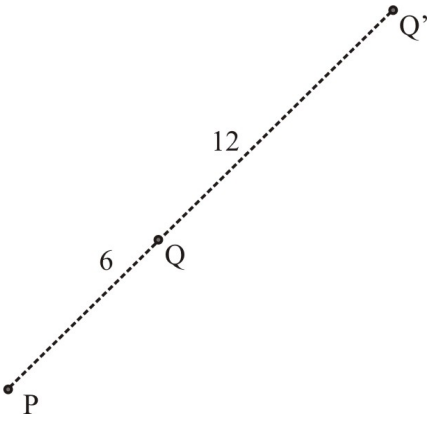


Point Q is 6 units from P. To find the image of point Q, we go $3 \times 6 = 18$ units from P along $\overline{PQ}$ to locate Q', the image of Q. Point Q' is three times as far (18 units) from P as Q is (6 units), and P, Q, and Q' are collinear.

Note: The scale factor is 3. The length from P to Q' is *"stretched"* three times as long as the length from P to Q.

Example 2

The center of dilation is P, and the scale factor is $\frac{1}{3}$.



Point Q is 6 units from P, as in example 1. To find the image of point Q, we go $\frac{1}{3} \times 6 = 2$ units from P along $\overline{PQ}$ to locate Q', the image of Q. Point Q' is $\frac{1}{3}$ times as far (2 units) from P as Q is (6 units), and P, Q, and Q' are collinear.

Note: The scale factor is $\frac{1}{3}$. The length from P to Q' *"shrinks"* to $\frac{1}{3}$ times as long as the length from P to Q.

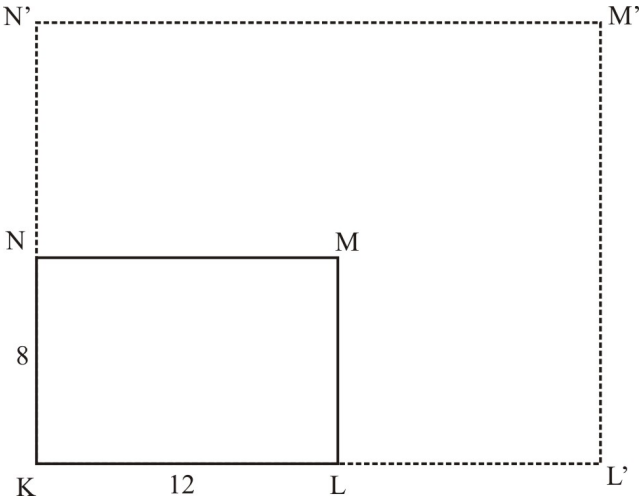
Example 3

$KLMN$ is a rectangle. What are the length, width, perimeter, and area of $KLMN$?

Length $= KL = 12$ Width $= KN = 8$

Perimeter $= 12 + 8 + 12 + 8 = 40$ Area $=$ length $\times$ width $= 12 \cdot 8 = 96$

The center of a dilation is K, and the scale factor is 2. What are the length, width, perimeter, and area of $K'L'M'N'$?



Point K' is the same as point K. LL' is 12, and NN' is 8.

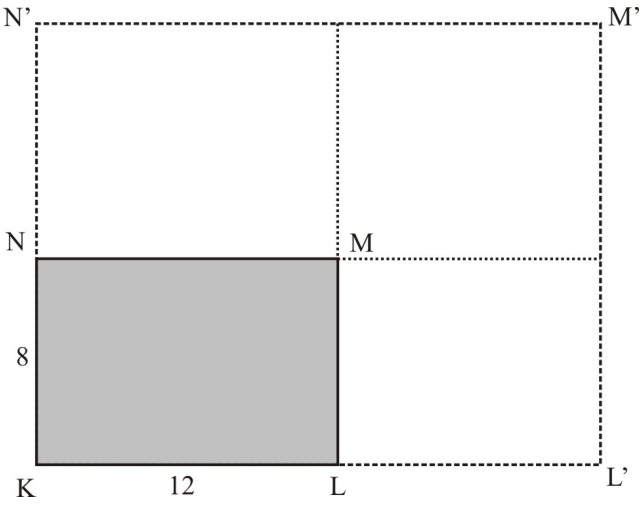
In $K'L'M'N'$:

Length $= KL' = 24$ Width $= KN' = 16$

Perimeter $= 24 + 16 + 24 + 16 = 80$ Area $=$ length $\times$ width $= 24 \cdot 16 = 384$

Note: The perimeter of $K'L'M'N'$ is 2 times the perimeter of $KLMN$, but the area of $K'L'M'N'$ is 4 times the area of $KLMN$.

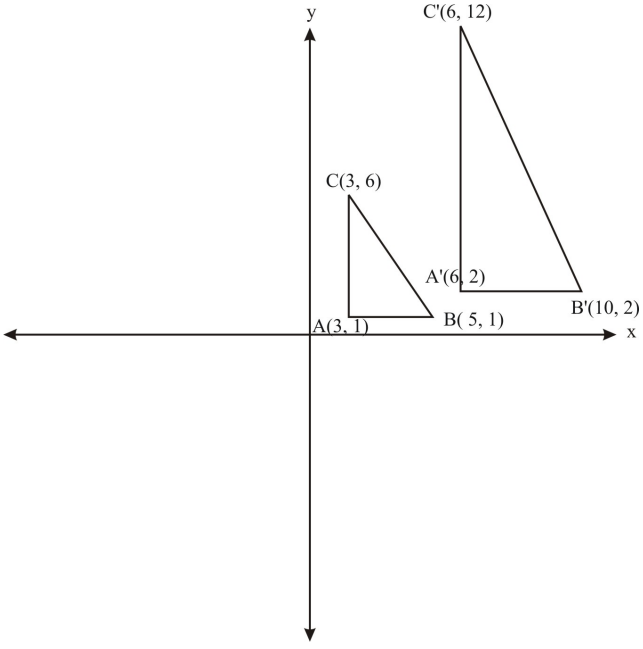
As the following diagram shows, four rectangles congruent to $KLMN$ fit exactly into $K'L'M'N'$.



Coordinate Notation for Dilations

We can work with dilations on a coordinate grid. To simplify our work, we'll study dilations that have their center of dilation at the origin.

Triangle ABC in the diagram below is dilated with scale factor 2.



Triangle $A'B'C'$ is the image of $\triangle ABC$.

Notice that each side of $\triangle A'B'C'$ is 2 times as long as the corresponding side of $\triangle ABC$. Notice also that A, A', and the origin are collinear. Thus is also true of B, B', and the origin,

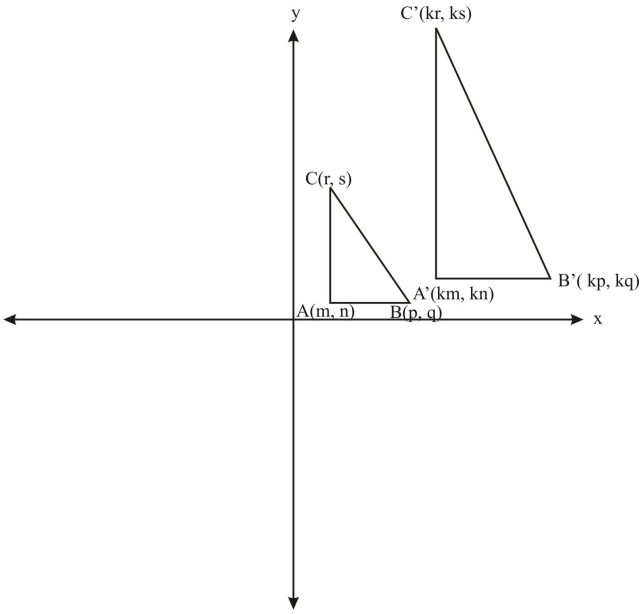
and of C, C', and the origin.

This leads to the following generalization

Generalization: Points P, P' (the image of P), and the origin are collinear for any point P in a dilation. You can prove the generalization in the Lesson Exercises.

How do we know that a dilation is a similarity transformation? We would have to establish that lengths of segments are proportional and that angles are congruent. Let's attack these requirements through the distance formula and slopes.

Let $A(m, n), B(p, q)$, and $C(r, s)$ be points in a coordinate grid. Let a dilation have center at the origin and scale factor k.



Part 1: Proportional Side Lengths

Let's look at the lengths of two segments, $\overline{AB}$, and $\overline{A'B'}$.

According to the distance formula,

$$AB = \sqrt{(p-m)^2 + (q-n)^2}$$

and

$$\begin{aligned} A'B' &= \sqrt{(kp-km)^2+(kq-kn)^2} \\ &= \sqrt{(k(p-m))^2+(k(q-n))^2} \\ &= \sqrt{k^2(p-m)^2+k^2(q-n)^2} \\ &= \sqrt{k^2((p-m)^2+(q-n)^2)} \\ &= k\sqrt{(p-m)^2+(q-n)^2)} \\ &= kAB \end{aligned}$$

What does this say about a segment and its image in a dilation? It says that the image of a segment is another segment k times the length of the original segment. If a polygon had several sides, each side of the image polygon would be k times the length of its corresponding side in the original polygon.

Conclusion: If a polygon is dilated, the corresponding sides of the image polygon and the original polygon are proportional. So half the battle is over.

Part 2: Congruent Angles

Let's look at the slopes of the sides of two angles, $\angle CAB$ and $\angle C'A'B'$.

slope of $\overline{AC} = \frac{s-n}{r-m}$

slope of $\overline{AB} = \frac{q-n}{p-m}$

slope of $\overline{A'C'} = \frac{ks-kn}{kr-km} = \frac{k(s-n)}{k(r-m)} = \frac{s-n}{r-m}$

slope of $\overline{A'B'} = \frac{kq-kn}{kp-km} = \frac{k(q-n)}{k(p-m)} = \frac{q-n}{p-m}$

Since $\overline{AC}$ and $\overline{A'C'}$ have the same slope, they are parallel. The same is true for $\overline{AB}$ and $\overline{A'B'}$. We know that if the sides of two angles are parallel, then the angles are congruent. This gives us: $\angle CAB \cong \angle C'A'B'$

Conclusion: If a polygon is dilated, the corresponding angles of the image polygon and the original polygon are congruent. So the battle is now over.

Final Conclusion: If a polygon is dilated, the original polygon and the image polygon are similar, because they have proportional side lengths and congruent angles. *A dilation is a similarity transformation.*

Lesson Summary

Dilations round out our study of geometric transformations. Unlike translations, rotations, and reflections, dilations are not congruence transformations. They are similarity transformations. If a dilation is applied to a polygon, the image is a similar polygon.

Points to Consider

We limited our study of dilations to those that have positive scale factors. To explore further, you might experiment with *negative* scale factors.

Tech Note - Geometry Software

Use your geometry software to explore dilations with negative scale factors.

Exploration 1

- Plot two points.
- Select one of the points as the center of dilation.
- Use 2 for the scale factor.
- Find the image of the other point.

Repeat, but use a different value for the scale factor.

What seems to be true about the two images?

Exploration 2

- Draw a triangle.
- Select a point as the center of dilation. Use one vertex of the triangle, or draw another point for the center.
- Use 2 for the scale factor.
- Find the image of the triangle.
- Repeat, but use a different value for the scale factor.

What seems to be true about the two images?

You can experiment further with different figures, centers, and scale factors.

Can you reach any conclusions about images when the scale factor is negative?

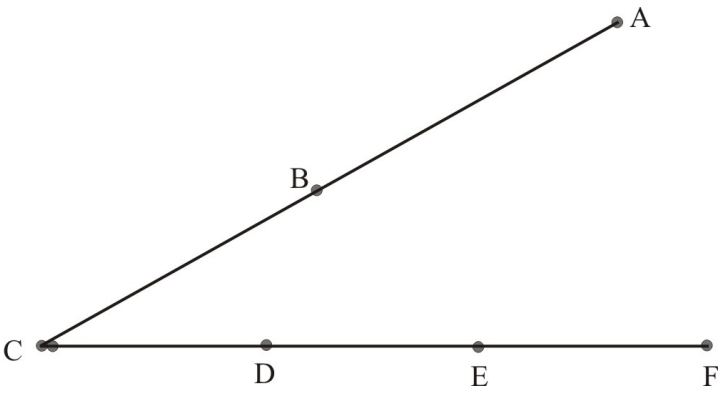
You may have noticed that if point A is dilated, the center is B, and the scale factor is $k, k > 0$, then the image of A is on the same side of B as A is. If the scale factor is $-k$ then the image of A is on the opposite side of B. You may have also also noticed that a dilation with a negative scale factor is equivalent to a dilation with a positive scale factor followed by a "reflection in a point," where the point is the center of dilation.

This lesson brings our study of similar figures almost to a close. We'll revisit similar figures once more in Chapter 10, where we analyze the perimeter and area of similar polygons. Some writers have used similarity concepts to explain why living things are the "right size" and why, for example, there are no 20 foot − tall human giants!

Review Questions

Use the diagram below for exercises 1 - 10.

$AB = BC = 30$ and $CD = DE = EF = 20$

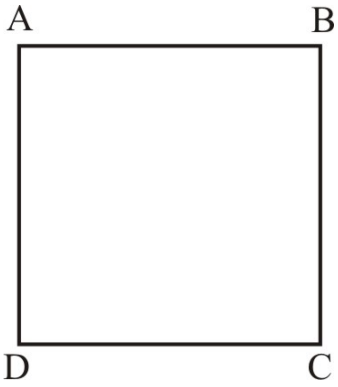


A dilation has the indicated center and scale factor. Complete the table.

Table 7.12:

Center	Scale Factor	Given Point	Image of Given Point
C	2	B	?
A	0.5	C	?
C	3	D	?
E	2	?	C
F	$\frac{1}{3}$	C	?
B	1	?	A
C	$\frac{2}{3}$	F	?
C	?	E	D
C	?	A	Midpoint of $\overline{AB}$
F	$\frac{5}{6}$	?	Midpoint of $\overline{CD}$

2. Copy the square shown below. Draw the image of the square for a dilation with center at the intersection of $\overline{AC}$ and $\overline{BD}$ scale factor 2.



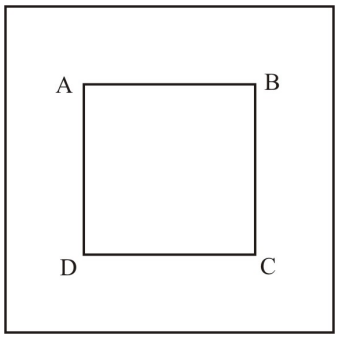
3. A given dilation is a *congruence* transformation. What is the scale factor of the dilation?
4. Imagine a dilation with a scale factor of 0. Describe the image of a given point for this dilation.
5. Let $A(m, n)$ and $B(km, kn)$ be two points in a coordinate grid, $m \neq 0$. Prove that A and B are collinear.
6. A dilation has its center at the origin and a scale factor of 3. Let A be the point . If A' is the image of A, and A'' is the image of A', what are the coordinates of A''?

Review Answers

Table 7.13:

Center	Scale Factor	Given Point	Image of Given Point	Answer
C	2	B	?	A
A	0.5	C	?	B
C	3	D	?	F
E	2	?	C	D
F	$\frac{1}{3}$	C	?	E
B	1	?	A	A
C	$\frac{2}{3}$	F	?	E
C	?	E	D	0.5 or $\frac{1}{2}$
C	?	A	Midpoint of $\overline{AB}$	0.75 or $\frac{3}{4}$
F	$\frac{5}{6}$	?	Midpoint of $\overline{CD}$	C

2. Small square centered in big square, each side of big square 2 times side of small square



3. 1
4. The image of any point is the point that is the center of dilation.
5. Let O be the origin $(0, 0)$.

$$\text{slope of } \overline{OA} = \frac{n-0}{m-0} = \frac{n}{m}$$
$$\text{slope of } \overline{OB} = \frac{kn-0}{km-0} = \frac{kn}{km} = \frac{n}{m}$$
$$\text{slope of } \overline{AB} = \frac{kn-n}{km-m} = \frac{n(k-1)}{m(k-1)} = \frac{n}{m}, k \neq 1$$

Since the segments have common endpoints and the same slope, they are collinear.

6. $(45, -18)$

7.8 Self-Similarity (Fractals)

Learning Objectives

- Appreciate the concept of self-similarity.
- Extend the pattern in a self-similar figure.

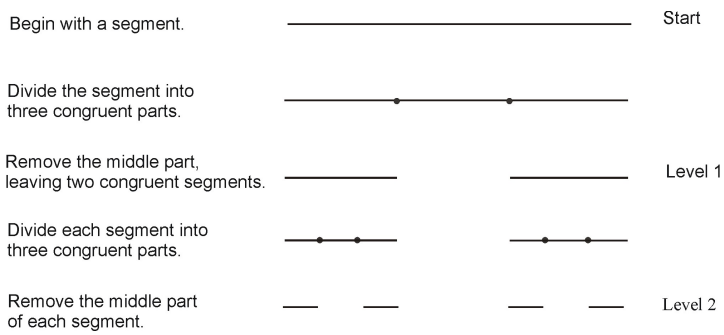
Introduction

In this lesson you'll learn about patterns called **fractals** that have self-similarity. Instead of using a formal definition, we'll work with a few examples that give the idea of self-similarity. In each example you will be able to see that later stages in a pattern have a similarity relationship to the original figure.

Example 1

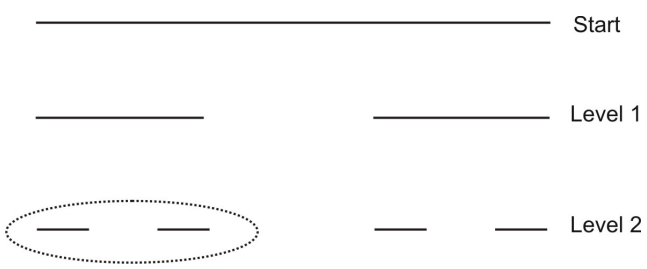
The Cantor Set

The pattern in the diagram below is called the **Cantor Set**, named for a creative mathematician of the late 1800s.



The pattern continues. Now let's see why this pattern is called "self-similar."

Look at the circled part of the pattern.

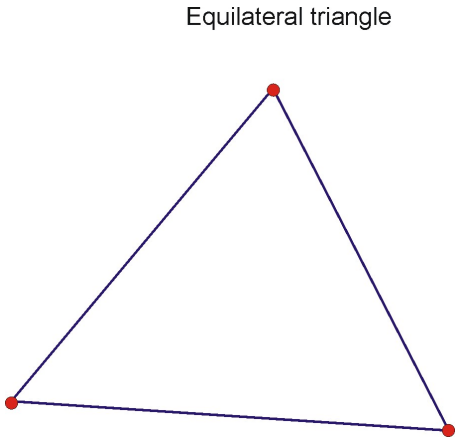


You can see that each part of Level 2 is similar to Level 1 with a scale factor of $\frac{1}{3}$. The same relationship continues as each level is created from the level before it.

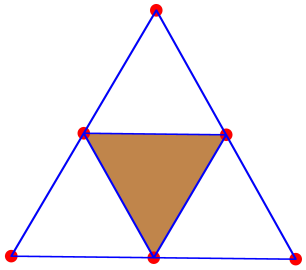
Example 2

Sierpinski Triangle

To construct a Sierpinski Triangle, begin with an equilateral triangle. (Actually, any triangle could be used.) This is the Start level.



Then connect the midpoints of the sides of the triangle. Shade in the central triangle.



This is Level 1.

Now repeat this process to create Level 2:

- Connect the midpoints of the sides of each unshaded triangle to form smaller triangles.
- Shade in each central triangle.

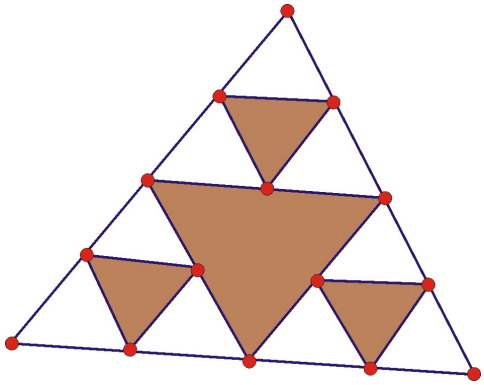


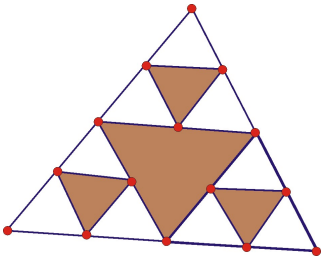
Figure 7.1

This is Level 2.

The pattern continues, as shown below:

To view some great examples of Sierpinski Triangles visit the following link:

Now let's see how the Sierpinski Triangle is self-similar.



Look at the triangle that is outlined in the diagram above. Could you prove that the outlined pattern is similar to the pattern of Level 1? Because of this relationship, the Sierpinski Triangle is self-similar.

Tech Note - Geometry Software

Use geometry software to create the next level, or levels, of the Sierpinski Triangle.

Lesson Summary

Fractals and self-similarity are fairly recent developments in geometry. The patterns are interesting on their own, and they have been found to have applications in the study of many natural and human-made fields. Successive levels of a fractal pattern are all similar to the preceding levels.

Points to Consider

You may want to learn more about fractals. Use a search engine to find information about fractals on the Internet.

Review Questions

Use the Cantor Set to answer questions 1-6.

1. How many segments are there in Level 3?
2. If the segment in the Start level is 1 unit long, how long is each segment in Level 2?
3. How many segments are there in Level 4?
4. How many segments are there in Level 10?
5. How many segments are there in Level n?
6. If the segment in the Start level is S units long, how long is each segment in Level n?

Use the Sierpinski Triangle to answer questions 7-13.

7. How many unshaded triangles are there in Level 2?
8. How many unshaded triangles are there in Level 3?
9. How many unshaded triangles are there in Level n?

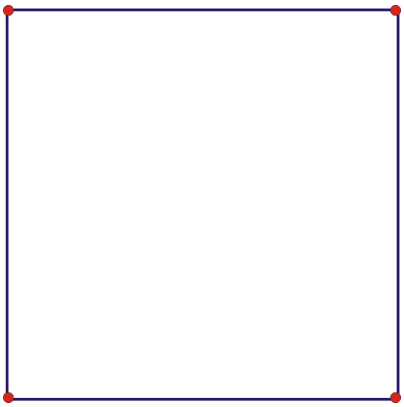
Suppose the area of the Start level triangle is 1.

10. What is the total area of the unshaded part of Level 1?

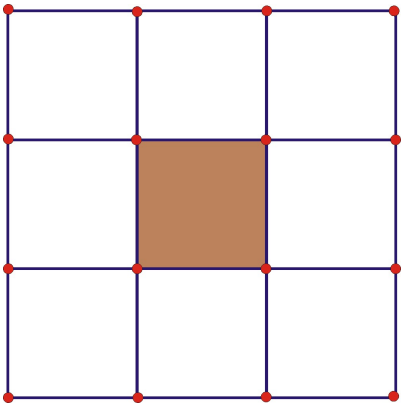
11. What is the total area of the unshaded part of Level 2?
12. What is the total area of the unshaded part of Level n?
13. Explain how you know that the outlined part of Level 2 is similar to Level 1.

Tech Note - Geometry Software

Use geometry software to create the next level of the fractal pattern shown below.



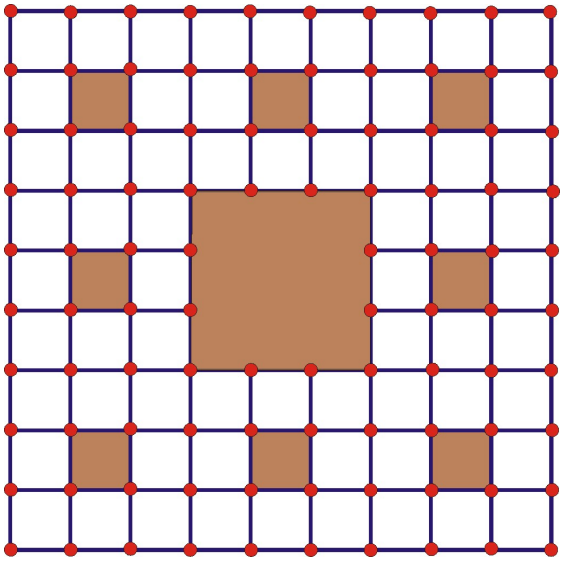
all lines should be straight and all angles right



all lines should be straight and all angles right

Review Answers

1. 8
2. $\frac{1}{9}$
3. 16
4. 1024
5. 2^n
6. $\frac{S}{3^n}$

7. 9
8. 27
9. 3^n
10. $\frac{3}{4}$
11. $\frac{9}{16}$
12. $\left(\frac{3}{4}\right)^2$
13. The midsegments of a triangle divide it into four congruent triangles, each of which is similar to the original triangle.
14. 

All lines should be straight and all angles right.

Image Sources

(1) *Sierpinski Triangle*.

Chapter 8

Right Triangle Trigonometry

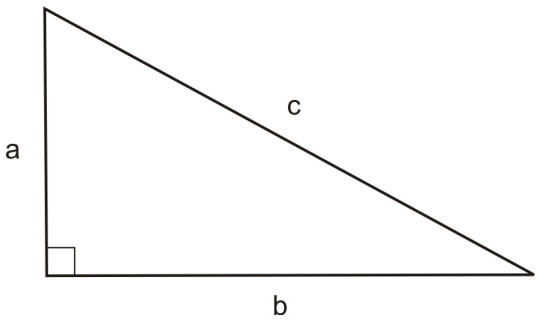
8.1 The Pythagorean Theorem

Learning Objectives

- Identify and employ the Pythagorean Theorem when working with right triangles.
- Identify common Pythagorean triples.
- Use the Pythagorean Theorem to find the area of isosceles triangles.
- Use the Pythagorean Theorem to derive the distance formula on a coordinate grid.

Introduction

The triangle below is a right triangle.



The sides labeled a and b are called the **legs** of the triangle and they meet at the right angle. The third side, labeled c is called the **hypotenuse.** The hypotenuse is opposite the right angle. The hypotenuse of a right triangle is also the longest side.

The Pythagorean Theorem states that the length of the hypotenuse squared will equal the

sum of the squares of the lengths of the two legs. In the triangle above, the sum of the squares of the legs is $a^2 + b^2$ and the square of the hypotenuse is c^2.

The Pythagorean Theorem: Given a right triangle with legs whose lengths are a and b and a hypotenuse of length c,

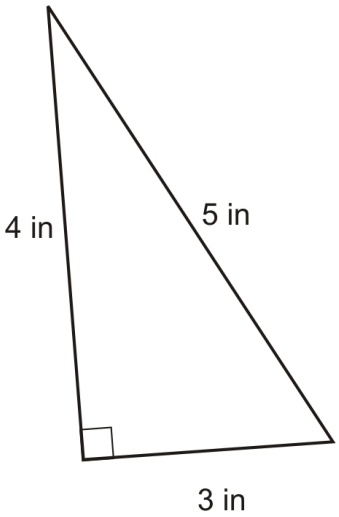
$$a^2 + b^2 = c^2$$

Be careful when using this theorem—you must make sure that the legs are labeled a and b and the hypotenuse is labeled c to use this equation. A more accurate way to write the Pythagorean Theorem is:

$$(\text{leg}_1)^2 + (\text{leg}_2)^2 = \text{hypotenuse}^2$$

Example 1

Use the side lengths of the following triangle to test the Pythagorean Theorem.



The legs of the triangle above are 3 inches and 4 inches . The hypotenuse is 5 inches . So, $a = 3, b = 4$, and $c = 5$. We can substitute these values into the formula for the Pythagorean Theorem to verify that the relationship works:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

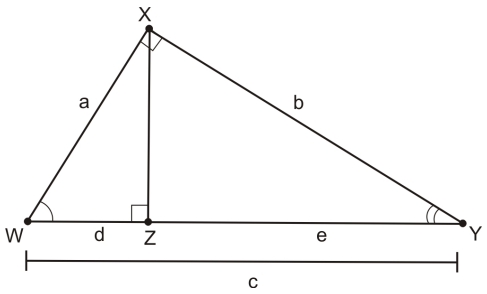
Since both sides of the equation equal 25, the equation is true. Therefore, the Pythagorean Theorem worked on this right triangle.

Proof of the Pythagorean Theorem

There are many ways to prove the Pythagorean Theorem. One of the most straightforward ways is to use similar triangles. Start with a right triangle and construct an altitude from the right angle to the opposite sides. In the figure below, we can see the following relationships:

Proof.

- Given: $\triangle WXY$ as shown in the figure below
- Prove: $a^2 + b^2 = c^2$



First we start with a triangle similarity statement:

$$\triangle WXY \sim \triangle WZX \sim \triangle XZY$$

These are all true by the AA triangle similarity postulate.
Now, using similar triangles, we can set up the following proportion:

$$\frac{d}{a} = \frac{a}{c}$$
$$a^2 = dc$$

and

$$\frac{e}{b} = \frac{b}{c}$$
$$b^2 = ec$$

Putting these equations together by using substitution,

$$a^2 + b^2 = dc + ec$$

factoring the right hand side,

$$a^2 + b^2 = c(d + e)$$

but notice $d + e = c$, so this becomes

$$a^2 + b^2 = c(c)$$

$$a^2 + b^2 = c^2.\blacklozenge$$

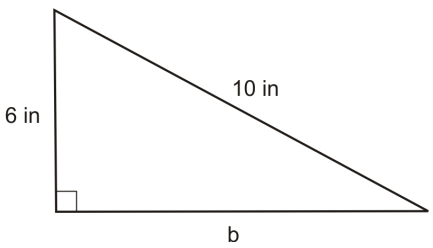
We have finished proving the Pythagorean Theorem. There are hundreds of other ways to prove the Pythagorean Theorem and one of those alternative proofs is in the exercises for this section.

Making Use of the Pythagorean Theorem

As you know from algebra, if you have one unknown variable in an equation, you can solve to find its value. Therefore, if you know the lengths of two out of three sides in a right triangle, you can use the Pythagorean Theorem to find the length of the missing side, whether it is a leg or a hypotenuse. Be careful to use inverse operations properly and avoid careless mistakes.

Example 2

What is the length of b in the triangle below?



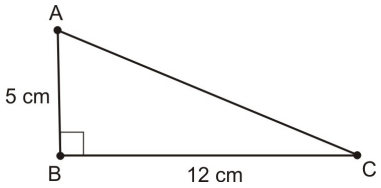
Use the Pythagorean Theorem to find the length of the missing leg, b. Set up the equation $a^2 + b^2 = c^2$, letting $a = 6$ and $b = 10$. Be sure to simplify the exponents and roots carefully, remember to use inverse operations to solve the equation, and always keep both sides of the equation balanced.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + b^2 &= 10^2 \\ 36 + b^2 &= 100 \\ 36 + b^2 - 36 &= 100 - 36 \\ b^2 &= 64 \\ \sqrt{b^2} &= \sqrt{64} \\ b &= \pm 8 \\ b &= 8 \end{aligned}$$

In algebra you learned that $\sqrt{x^2} = \pm x$ because, for example, $(5)^2 = (-5)^2 = 25$. However, in this case (and in much of geometry), we are only interested in the positive solution to $b = \sqrt{64}$ because geometric lengths are positive. So, in example 2, we can disregard the solution $b = -8$, and our final answer is $b = 8$ inches.

Example 3

Find the length of the missing side in the triangle below.



Use the Pythagorean Theorem to set up an equation and solve for the missing side. Let $a = 5$ and $b = 12$.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= c^2 \\ 25 + 144 &= c^2 \\ 169 &= c^2 \\ \sqrt{169} &= \sqrt{c^2} \\ 13 &= c \end{aligned}$$

So, the length of the missing side is 13 centimeters.

Using Pythagorean Triples

In example 1, the sides of the triangle were $3, 4$, and 5. This combination of numbers is referred to as a **Pythagorean triple**. A Pythagorean triple is three numbers that make the Pythagorean Theorem true and they are *integers* (whole numbers with no decimal or fraction part). Throughout this chapter, you will use other Pythagorean triples as well. For instance, the triangle in example 2 is proportional to the same ratio of $3 : 4 : 5$. If you divide the lengths of the triangle in example 2 $(6, 8, 10)$ by two, you find the same proportion—$3 : 4 : 5$. Whenever you find a Pythagorean triple, you can apply those ratios with greater factors as well. Finally, take note of the side lengths of the triangle in example 3—$5 : 12 : 13$. This, too, is a Pythagorean triple. You can extrapolate that this ratio, multiplied by greater factors, will also yield numbers that satisfy the Pythagorean Theorem.

There are infinitely many Pythagorean triples, but a few of the most common ones and their multiples are:

Table 8.1:

Triple	$\times 2$	$\times 3$	$\times 4$
$3 - 4 - 5$	$6 - 8 - 10$	$9 - 12 - 15$	$12 - 16 - 20$
$5 - 12 - 13$	$10 - 24 - 26$	$15 - 36 - 39$	$20 - 48 - 52$
$7 - 24 - 25$	$14 - 48 - 50$	$21 - 72 - 75$	$28 - 96 - 100$
$8 - 15 - 17$	$16 - 30 - 34$	$24 - 45 - 51$	$32 - 60 - 68$

Area of an Isosceles Triangle

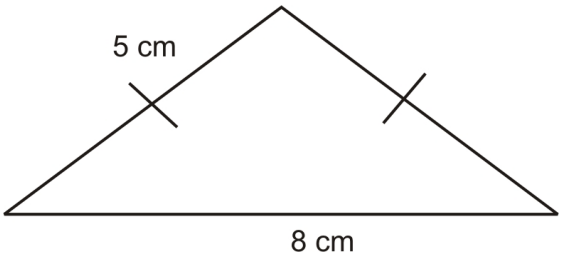
There are many different applications of the Pythagorean Theorem. One way to use The Pythagorean Theorem is to identify the heights in isosceles triangles so you can calculate the area. The area of a triangle is half of the product of its base and its height (also called altitude). This formula is shown below.

$$A = \frac{1}{2}bh$$

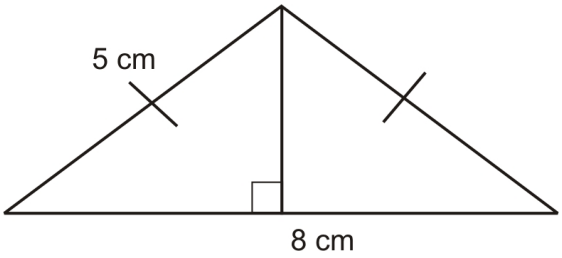
If you are given the base and the sides of an isosceles triangle, you can use the Pythagorean Theorem to calculate the height. Recall that the height (altitude) of a triangle is the length of a segment from one angle in the triangle perpendicular to the opposite side. In this case we focus on the altitude of isosceles triangles going from the vertex angle to the base.

Example 4

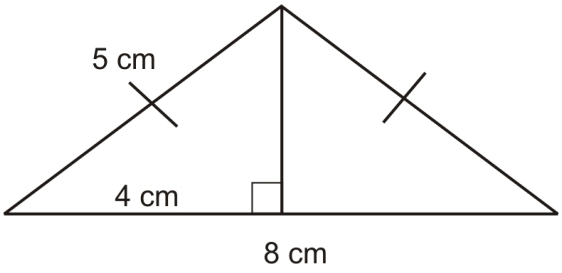
What is the height of the triangle below?



To find the area of this isosceles triangle, you will need to know the height in addition to the base. Draw in the height by connecting the vertex of the triangle with the base at a right angle.



Since the triangle is isosceles, the altitude will bisect the base. That means that it will divide it into two congruent, or equal parts. So, you can identify the length of one half of the base as 4 centimeters.



If you look at the smaller triangle now inscribed in the original shape, you'll notice that it is a right triangle with one leg 4 and hypotenuse 5. So, this is a $3 : 4 : 5$ triangle. If the leg is 4 cm and the hypotenuse is 5 cm, the missing leg must be 3 cm. So, the height of the isosceles triangle is 3 cm.

Use this information along with the original measurement of the base to find the area of the entire isosceles triangle.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(8)(3) \\ &= \frac{1}{2}(24) \\ &= 12 \end{aligned}$$

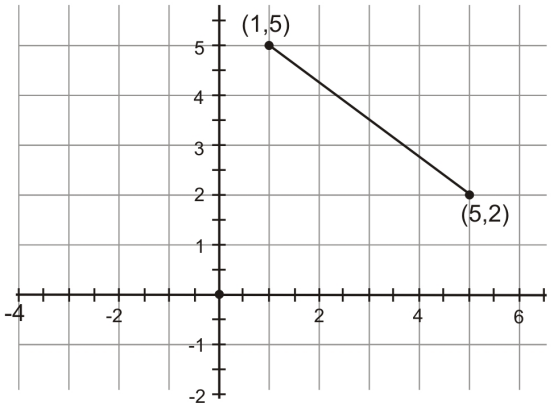
The area of the entire isosceles triangle is 12 cm^2.

The Distance Formula

You have already learned that you can use the Pythagorean Theorem to understand different types of right triangles, find missing lengths, and identify Pythagorean triples. You can also apply the Pythagorean Theorem to a coordinate grid and learn how to use it to find distances between points.

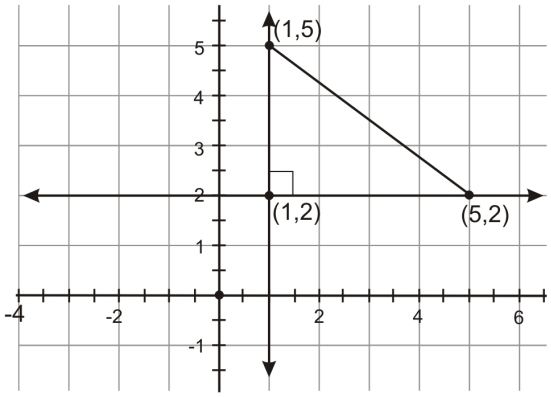
Example 5

Look at the points on the grid below.



Find the length of the segment connecting $(1, 5)$ *and* $(5, 2)$.

The question asks you to identify the length of the segment. Because the segment is not parallel to either axis, it is difficult to measure given the coordinate grid. However, it is possible to think of this segment as the hypotenuse of a right triangle. Draw a vertical line at $x = 1$ and a horizontal line at $y = 2$ and find the point of intersection. This point represents the third vertex in the right triangle.



You can easily count the lengths of the legs of this triangle on the grid. The vertical leg extends from $(1, 2)$ to $(1, 5)$, so it is $|5 - 2| = |3| = 3$ units long. The horizontal leg extends from $(1, 2)$ to $(5, 2)$, so it is $|5 - 1| = |4| = 4$ units long. Use the Pythagorean Theorem with these values for the lengths of each leg to find the length of the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ \sqrt{25} &= \sqrt{c^2} \\ 5 &= c \end{aligned}$$

The segment connecting $(1, 5)$ and $(5, 2)$ is 5 units long.

Mathematicians have simplified this process and created a formula that uses these steps to find the distance between any two points in the coordinate plane. If you use the distance formula, you don't have to draw the extra lines.

Distance Formula: Give points (x_1, y_1) and (x_2, y_2), the length of the segment connecting those two points is $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 6

Use the distance formula $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *to find the distance between the points* $(1, 5)$ *and* $(5, 2)$ *on a coordinate grid.*

You already know from example 1 that the distance will be 5 units, but you can practice using the distance formula to make sure it works. In this formula, substitute 1 for x_1, 5 for y_1, 5 for x_2, and 2 for y_2 because $(1, 5)$ and $(5, 2)$ are the two points in question.

$$\begin{aligned} D &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5-1)^2 + (2-5)^2} \\ &= \sqrt{(4)^2 + (-3)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Now you see that no matter which method you use to solve this problem, the distance between $(1, 5)$ and $(5, 2)$ on a coordinate grid is 5 units.

Lesson Summary

In this lesson, we explored how to work with different radical expressions both in theory and in practical situations. Specifically, we have learned:

- How to identify and employ the Pythagorean Theorem when working with right triangles.
- How to identify common Pythagorean triples.
- How to use the Pythagorean Theorem to find the area of isosceles triangles.
- How to use the Pythagorean Theorem to derive the distance formula on a coordinate grid.

These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply the Pythagorean Theorem to mathematical situations.

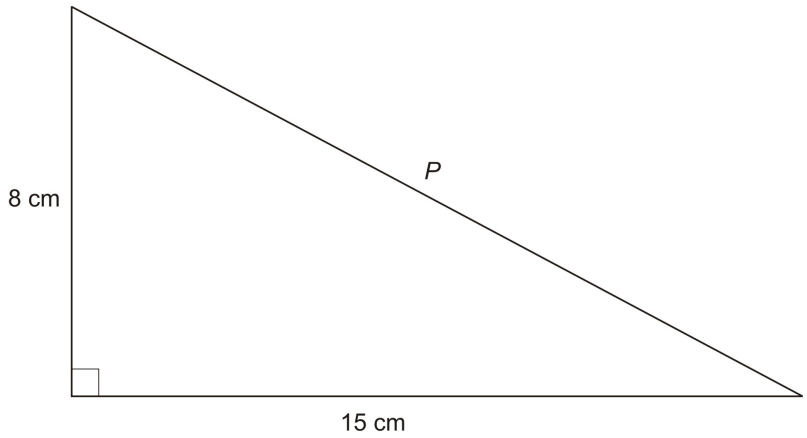
Points to Consider

Now that you have learned the Pythagorean Theorem, there are countless ways to apply it. Could you use the Pythagorean Theorem to prove that a triangle contained a right angle if you did not have an accurate diagram?

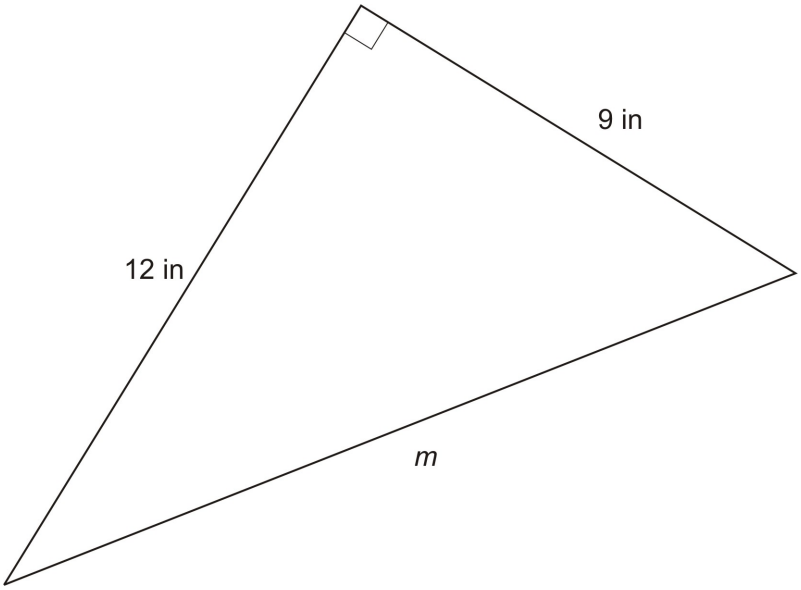
Review Questions

1. What is the distance between $(-5, -5)$ and $(-2, -1)$?
2. Do the numbers 12, 16, and 20 make a Pythagorean triple?

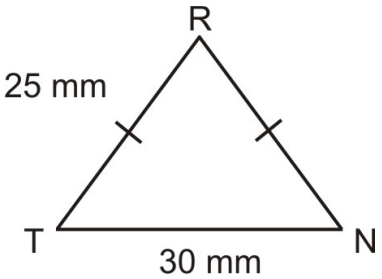
3. What is the length of p in the triangle below?



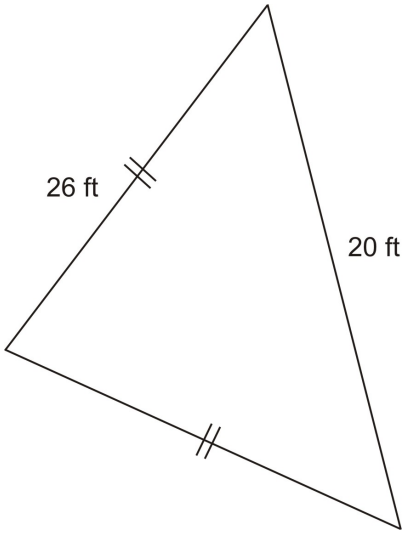
4. Do the numbers $13, 26$, and 35 make a Pythagorean triple?
5. What is the distance between $(1, 9)$ and $(9, 4)$?
6. What is the length of m in the triangle below?



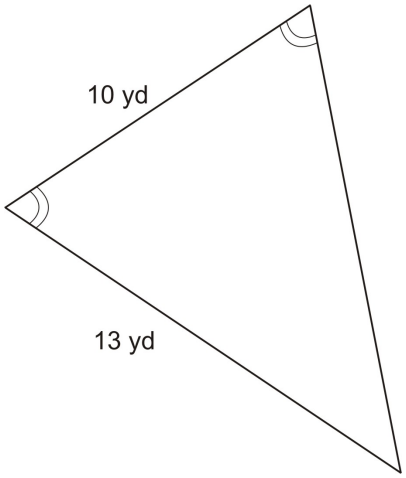
7. What is the distance between $(-3, 7)$ and $(6, 5)$?
8. What is the area of $\triangle TRN$ below?



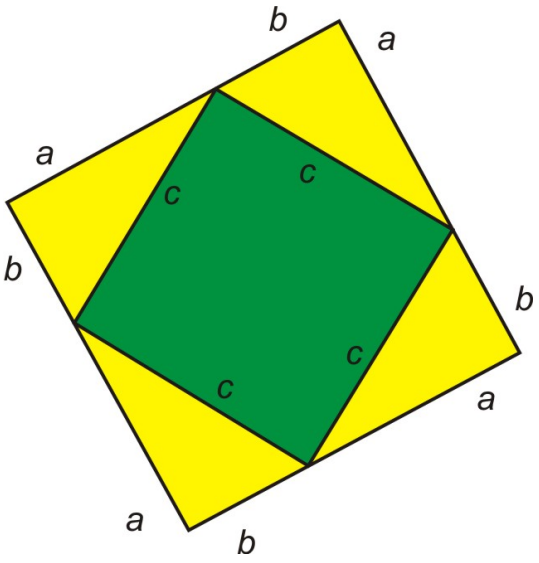
9. What is the area of the triangle below?



10. What is the area of the triangle below?



11. An alternative proof of the Pythagorean Theorem uses the area of a square. The diagram below shows a square with side lengths $a + b$, and an inner square with side lengths c. Use the diagram below to prove $a^2 + b^2 = c^2$.Hint: Find the area of the inner square in two ways: once directly, and once by finding the area of the larger square and subtracting the area of each triangle.



Review Answers

1. 5
2. yes
3. 17 inches
4. no
5. $\sqrt{89}$
6. 15 inches
7. $\sqrt{85}$
8. 300 squaremillimeters
9. 240 squarefeet
10. 60 squareyards
11. **Proof**. The plan is, we will find the area of the green square in two ways. Since those two areas must be equal, we can set those areas equal to each other. For the inner square (in green), we can directly compute the area: Area of inner square $= c^2$.
    Now, the area of the large, outer square is $(a+b)^2$. Don't forget to multiply this binomial carefully!

$$\begin{aligned} \text{area} &= (a+b)^2 \\ &= (a+b)(a+b) \\ &= a^2 + 2ab + b^2 \end{aligned}$$

The area of each small right triangle (in yellow) is

$$\text{area} = \frac{1}{2}ab$$

.

Since there are four of those right triangles, we have the combined area

$$4\left(\frac{1}{2}ab\right) = 2ab$$

Finally, subtract the area of the four yellow triangles from the area of the larger square, and we are left with

$$\text{large square} - \text{four triangles} = \text{area of inner square}$$
$$a^2 + 2ab + b^2 - 2ab = a^2 + b^2$$

Putting together the two different ways for finding the area of the inner square, we have $a^2 + b^2 = c^2$.

8.2 Converse of the Pythagorean Theorem

Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Identify acute triangles from side measures.
- Identify obtuse triangles from side measures.
- Classify triangles in a number of different ways.

Converse of the Pythagorean Theorem

In the last lesson, you learned about the Pythagorean Theorem and how it can be used. As you recall, it states that the sum of the squares of the legs of any right triangle will equal the square of the hypotenuse. If the lengths of the legs are labeled a and b, and the hypotenuse is c, then we get the familiar equation:

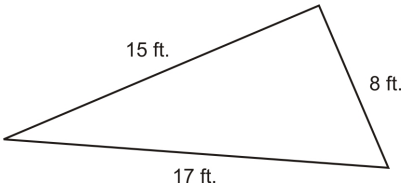
$$a^2 + b^2 = c^2$$

The **Converse of the Pythagorean Theorem** is also true. That is, if the lengths of three sides of a triangle make the equation $a^2 + b^2 = c^2$ true, then they represent the sides of a right triangle.

With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle's angle measurements.

Example 1

Does the triangle below contain a right angle?



This triangle does not have any right angle marks or measured angles, so you cannot assume you know whether the triangle is acute, right, or obtuse just by looking at it. Take a moment to analyze the side lengths and see how they are related. Two of the sides (15 and 17) are relatively close in length. The third side (8) is about half the length of the two longer sides.

To see if the triangle might be right, try substituting the side lengths into the Pythagorean Theorem to see if they makes the equation true. *The hypotenuse is always the longest side,* so 17 should be substituted for c. The other two values can represent a and b and the order is not important.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 8^2 + 15^2 &= 17^2 \\ 64 + 225 &= 289 \\ 289 &= 289 \end{aligned}$$

Since both sides of the equation are equal, these values satisfy the Pythagorean Theorem. Therefore, the triangle described in the problem is a right triangle.

In summary, example 1 shows how you can use the converse of the Pythagorean Theorem. The Pythagorean Theorem states that in a right triangle with legs a and b, and hypotenuse c, $a^2 + b^2 = c^2$. The converse of the Pythagorean Theorem states that if $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Identifying Acute Triangles

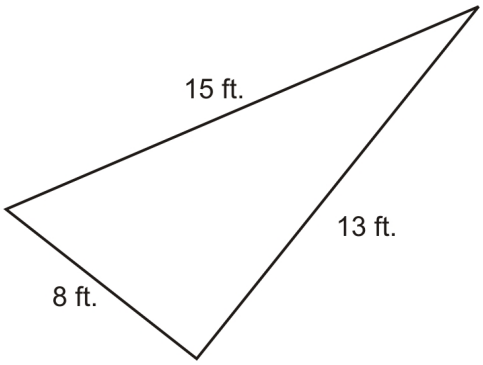
Using the converse of the Pythagorean Theorem, you can identify whether triangles contain a right angle or not. However, if a triangle does not contain a right angle, you can still learn more about the triangle itself by using the formula from Pythagorean Theorem. If the sum of the squares of the two shorter sides of a triangle is **greater** than the square of the longest side, the triangle is **acute** (all angles are less than 90°). In symbols, if $a^2 + b^2 > c^2$ then the triangle is acute.

Identifying the "shorter" and "longest" sides may seem ambiguous if sides have the same

length, but in this case any ordering of equal length sides leads to the same result. For example, an equilateral triangle always satisfies $a^2 + b^2 > c^2$ and so is acute.

Example 2

Is the triangle below acute or right?



The two shorter sides of the triangle are 8 and 13. The longest side of the triangle is 15. First find the sum of the squares of the two shorter legs.

$$\begin{aligned} 8^2 + 13^2 &= c^2 \\ 64 + 169 &= c^2 \\ 233 &= c^2 \end{aligned}$$

The sum of the squares of the two shorter legs is 233. Compare this to the square of the longest side, 15.

$$15^2 = 225$$

The square of the longest side is 225. Since $8^2 + 13^2 = 233 \neq 255 = 15^2$, this triangle is not a right triangle. Compare the two values to identify which is greater.

$$233 > 225$$

The sum of the square of the shorter sides is greater than the square of the longest side. Therefore, this is an acute triangle.

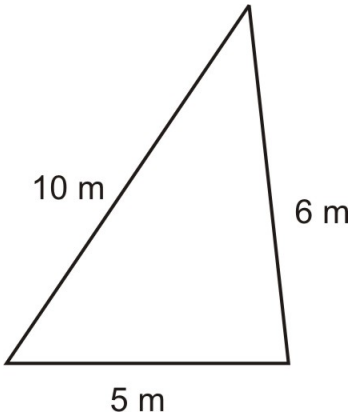
Identifying Obtuse Triangles

As you have probably figured out, you can prove a triangle is **obtuse** (has one angle larger than 90°) by using a similar method. Find the sum of the squares of the two shorter sides in

a triangle. If this value is less than the square of the longest side, the triangle is obtuse. In symbols, if $a^2 + b^2 < c^2$, then the triangle is obtuse. You can solve this problem in a manner almost identical to example 2 above.

Example 3

Is the triangle below acute or obtuse?



The two shorter sides of the triangle are 5 and 6. The longest side of the triangle is 10. First find the sum of the squares of the two shorter legs.

$$\begin{aligned} a^2 + b^2 &= 5^2 + 6^2 \\ &= 25 + 36 \\ &= 61 \end{aligned}$$

The sum of the squares of the two shorter legs is 61. Compare this to the square of the longest side, 10.

$$10^2 = 100$$

The square of the longest side is 100. Since $5^2 + 6^2 \neq 100^2$, this triangle is not a right triangle. Compare the two values to identify which is greater.

$$61 < 100$$

Since the sum of the square of the shorter sides is less than the square of the longest side, this is an obtuse triangle.

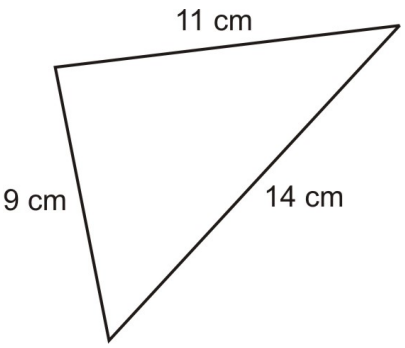
Triangle Classification

Now that you know the ideas presented in this lesson, you can classify any triangle as right, acute, or obtuse given the length of the three sides. Begin by ordering the sides of the triangle from smallest to largest, and substitute the three side lengths into the equation given by the Pythagorean Theorem using $a \leq b < c$. Be sure to use the longest side for the hypotenuse.

- If $a^2 + b^2 = c^2$, the figure is a right triangle.
- If $a^2 + b^2 > c^2$, the figure is an acute triangle.
- If $a^2 + b^2 < c^2$, the figure is an obtuse triangle.

Example 4

Classify the triangle below as right, acute, or obtuse.



The two shorter sides of the triangle are 9 and 11. The longest side of the triangle is 14. First find the sum of the squares of the two shorter legs.

$$\begin{aligned} a^2 + b^2 &= 9^2 + 11^2 \\ &= 81 + 121 \\ &= 202 \end{aligned}$$

The sum of the squares of the two shorter legs is 202. Compare this to the square of the longest side, 14.

$$14^2 = 196$$

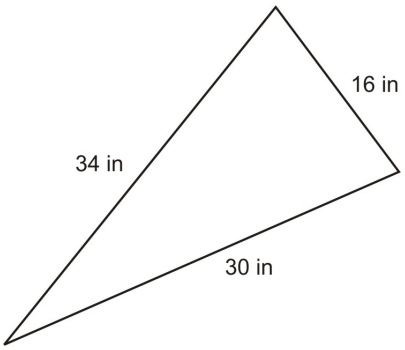
The square of the longest side is 196. Therefore, the two values are not equal, $a^2 + b^2 \neq c^2$ and this triangle is not a right triangle. Compare the two values, $a^2 + b^2$ and c^2 to identify which is greater.

$$202 > 196$$

Since the sum of the square of the shorter sides is greater than the square of the longest side, this is an acute triangle.

Example 5

Classify the triangle below as right, acute, or obtuse.



The two shorter sides of the triangle are 16 and 30. The longest side of the triangle is 34. First find the sum of the squares of the two shorter legs.

$$\begin{aligned} a^2 + b^2 &= 16^2 + 30^2 \\ &= 256 + 900 \\ &= 1156 \end{aligned}$$

The sum of the squares of the two legs is $1,156$. Compare this to the square of the longest side, 34.

$$c^2 = 34^2 = 1156$$

The square of the longest side is $1,156$. Since these two values are equal, $a^2 + b^2 = c^2$, and this is a right triangle.

Lesson Summary

In this lesson, we explored how to work with different radical expressions both in theory and in practical situations. Specifically, we have learned:

- How to use the converse of the Pythagorean Theorem to prove a triangle is right.
- How to identify acute triangles from side measures.
- How to identify obtuse triangles from side measures.
- How to classify triangles in a number of different ways.

These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to apply the Pythagorean Theorem and its converse to mathematical situations.

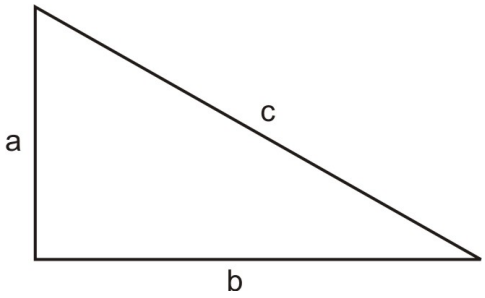
Points to Consider

Use the Pythagorean Theorem to explore relationships in common right triangles. Do you find that the sides are proportionate?

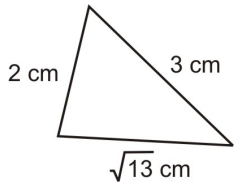
Review Questions

Solve each problem.

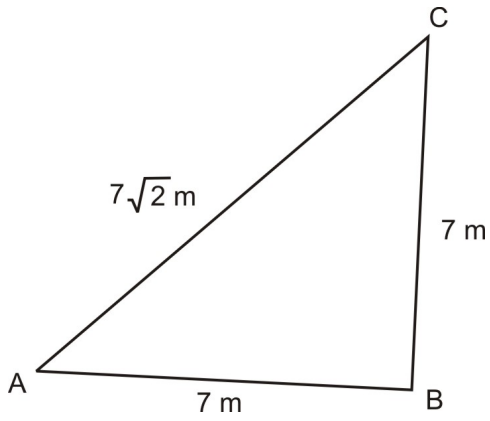
For exercises 1-8, classify the following triangle as acute, obtuse, or right based on the given side lengths. Note, the figure is not to scale.



1. $a = 9$ in, $b = 12$ in, $c = 15$ in
2. $a = 7$ cm, $b = 7$ cm, $c = 8$ cm
3. $a = 4$ m, $b = 8$ m, $c = 10$ m
4. $a = 10$ ft, $b = 22$ ft, $c = 23$ ft
5. $a = 21$ cm, $b = 28$ cm, $c = 35$ cm
6. $a = 10$ ft, $b = 12$ ft, $c = 14$ ft
7. $a = 15$ m, $b = 18$ m, $c = 30$ m
8. $a = 5$ in, $b = \sqrt{75}$ in, $c = 110$ in
9. In the triangle below, which sides should you use for the legs (usually called sides a , and b) and the hypotenuse (usually called side c), in the Pythagorean theorem? How do you know?



10.



(a) $m\angle A =$
(b) $m\angle B =$

Review Answers

1. Right
2. Acute
3. Obtuse
4. Acute
5. Right
6. Acute
7. Obtuse
8. Obtuse
9. The side with length $\sqrt{13}$ should be the hypotenuse since it is the longest side. The order of the legs does not matter
10. $m\angle A = 45°, m\angle B = 90°$

8.3 Using Similar Right Triangles

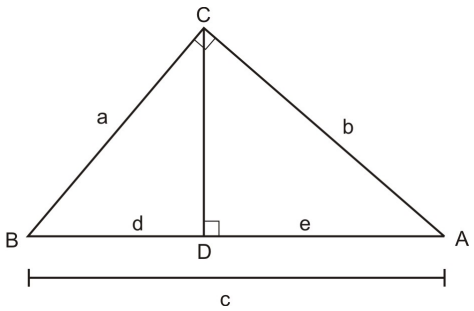
Learning Objectives

- Identify similar triangles inscribed in a larger triangle.
- Evaluate the geometric mean of various objects.
- Identify the length of an altitude using the geometric mean of a separated hypotenuse.

- Identify the length of a leg using the geometric mean of a separated hypotenuse.

Introduction

In this lesson, you will study figures **inscribed,** or drawn within, existing triangles. One of the most important types of lines drawn within a right triangle is called an **altitude.** Recall that the altitude of a triangle is the perpendicular distance from one vertex to the opposite side. By definition each leg of a right triangle is an altitude. We can find one more altitude in a right triangle by adding an auxiliary line segment that connects the vertex of the right angle with the hypotenuse, forming a new right angle.



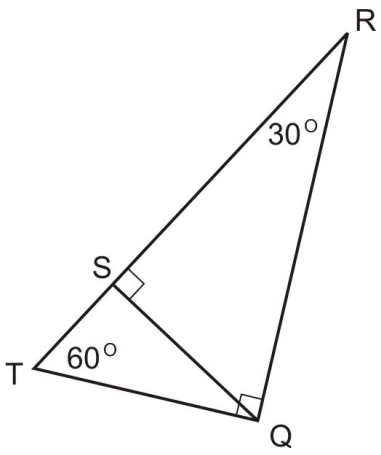
You may recall this is the figure that we used to prove the Pythagorean Theorem. In right triangle ABC above, the segment $\overline{CD}$ is an altitude. It begins at angle C, which is a right angle, and it is perpendicular to the hypotenuse $\overline{AB}$. In the resulting figure, we have three right triangles, and all of them are similar.

Inscribed Similar Triangles

You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length. In other words, similar figures are the same shape, but different sizes. To prove that two triangles are similar, it is sufficient to prove that all angle measures are congruent (note, this is NOT true for other polygons. For example, both squares and "long" rectangles have all 90° angles, but they are not similar). Use logic, and the information presented above to complete Example 1.

Example 1

Justify the statement that $\triangle TQR \sim \triangle TSQ \sim \triangle QSR$.



In the figure above, the big triangle $\triangle TQR$ is a right triangle with right angle $\angle Q$ and $m\angle R = 30°$ and $m\angle T = 60°$. So, if $\triangle TQR, \triangle TSQ$, and $\triangle QSR$ are similar, they will all have angles of $30°, 60°$, and $90°$.

First look at $\triangle TSQ$. $m\angle QST = 90°$, and $m\angle T = 60°$. Since the sum of the three angles in a triangle always equals 180°, the missing angle, $\angle TQS$, must measure 30°, since $30+60+90 = 180$. Lining up the congruent angles, we can write $\triangle TQR \sim \triangle TSQ$.

Now look at $\triangle QRS.\angle QSR$ has a measure of 90°, and $m\angle R = 30°$. Since the sum of the three angles in a triangle always equals 180°, the missing angle, $\angle RQS$, must measure 60°, since $30 + 60 + 90 = 180$. Now, since the triangles have congruent corresponding angles, $\triangle QSR$ and $\triangle TQR$ are similar.

Thus, $\triangle TQR \sim \triangle TSQ \sim \triangle QSR$. Their angles are congruent and their sides are proportional.

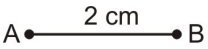
Note that you must be very careful to match up corresponding angles when writing triangle similarity statements. Here we should write $\triangle TQR \sim \triangle TSQ \sim \triangle QSR$. This example is challenging because the triangles are overlapping.

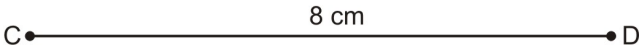
Geometric Means

When someone asks you to find the average of two numbers, you probably think of the arithmetic mean (average). Chances are good you've worked with arithmetic means for many years, but the concept of a **geometric mean** may be new. An arithmetic mean is found by dividing the sum of a set of numbers by the number of items in the set. Arithmetic means are used to calculate overall grades and many other applications. The big idea behind the arithmetic mean is to find a "measure of center" for a group of numbers.

A geometric mean applies the same principles, but relates specifically to size, length, or measure. For example, you may have two line segments as shown below. Instead of adding and dividing, you find a geometric mean by multiplying the two numbers, then finding the

square root of the product.





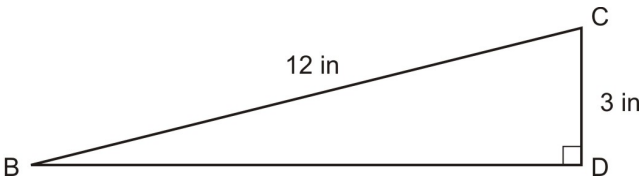
To find the geometric mean of these two segments, multiply the lengths and find the square root of the product.

$$\begin{aligned} \text{mean} &= \sqrt{8 \cdot 2} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

So, the geometric mean of the two segments would be a line segment that is 4 cm in length. Use these concepts and strategies to complete example 2.

Example 2

In $\triangle BCD$ below, what is the geometric mean of BC and CD?



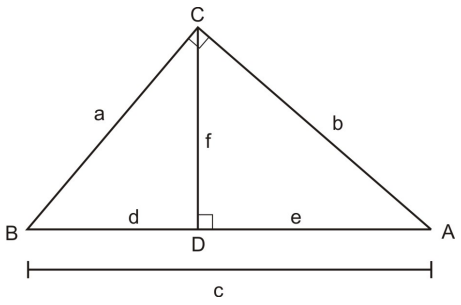
When finding a geometric mean, you first find the product of the items involved. In this case, segment BC is 12 inches and segment CD is 3 inches. Then find the square root of this product.

$$\begin{aligned} \text{mean} &= \sqrt{12 \cdot 3} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

So, the geometric mean of BC and CD in $\triangle BCD$ is 6 inches.

Altitude as Geometric Mean

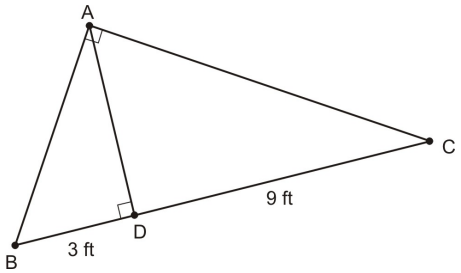
In a right triangle, the length of the altitude from the right angle to the hypotenuse is the geometric mean of the lengths of the two segments of the hypotenuse. In the diagram below we can use $\triangle BDC \sim \triangle CDA$ to create the proportion $\frac{d}{f} = \frac{f}{e}$. Solving for f, $f = \sqrt{d \cdot e}$.



You can use this relationship to find the length of the altitude if you know the length of the two segments of the divided hypotenuse.

Example 3

What is the length of the altitude $\overline{AD}$ in the triangle below?



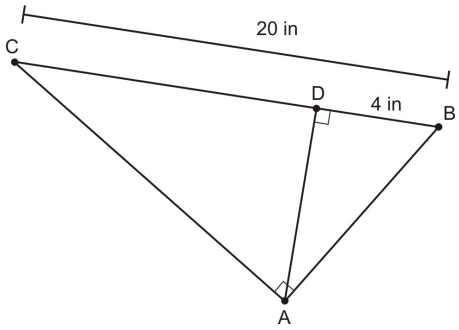
To find the altitude of this triangle, find the geometric mean of the two segments of the hypotenuse. In this case, you need to find the geometric mean of 9 and 3. To find the geometric mean, find the product of the two numbers and then take its square root.

$$\begin{aligned} \text{mean} &= \sqrt{9 \cdot 3} \\ &= \sqrt{27} \\ &= 3\sqrt{3} \end{aligned}$$

So $AD = 3\sqrt{3}$ feet, or approximately $3(1.732) = 5.2$ feet.

Example 4

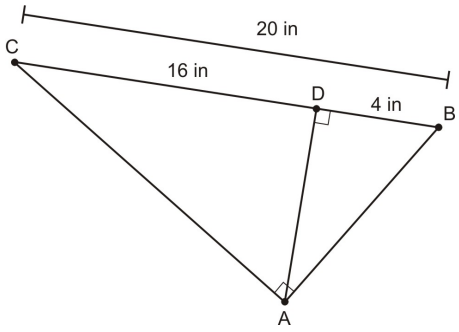
What is the length of the altitude in the triangle below?



The altitude of this triangle is $\overline{AD}$. Remember the altitude does not always go "down"! To find AD, find the geometric mean of the two segments of the hypotenuse. Make sure that you fill in missing information in the diagram. You know that the whole hypotenuse, $\overline{CB}$ is 20 inches long and $BD = 4$ inches, but you need to know CD, the length of the longer subsection of $\overline{CB}$, to find the geometric mean. To do this, subtract.

$$\begin{aligned} CD &= CB - DB \\ &= 20 - 4 \\ &= 16 \end{aligned}$$

So $CD = 16$ inches. Write this measurement on the diagram to keep track of your work.



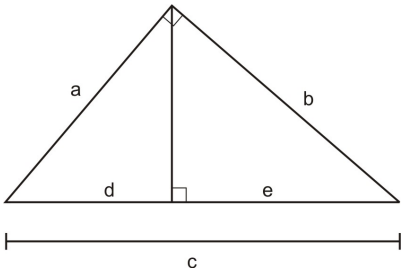
Now find the geometric mean of 16 and 4 to identify the length of the altitude.

$$\begin{aligned} AD &= \sqrt{16 \cdot 4} \\ &= \sqrt{64} \\ &= 8 \end{aligned}$$

The altitude of the triangle will measure 8 inches.

Leg as Geometric Mean

Just as we used similar triangles to create a proportion using the altitude, the lengths of the legs in right triangles can also be found with a geometric mean with respect to the hypotenuse. The length of one leg in a right triangle is the geometric mean of the adjacent segment and the entire hypotenuse. The diagram below shows the relationships.

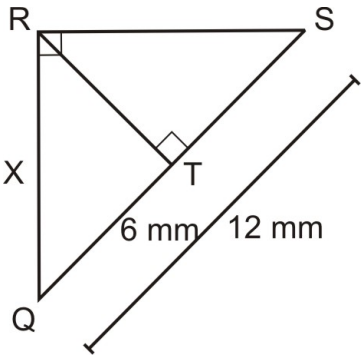


$$a = \sqrt{d \cdot c}$$
$$b = \sqrt{e \cdot c}$$

You can use this relationship to find the length of the leg if you know the length of the two segments of the divided hypotenuse.

Example 5

What is the length of x in the triangle below?



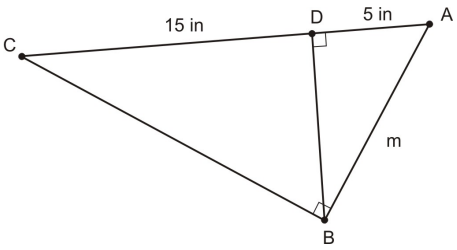
To find x, the leg of the large right triangle, find the geometric mean of the adjacent segments of the hypotenuse and the entire hypotenuse. In this case, you need to find the geometric mean of 6 and 12. To find the geometric mean, find the product of the two numbers and then take the square root of that product.

$$x = \sqrt{6 \cdot 12}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2}$$

So, $x = 6\sqrt{2}$ millimeters or approximately 8.49 millimeters.

Example 6

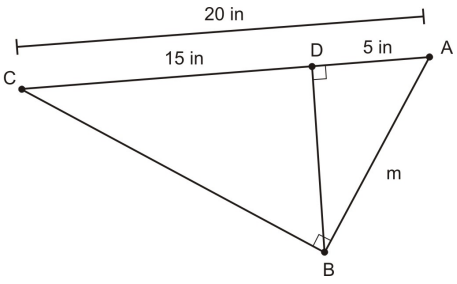
If $m = AB$, what is the value m in the triangle below?



To find m in this triangle, find the geometric mean of the adjacent segment of the hypotenuse and the entire hypotenuse. Make sure that you fill in missing information in the diagram. You know that the two shorter sections of the hypotenuse are 15 inches and 5 inches, but you need to know the length of the entire hypotenuse to find the geometric mean. To do this, add.

$$AD + DC = AC$$
$$5 + 15 = 20$$

So, $AC = 20$ inches. Write this measurement on the diagram to keep track of your work.



Now find the geometric mean of 20 and 5 to identify the length of the altitude.

$$\begin{aligned} m &= \sqrt{20 \cdot 5} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

So, $m = 10$ inches.

Lesson Summary

In this lesson, we explored how to work with different radicals both in theory and in practical situations. Specifically, we have learned:

- How to identify similar triangles inscribed in a larger triangle.
- How to evaluate the geometric mean of various objects.
- How to identify the length of an altitude using the geometric mean of a separated hypotenuse.
- How to identify the length of a leg using the geometric mean of a separated hypotenuse.

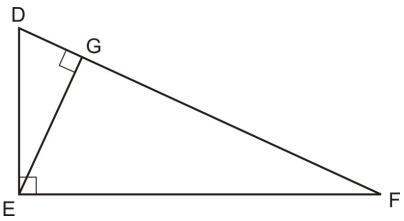
These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to find relationships between sides and angles in triangles.

Points to Consider

How can you use the Pythagorean Theorem to identify other relationships between sides in triangles?

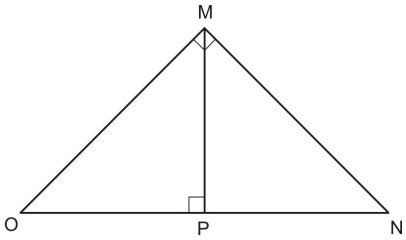
Review Questions

1. Which triangles in the diagram below are similar?

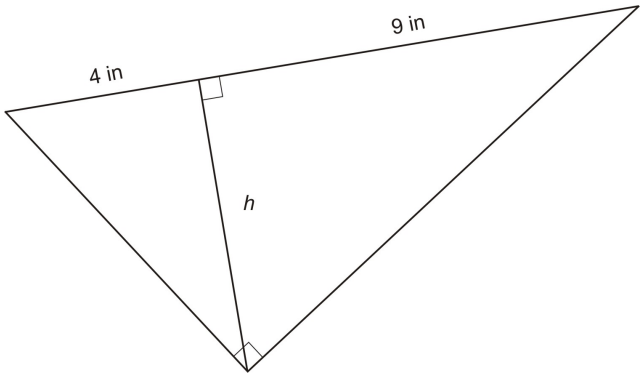


2. What is the geometric mean of two line segments that are 1 and 4 inches, respectively?
3. What is the geometric mean of two line segments that are 3 cm each?

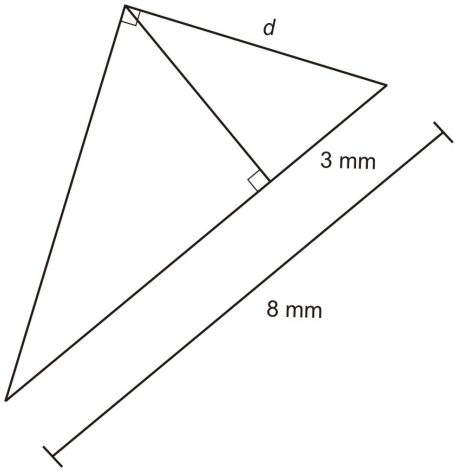
4. Which triangles in the diagram below are similar?



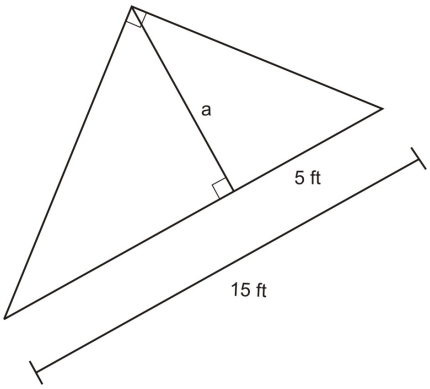
5. What is the length of the altitude, h, in the triangle below?



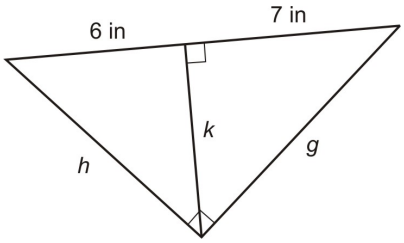
6. What is the length of d in the triangle below?



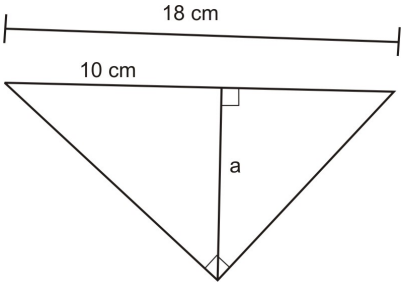
7. What is the geometric mean of two line segments that are 4 yards and 8 yards, respectively?
8. What is the length of the altitude in the triangle below?



Use the following diagram from exercises 9-11:



9. $g =$ _____
10. $h =$ _____
11. $k =$ _____ (for an extra challenge, find k in two different ways)
12. What is the length of the altitude in the triangle below?



Review Answers

1. Triangles DEF, EGF, and DGE are all similar
2. 2 inches
3. 3 cm
4. Triangles MNO, PNM, and PMO are all similar.
5. 6 inches
6. $2\sqrt{6}$ mm, or approximately 4.9 mm
7. $4\sqrt{2}$ yards, or approximately 5.66 yards
8. $5\sqrt{2}$ feet, or approximately 7.07 feet
9. $g = \sqrt{91}$ inches, or approximately 9.54 inches

10. $h = \sqrt{78}$ inches or approximately 8.83 inches
11. $k = \sqrt{42}$ inches or approximately 6.48 inches. One way to find k is with the geometric mean: $k = \sqrt{6}.7 = \sqrt{42}$ inches. Alternatively, using the answer from 9 and one of the smaller right triangles, $k = \sqrt{(\sqrt{91})^2 - (7)^2} = \sqrt{91 - 49} = \sqrt{42}$ inches
12. .

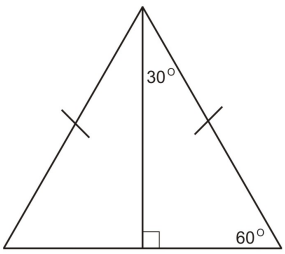
8.4 Special Right Triangles

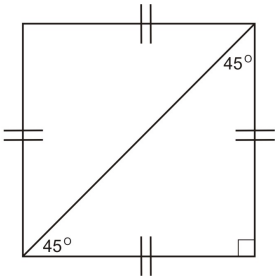
Learning Objectives

- Identify and use the ratios involved with right isosceles triangles.
- Identify and use the ratios involved with $30° - 60° - 90°$ triangles.
- Identify and use ratios involved with equilateral triangles.
- Employ right triangle ratios when solving real-world problems.

Introduction

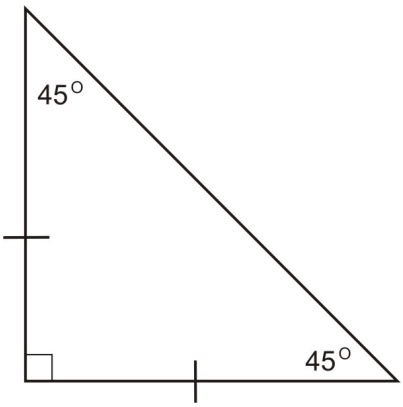
What happens when you cut an equilateral triangle in half using an altitude? You get two right triangles. What about a square? If you draw a diagonal across a square you also get two right triangles. These two right triangles are *special special right triangles* called the $30° - 60° - 90°$ and the $45° - 45° - 90°$ right triangles. They have unique properties and if you understand the relationships between the sides and angles in these triangles, you will do well in geometry, trigonometry, and beyond.





Right Isosceles Triangles

The first type of right triangle to examine is **isosceles.** As you know, isosceles triangles have two sides that are the same length. Additionally, the base angles of an isosceles triangle are congruent as well. An isosceles **right** triangle will always have base angles that each measure $45°$ and a vertex angle that measures $90°$.

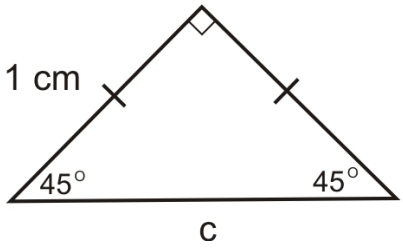


Don't forget that the base angles are the angles across from the congruent sides. They don't have to be on the bottom of the figure.

Because the angles of all $45° - 45° - 90°$ triangles will, by definition, remain the same, all $45° - 45° - 90°$ triangles are **similar,** so their sides will always be proportional. To find the relationship between the sides, use the Pythagorean Theorem.

Example 1

The isosceles right triangle below has legs measuring 1 centimeter.



Use the Pythagorean Theorem to find the length of the hypotenuse.

Since the legs are 1 centimeter each, substitute 1 for both a and b, and solve for c:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + 1^2 &= c^2 \\ 1 + 1 &= c^2 \\ 2 &= c^2 \\ \sqrt{2} &= \sqrt{c^2} \\ c &= \sqrt{2} \end{aligned}$$

In this example $c = \sqrt{2}$ cm.

What if each leg in the example above was 5 cm? Then we would have

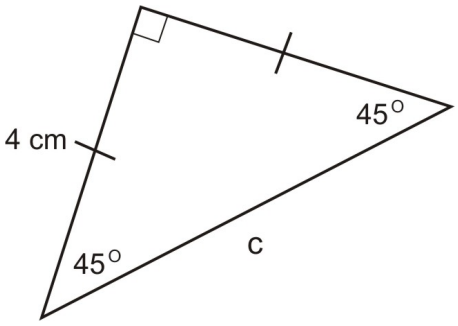
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 5^2 &= c^2 \\ 25 + 25 &= c^2 \\ 50 &= c^2 \\ \sqrt{50} &= \sqrt{c^2} \\ c &= 5\sqrt{2} \end{aligned}$$

If each leg is 5 cm, then the hypotenuse is $5\sqrt{2}$ cm.

When the length of each leg was 1, the hypotenuse was $1\sqrt{2}$. When the length of each leg was 5, the hypotenuse was $5\sqrt{2}$. Is this a coincidence? No. Recall that the legs of all $45° - 45° - 90°$ triangles are proportional. The hypotenuse of an isosceles right triangle will always equal the product of the length of one leg and $\sqrt{2}$. Use this information to solve the problem in example 2.

Example 2

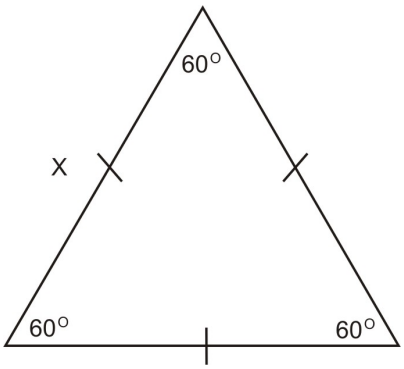
What is the length of the hypotenuse in the triangle below?



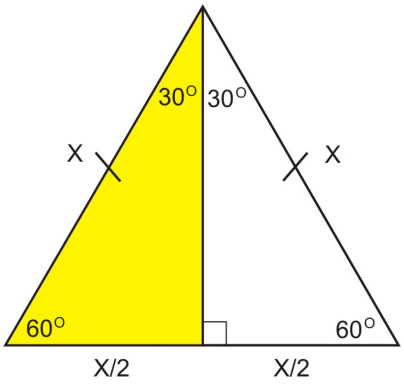
Since the length of the hypotenuse is the product of one leg and $\sqrt{2}$, you can easily calculate this length. One leg is 4 inches, so the hypotenuse will be $4\sqrt{2}$ inches, or about 5.66 inches.

Equilateral Triangles

Remember that an equilateral triangle has sides that all have the same length. Equilateral triangles are also equiangular—all angles have the same measure. In an equilateral triangle, all angles measure exactly 60°.



Notice what happens when you divide an equilateral triangle in half.



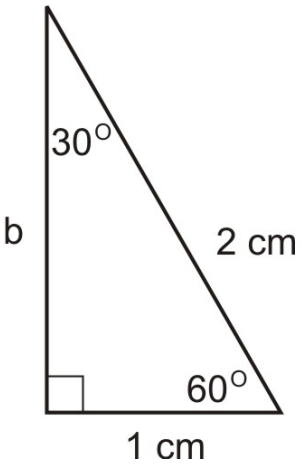
When an equilateral triangle is divided into two equal parts using an altitude, each resulting right triangle is a $30° - 60° - 90°$ triangle. The hypotenuse of the resulting triangle was the side of the original, and the shorter leg is half of an original side. This is why the hypotenuse is always twice the length of the shorter leg in a $30° - 60° - 90°$ triangle. You can use this information to solve problems about equilateral triangles.

30º-60º-90º Triangles

Another important type of right triangle has angles measuring 30°, 60°, and 90°. Just as you found a constant ratio between the sides of an isosceles right triangle, you can find constant ratios here as well. Use the Pythagorean Theorem to discover these important relationships.

Example 3

Find the length of the missing leg in the following triangle. Use the Pythagorean Theorem to find your answer.



Just like you did for $45° - 45° - 90°$ triangles, use the Pythagorean theorem to find the missing side. In this diagram, you are given two measurements: the hypotenuse (c) is 2 cm and the shorter leg (a) is 1 cm. Find the length of the missing leg (b).

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \\ 1 + b^2 &= 4 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

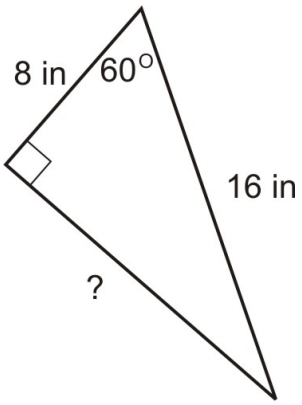
You can leave the answer in radical form as shown, or use your calculator to find the approximate value of $b \approx 1.732$ cm.

On your own, try this again using a hypotenuse of 6 feet. Recall that since the $30° - 60° - 90°$ triangle comes from an equilateral triangle, you know that the length of the shorter leg is half the length of the hypotenuse.

Now you should be able to identify the constant ratios in $30° - 60° - 90°$ triangles. The hypotenuse will always be **twice** the length of the shorter leg, and the longer leg is always the product of the length of the shorter leg and $\sqrt{3}$. In ratio form, the sides, in order from shortest to longest are in the ratio $x : x\sqrt{3} : 2x$.

Example 4

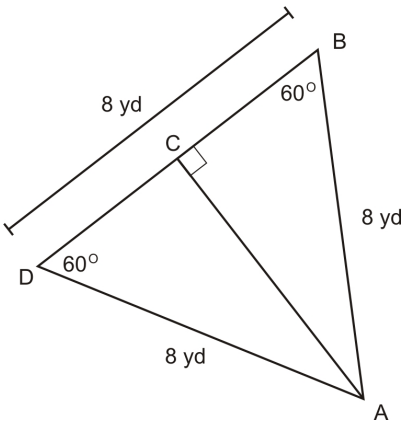
What is the length of the missing leg in the triangle below?



Since the length of the longer leg is the product of the shorter leg and $\sqrt{3}$, you can easily calculate this length. The short leg is 8 inches, so the longer leg will be $8\sqrt{3}$ inches, or about 13.86 inches.

Example 5

What is AC below?



To find the length of segment $\overline{AC}$, identify its relationship to the rest of the triangle. Since it is an altitude, it forms two congruent triangles with angles measuring 30°, 60°, and 90°. So, AC will be the product of BC (the shorter leg) and $\sqrt{3}$.

$$AC = BC\sqrt{3}$$
$$= 4\sqrt{3}$$

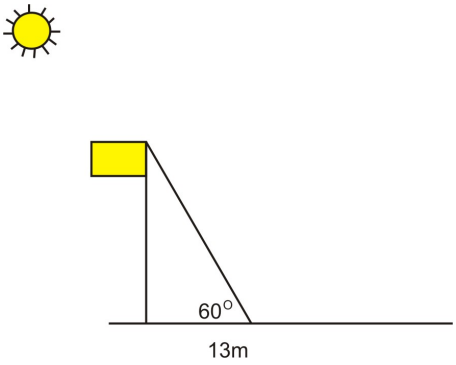
$AC = 4\sqrt{3}$ yards , or approximately 6.93 yards.

Special Right Triangles in the Real World

You can use special right triangles in many real-world contexts. Many real-life applications of geometry rely on special right triangles, so being able to recall and use these ratios is a way to save time when solving problems.

Example 6

The diagram below shows the shadow a flagpole casts at a certain time of day.



If the length of the shadow cast by the flagpole is 13 m*, what is the height of the flagpole and the length of the hypotenuse of the right triangle shown?*

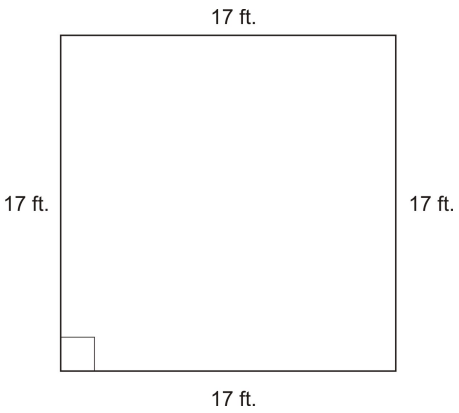
The wording in this problem is complicated, but you only need to notice a few things. You can tell in the picture that this triangle has angles of $30°, 60°$, and $90°$ (This assumes that the flagpole is perpendicular to the ground, but that is a safe assumption). The height of the flagpole is the longer leg in the triangle, so use the special right triangle ratios to find the length of the hypotenuse.

The longer leg is the product of the shorter leg and $\sqrt{3}$. The length of the shorter leg is given as 13 meters, so the height of the flagpole is $13\sqrt{3}$ m.

The length of the hypotenuse is the hypotenuse of a $30° - 60° - 90°$ triangle. It will always be twice the length of the shorter leg, so it will equal $13 \cdot 2$, or 26 meters.

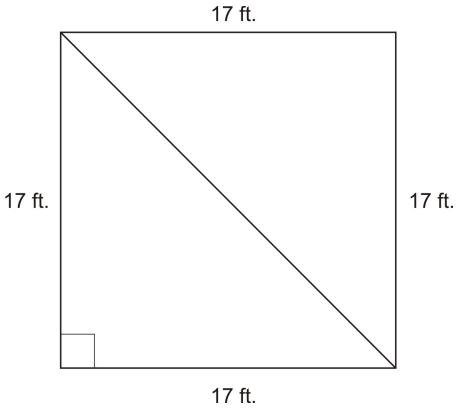
Example 7

Antonio built a square patio in his backyard.



He wants to make a water pipe for flowers that goes from one corner to another, diagonally. How long will that pipe be?

The first step in a word problem of this nature is to add important information to the drawing. Because the problem asks you to find the length from one corner to another, you should draw that segment in.



Once you draw the diagonal path, you can see how triangles help answer this question. Because both legs of the triangle have the same measurement (17 feet), this is an isosceles right triangle. The angles in an isosceles right triangle are $45°, 45°$, and $90°$.

In an isosceles right triangle, the hypotenuse is always equal to the product of the length of one leg and $\sqrt{2}$. So, the length of Antonio's water pipe will be the product of 17 and $\sqrt{2}$, or $17\sqrt{2} \approx 17(1.414)$ feet. This value is approximately equal to 24.04 feet.

Lesson Summary

In this lesson, we explored how to work with different radicals both in theory and in practical situations. Specifically, we have learned:

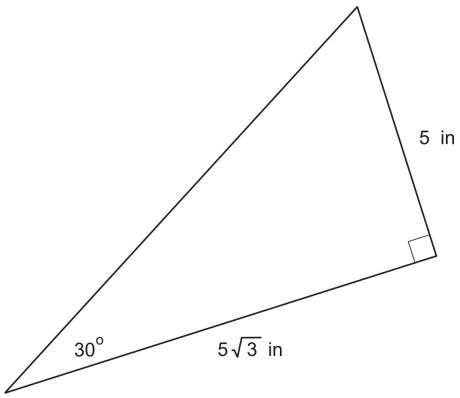
- How to identify and use the ratios involved with right isosceles triangles.

- How to identify and use the ratios involved with $30° - 60° - 90°$ triangles.
- How to identify and use ratios involved with equilateral triangles.
- How to employ right triangle ratios when solving real-world problems.

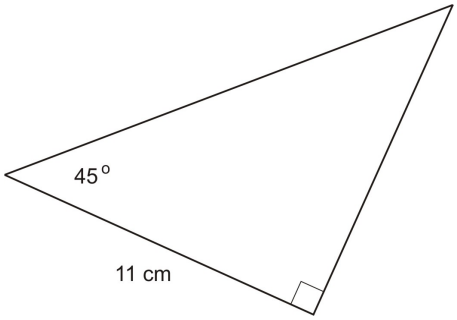
These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to find relationships between sides and angles in triangles.

Review Questions

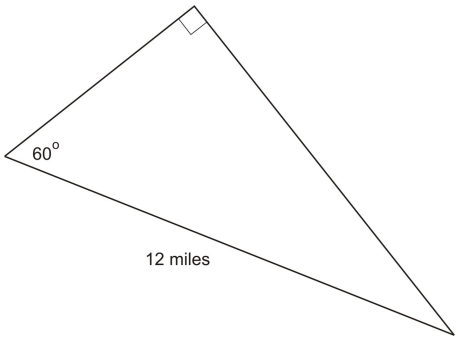
1. Mildred had a piece of scrap wood cut into an equilateral triangle. She wants to cut it into two smaller congruent triangles. What will be the angle measurement of the triangles that result?
2. Roberto has a square pizza. He wants to cut two congruent triangles out of the pizza without leaving any leftovers. What will be the angle measurements of the triangles that result?
3. What is the length of the hypotenuse in the triangle below?



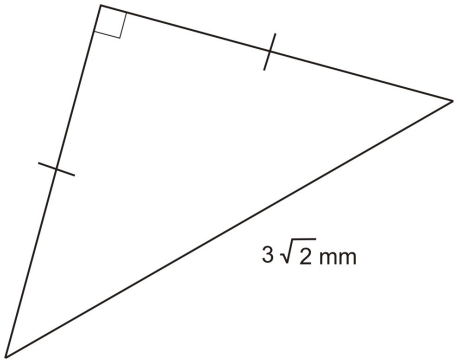
4. What is the length of the hypotenuse in the triangle below?



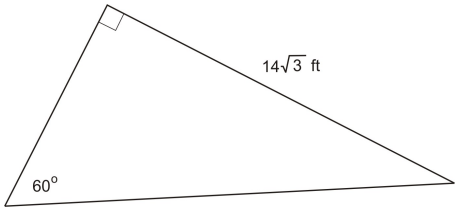
5. What is the length of the longer leg in the triangle below?



6. What is the length of one of the legs in the triangle below?



7. What is the length of the shorter leg in the triangle below?



8. A square window has a diagonal of $5\sqrt{2}$ feet. What is the length of one of its sides?
9. A square block of foam is cut into two congruent wedges. If a side of the original block was 3 feet, how long is the diagonal cut?
10. They wants to find the area of an equilateral triangle but only knows that the length of one side is 6 inches. What is the height of Thuy's triangle? What is the area of the triangle?

Review Answers

1. $30°, 60°$, and $90°$
2. $45°, 45°$, and $90°$
3. 10
4. $11\sqrt{2}$ cm or approx. 15.56 cm
5. $6\sqrt{3}$ miles or approx. 10.39 miles
6. 3 mm

7. 14 feet
8. 5 feet
9. $3\sqrt{2}$ feet or approx. 4.24 feet
10. $3\sqrt{3}$ inches or approx. 5.2 in. The area is $9\sqrt{3} \approx 15.59$ inches^2

8.5 Tangent Ratio

Learning Objectives

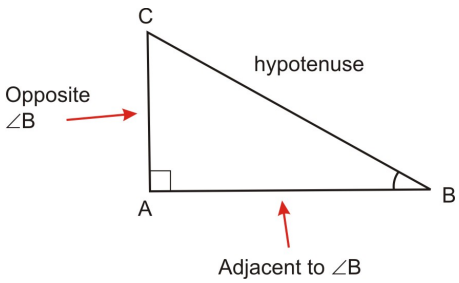
- Identify the different parts of right triangles.
- Identify and use the tangent ratio in a right triangle.
- Identify complementary angles in right triangles.
- Understand tangent ratios in special right triangles.

Introduction

Now that you are familiar with right triangles, the ratios that relate the sides, as well as other important applications, it is time to learn about trigonometric ratios. Trigonometric ratios show the relationship between the sides of a triangle and the angles inside of it. This lesson focuses on the tangent ratio.

Parts of a Triangle

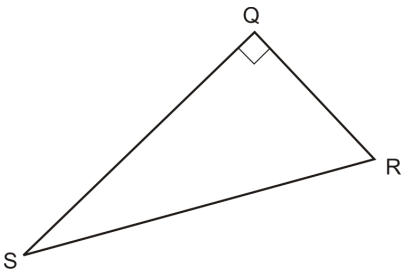
In trigonometry, there are a number of different labels attributed to different sides of a right triangle. They are usually in reference to a specific angle. The hypotenuse of a triangle is always the same, but the terms **adjacent** and **opposite** depend on which angle you are referencing. A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle.



In the triangle shown above, segment $\overline{AB}$ is adjacent to $\angle B$, and segment $\overline{AC}$ is opposite to $\angle B$. Similarly, $\overline{AC}$ is adjacent to $\angle x$, and $\overline{AB}$ is opposite $\angle C$. The hypotenuse is always $\overline{BC}$.

Example 1

Examine the triangle in the diagram below.



Identify which leg is adjacent to $\angle R$, opposite to $\angle R$, and the hypotenuse.

The first part of the question asks you to identify the leg adjacent to $\angle R$. Since an adjacent leg is the one that helps to form the angle and is not the hypotenuse, it must be $\overline{QR}$. The next part of the question asks you to identify the leg opposite $\angle R$. Since an opposite leg is the leg that does not help to form the angle, it must be $\overline{QS}$. The hypotenuse is always opposite the right angle, so in this triangle the hypotenuse is segment $\overline{RS}$.

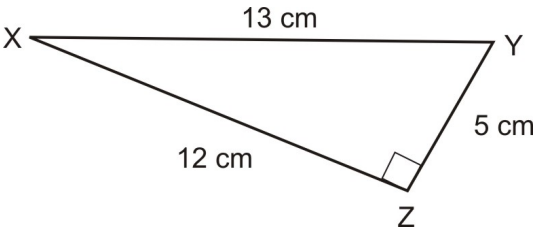
The Tangent Ratio

The first ratio to examine when studying right triangles is the tangent. The tangent of an angle is the ratio of the length of the opposite side to the length of the adjacent side. The hypotenuse is not involved in the tangent at all. Be sure when you find a tangent that you find the opposite and adjacent sides *relative to the angle in question.*

For an acute angle measuring x, we define $\tan x = \frac{\text{opposite}}{\text{adjacent}}$.

Example 2

What are the tangents of $\angle X$ and $\angle Y$ in the triangle below?



To find these ratios, first identify the sides opposite and adjacent to each angle.

$$\tan \angle X = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12} \approx 0.417$$
$$\tan \angle Y = \frac{\text{opposite}}{\text{adjacent}} = \frac{12}{5} = 2.4$$

So, the tangent of $\angle X$ is about 0.417 and the tangent of $\angle Y$ is 2.4.

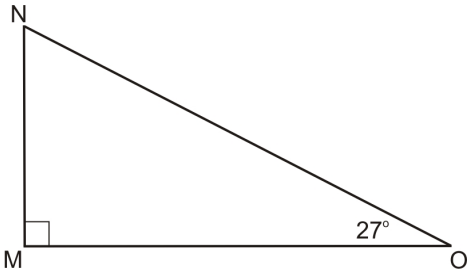
It is common to write $\tan X$ instead of $\tan \angle X$. In this text we will use both notations.

Complementary Angles in Right Triangles

Recall that in all triangles, the sum of the measures of all angles must be 180°. Since a right angle has a measure of 90°, the remaining two angles in a right triangle must be **complementary**. Complementary angles have a sum of 90°. This means that if you know the measure of one of the smaller angles in a right triangle, you can easily find the measure of the other. Subtract the known angle from 90° and you'll have the measure of the other angle.

Example 3

What is the measure of $\angle N$ in the triangle below?



To find $m\angle N$, you can subtract the measure of $\angle N$ from 90°.

$$\begin{aligned} m\angle N + m\angle O &= 90 \\ m\angle N &= 90 - m\angle O \\ m\angle N &= 90 - 27 \\ m\angle N &= 63 \end{aligned}$$

So, the measure of $\angle N$ is 63° since $\angle N$ and $\angle O$ are complementary.

Tangents of Special Right Triangles

It may help you to learn some of the most common values for tangent ratios. The table below shows you values for angles in special right triangles.

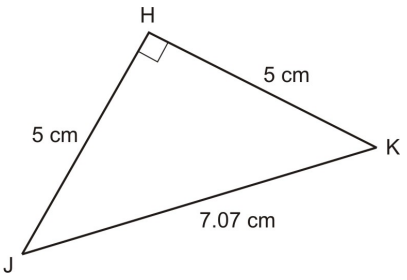
Table 8.2:

	30°	45°	60°
Tangent	$\frac{1}{\sqrt{3}} \approx 0.577$	$\frac{1}{1} = 1$	$\frac{\sqrt{3}}{1} \approx 1.732$

Notice that you can derive these ratios from the $30^\circ - 60^\circ - 90^\circ$ special right triangle. You can use these ratios to identify angles in a triangle. Work backwards from the ratio. If the ratio equals one of these values, you can identify the measurement of the angle.

Example 4

What is $m\angle J$ in the triangle below?



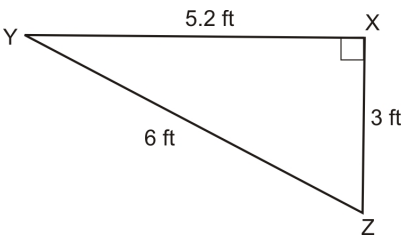
Find the tangent of $\angle J$ and compare it to the values in the table above.

$$\begin{aligned} \tan J &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{5}{5} \\ &= 1 \end{aligned}$$

So, the tangent of $\angle J$ is 1. If you look in the table, you can see that an angle that measures 45° has a tangent of 1. So, $m\angle J = 45^\circ$.

Example 5

What is $m\angle Z$ in the triangle below?



Find the tangent of $\angle Z$ and compare it to the values in the table above.

$$\begin{aligned}\tan Z &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{5.2}{3} \\ &= 1.7\bar{3}\end{aligned}$$

So, the tangent of $\angle Z$ is about 1.73. If you look in the table, you can see that an angle that measures 60° has a tangent of 1.732. So, $m\angle z \approx 60^\circ$.

Notice in this example that $\triangle XYZ$ is a $30^\circ - 60^\circ - 90^\circ$ triangle. You can use this fact to see that $XY = 5.2 \approx 3\sqrt{3}$.

Lesson Summary

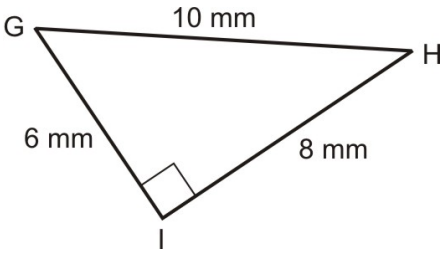
In this lesson, we explored how to work with different radical expressions both in theory and in practical situations. Specifically, we have learned:

- How to identify the different parts of right triangles.
- How to identify and use the tangent ratio in a right triangle.
- How to identify complementary angles in right triangles.
- How to understand tangent ratios in special right triangles.

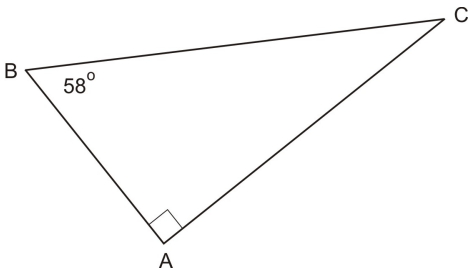
These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to find relationships between sides and angles in triangles.

Review Qusetions

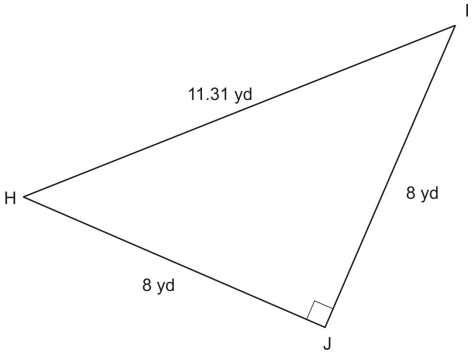
Use the following diagram for exercises 1-5.



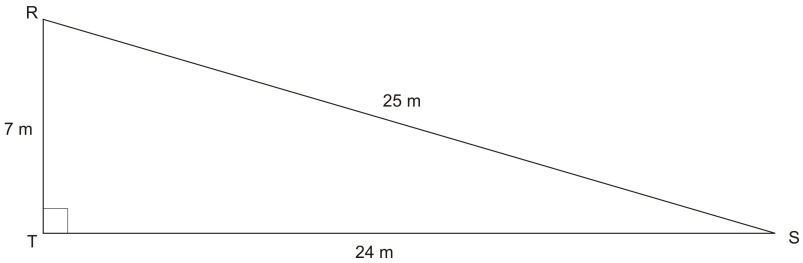
1. How long is the side opposite angle G?
2. How long is the side adjacent to angle G?
3. How long is the hypotenuse?
4. What is the tangent of $\angle G$?
5. What is the tangent of $\angle H$?
6. What is the measure of $\angle C$ in the diagram below?



7. What is the measure of $\angle H$ in the diagram below?

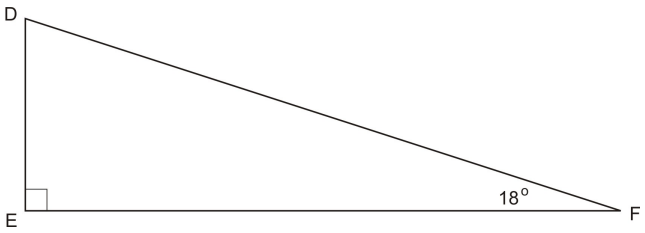


Use the following diagram for exercises 8-9.



8. What is the tangent of $\angle R$?
9. What is the tangent of $\angle S$?

10. What is the measure of $\angle E$ in the triangle below?



Review Answers

1. 8 mm
2. 6 mm
3. 10 mm
4. $\frac{8}{6} = 1.\overline{3}$
5. $\frac{6}{8} = \frac{3}{4} = 0.75$
6. 32°
7. 45°
8. $\frac{7}{24} = 0.292$
9. $\frac{24}{7} = 3.43$
10. 72°

8.6 Sine and Cosine Ratios

Learning Objectives

- Review the different parts of right triangles.
- Identify and use the sine ratio in a right triangle.
- Identify and use the cosine ratio in a right triangle.
- Understand sine and cosine ratios in special right triangles.

Introduction

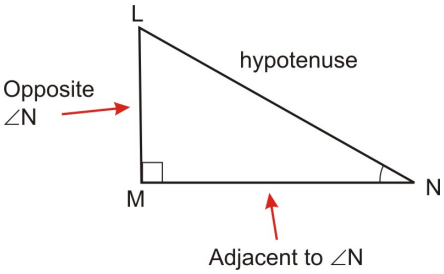
Now that you have some experience with tangent ratios in right triangles, there are two other basic types of trigonometric ratios to explore. The **sine** and **cosine ratios** relate opposite and adjacent sides of a triangle to the hypotenuse. Using these three ratios and a calculator or a table of trigonometric ratios you can solve a wide variety of problems!

Review: Parts of a Triangle

The sine and cosine ratios relate opposite and adjacent sides to the hypotenuse. You already learned these terms in the previous lesson, but they are important to review and commit to memory. The hypotenuse of a triangle is always opposite the right angle, but the terms adjacent and opposite depend on which angle you are referencing. A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle.

Example 1

Examine the triangle in the diagram below.



Identify which leg is adjacent to angle N, which leg is opposite to angle N, and which segment is the hypotenuse.

The first part of the question asks you to identify the leg adjacent to $\angle N$. Since an adjacent leg is the one that helps to form the angle and is not the hypotenuse, it must be $\overline{MN}$. The next part of the question asks you to identify the leg opposite $\angle N$. Since an opposite leg is the leg that does not help to form the angle, it must be $\overline{LM}$. The hypotenuse is always opposite the right angle, so in this triangle it is segment $\overline{LN}$.

The Sine Ratio

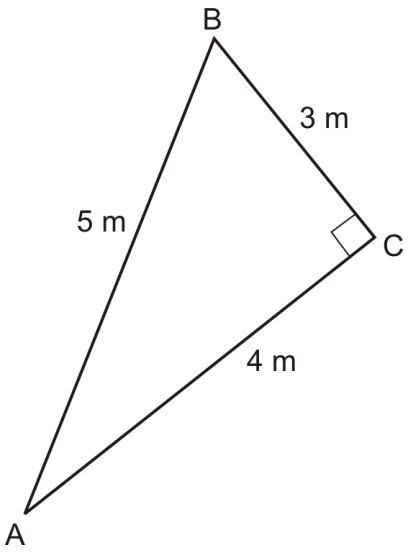
Another important trigonometric ratio is **sine**. A sine ratio must always refer to a particular angle in a right triangle. The sine of an angle is the ratio of the length of the leg opposite the angle to the length of the hypotenuse. Remember that in a ratio, you list the first item on top of the fraction and the second item on the bottom.

So, the ratio of the sine will be

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Example 2

What are $\sin A$ *and* $\sin B$ *in the triangle below?*



All you have to do to find the solution is build the ratio carefully.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$
$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

So, $\sin A = 0.6$ and $\sin B = 0.8$.

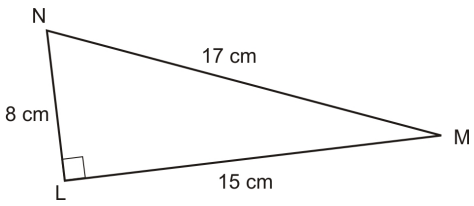
The Cosine Ratio

The next ratio to examine is called the **cosine**. The cosine is the ratio of the adjacent side of an angle to the hypotenuse. Use the same techniques you used to find sines to find cosines.

$$\cos(\text{angle}) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Example 3

What are the cosines of $\angle M$ and $\angle N$ in the triangle below?



To find these ratios, identify the sides adjacent to each angle and the hypotenuse. Remember that an adjacent side is the one that does create the angle and is not the hypotenuse.

$$\cos M = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17} \approx 0.88$$
$$\cos N = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{8}{17} \approx 0.47$$

So, the cosine of $\angle M$ is about 0.88 and the cosine of $\angle N$ is about 0.47.

Note that $\triangle LMN$ is NOT one of the special right triangles, but it is a right triangle whose sides are a Pythagorean triple.

Sines and Cosines of Special Right Triangles

It may help you to learn some of the most common values for sine and cosine ratios. The table below shows you values for angles in special right triangles.

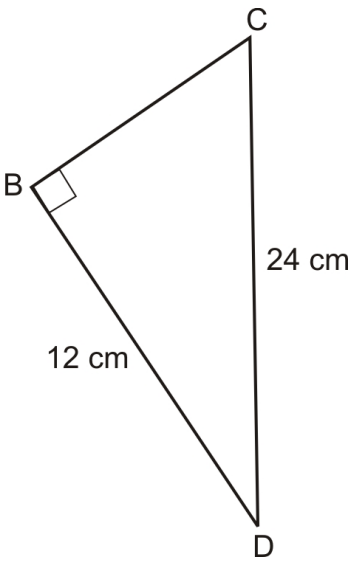
Table 8.3:

	30°	45°	60°
Sine	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{\sqrt{3}}{2} \approx 0.866$
Cosine	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{1}{2} = 0.5$

You can use these ratios to identify angles in a triangle. Work backwards from the ratio. If the ratio equals one of these values, you can identify the measurement of the angle.

Example 4

What is the measure of $\angle C$ in the triangle below?



Note: Figure is not to scale.

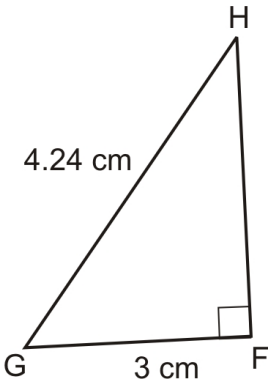
Find the sine of $\angle C$ and compare it to the values in the table above.

$$\begin{aligned}\sin C &= \frac{\text{opposite}}{\text{hypotenuse}} \\ &= \frac{12}{24} \\ &= 0.5\end{aligned}$$

So, the sine of $\angle C$ is 0.5. If you look in the table, you can see that an angle that measures 30° has a sine of 0.5. So, $m\angle C = 30^\circ$.

Example 5

What is the measure of $\angle G$ in the triangle below?



Find the cosine of $\angle G$ and compare it to the values in the previous table.

$$\begin{aligned}\cos G &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{3}{4.24} \\ &= 0.708\end{aligned}$$

So, the cosine of $\angle G$ is about 0.708. If you look in the table, you can see that an angle that measures 45° has a cosine of 0.707. So, $\angle G$ measures about 45°. This is a $45^\circ - 45^\circ - 90^\circ$ right triangle.

Lesson Summary

In this lesson, we explored how to work with different trigonometric ratios both in theory and in practical situations. Specifically, we have learned:

- The different parts of right triangles.
- How to identify and use the sine ratio in a right triangle.
- How to identify and use the cosine ratio in a right triangle.
- How to apply sine and cosine ratios in special right triangles.

These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to find relationships between sides and angles in triangles.

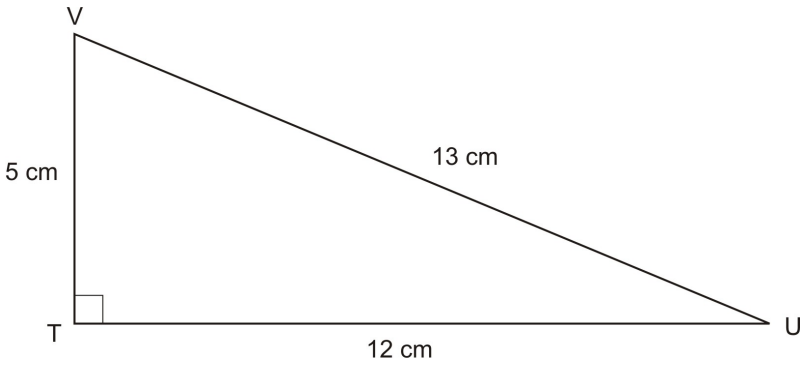
Points to Consider

Before you begin the next lesson, think about strategies you could use to simplify an equation that contains a trigonometric function.

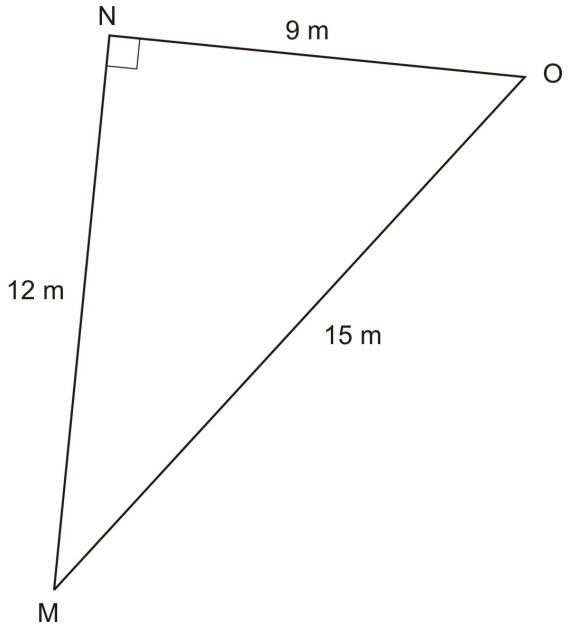
Note, you can only use the $\sin, \cos$, and $\tan$ ratios on the acute angles of a right triangle. For now it only makes sense to talk about the $\sin, \cos$, or $\tan$ ratio of an acute angle. Later in your mathematics studies you will redefine these ratios in a way that you can talk about $\sin, \cos$, and $\tan$ of acute, obtuse, and even negative angles.

Review Questions

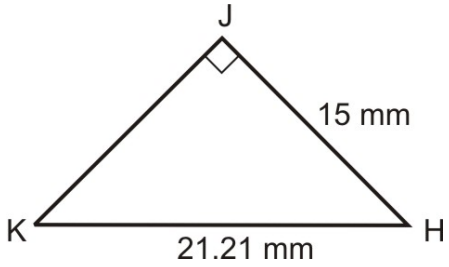
Use the following diagram for exercises 1-3.



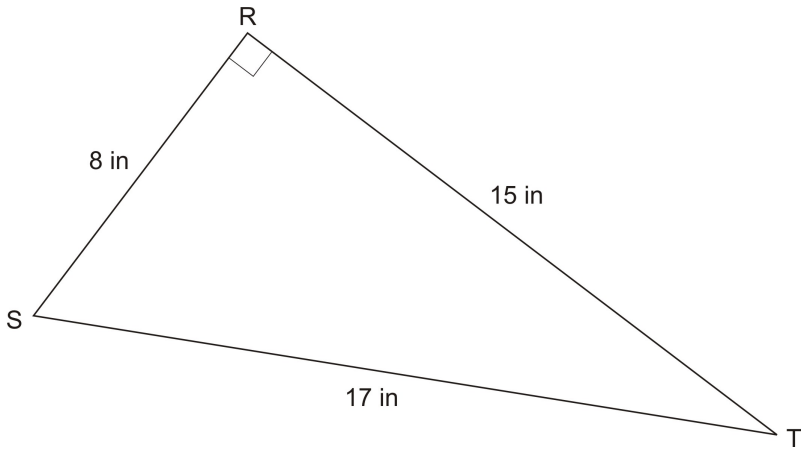
1. What is the sine of $\angle V$?
2. What is the cosine of $\angle V$?
3. What is the cosine of $\angle U$?Use the following diagram for exercises 4-6.



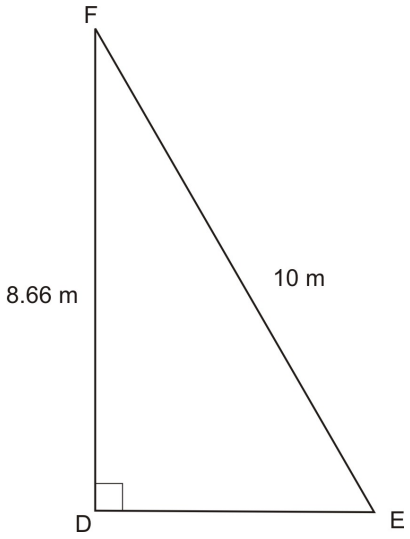
4. What is the sine of $\angle O$?
5. What is the cosine of $\angle O$?
6. What is the sine of $\angle M$?
7. What is the measure of $\angle H$ in the diagram below?



Use the following diagram for exercises 8-9.



8. What is the sine of $\angle S$?
9. What is the cosine of $\angle S$?
10. What is the measure of $\angle E$ in the triangle below?



Review Answers

1. $\frac{12}{13} \approx 0.923$ cm
2. $\frac{5}{13} \approx 0.385$ cm
3. $\frac{12}{13} \approx 0.923$ cm
4. $\frac{12}{15} = \frac{4}{5} = 0.8$ inches
5. $\frac{9}{15} = \frac{3}{5} = 0.6$ inches
6. $\frac{9}{15} = \frac{3}{5} = 0.6$ inches
7. 45°
8. $\frac{8}{17} \approx 0.471$
9. $\frac{15}{17} \approx 0.882$

10. 60°

8.7 Inverse Trigonometric Ratios

Learning Objectives

- Identify and use the arctangent ratio in a right triangle.
- Identify and use the arcsine ratio in a right triangle.
- Identify and use the arccosine ratio in a right triangle.
- Understand the general trends of trigonometric ratios.

Introduction

The word **inverse** is probably familiar to you often in mathematics, after you learn to do an operation, you also learn how to "undo" it. Doing the inverse of an operation is a way to undo the original operation. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities. You may also remember the term *additive inverse*, or a number that can be added to the original to yield a sum of 0. For example, 5 and -5 are additive inverses because $5 + (-5) = 0$.

In this lesson you will learn to use the inverse operations of the trigonometric functions you have studied thus far. You can use inverse trigonometric functions to find the measures of angles when you know the lengths of the sides in a right triangle.

Inverse Tangent

When you find the inverse of a trigonometric function, you put the word *arc* in front of it. So, the inverse of a tangent is called the arctangent (or arctan for short). Think of the arctangent as a tool you can use like any other inverse operation when solving a problem. If tangent tells you the ratio of the lengths of the sides opposite and adjacent to an angle, then arctan tells you the measure of an angle with a given ratio.

Suppose $\tan X = 0.65$. The arctangent can be used to find the measure of $\angle X$ on the left side of the equation.

$$\begin{aligned} \arctan(\tan X) &= \arctan(0.65) \\ m\angle X &= \arctan(0.65) \approx 33^\circ \end{aligned}$$

Where did that 33° come from? There are two basic ways to find an arctangent. Sometimes you will be given a table of trigonometric values and the angles to which they correspond. In this scenario, find the value that is closest to the one provided, and identify the corresponding angle.

Another, easier way of finding the arctangent is to use a calculator. The arctangent button may be labeled "arctan," "atan," or " $\tan^{-1}$." Either way, select this button, and input the value in question. In this case, you would press the arctangent button and enter 0.65 (or on some calculators, enter .65, then press "arctan"). The output will be the value of measure $\angle X$.

$$m\angle X = \arctan(0.65)$$
$$m\angle X \approx 33$$

$m\angle X$ is about 33°.

Example 1

Solve for $m\angle Y$ *if* $\tan Y = 0.384$

You can use the inverse of tangent, arctangent to find this value.

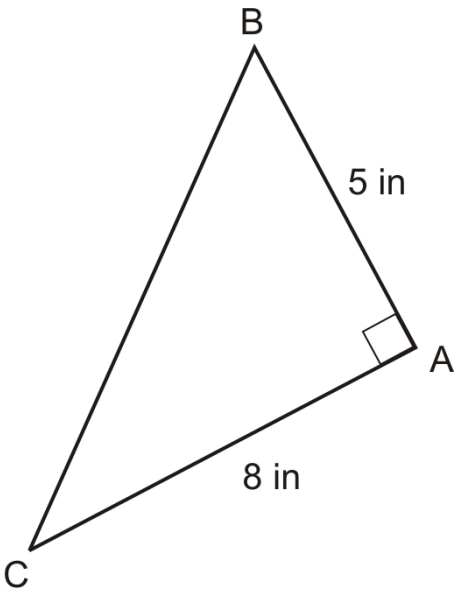
$$\arctan(\tan Y) = \arctan(0.384)$$
$$m\angle Y = \arctan(0.384)$$

Then use your calculator to find the arctangent of 0.384.

$$m\angle Y \approx 21^\circ$$

Example 2

What is $m\angle B$ *in the triangle below?*



First identify the proper trigonometric ratio related to $\angle B$ that can be found using the sides given. The tangent uses the opposite and adjacent sides, so it will be relevant here.

$$\begin{aligned}\tan B &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{8}{5} \\ &= 1.6\end{aligned}$$

Now use the arctangent to solve for the measure of $\angle B$.

$$\begin{aligned}\arctan(\tan B) &= \arctan(1.6) \\ m\angle B &= \arctan(1.6)\end{aligned}$$

Then use your calculator to find the arctangent of 1.6.

$$m\angle B \approx 58^\circ$$

Inverse Sine

Just as you used arctangent as the inverse operation for tangent, you can also use arcsine (shortened as arcsin) as the inverse operation for sine. The same rules apply. You can use it

to isolate a variable for an angle measurement, but you must perform the operation on both sides of the equation. When you know the arcsine value, use a table or a calculator to find the measure of the angle.

Example 3

Solve for $m\angle P$ *if* $\sin P = 0.891$

You can use the inverse of sine, arcsine to find this value.

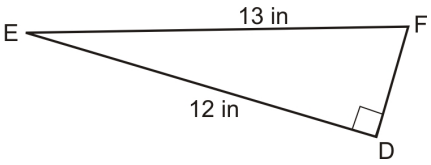
$$\begin{aligned} \arcsin(\sin P) &= \arcsin(0.891) \\ m\angle P &= \arcsin(0.891) \end{aligned}$$

Then use your calculator to find the arcsine of 0.891.

$$m\angle P \approx 63^\circ$$

Example 4

What is $m\angle F$ *in the triangle below?*



First identify the proper trigonometric ratio related to angle F that can be found using the sides given. The sine uses the opposite side and the hypotenuse, so it will be relevant here.

$$\begin{aligned} \sin F &= \frac{\text{opposite}}{\text{adjacent}} \\ \sin F &= \frac{12}{13} \\ \sin F &\approx 0.923 \end{aligned}$$

Now use the arcsine to isolate the value of angle F.

$$\begin{aligned} \arcsin(\sin F) &= \arcsin(0.923) \\ m\angle F &= \arcsin(0.923) \end{aligned}$$

Finally, use your calculator to find the arcsine of 0.923.

$$m\angle F \approx 67^\circ$$

Inverse Cosine

The last inverse trigonometric ratio is arccosine (often shortened to arccos). The same rules apply for arccosine as apply for all other inverse trigonometric functions. You can use it to isolate a variable for an angle measurement, but you must perform the operation on both sides of the equation. When you know the arccosine value, use a table or a calculator to find the measure of the angle.

Example 5

Solve for $m\angle Z$ *if* $\cos Z = 0.31$.

You can use the inverse of cosine, arccosine, to find this value.

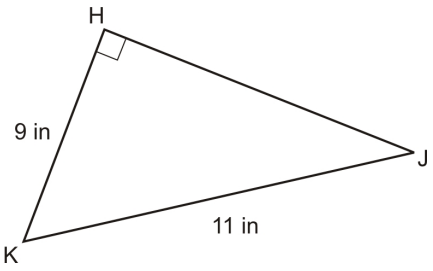
$$\begin{aligned} \arccos(\cos Z) &= \arccos(0.31) \\ m\angle Z &= \arccos(0.31) \end{aligned}$$

Then use your calculator to find the arccosine of 0.31.

$$m\angle Z \approx 72^\circ$$

Example 6

What is the measure of $\angle K$ *in the triangle below?*



First identify the proper trigonometric ratio related to $\angle K$ that can be found using the sides given. The cosine uses the adjacent side and the hypotenuse, so it will be relevant here.

$$\begin{aligned}\cos K &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{9}{11} \\ &= 0.818\end{aligned}$$

Now use the arccosine to isolate the value of $\angle K$.

$$\begin{aligned}\arccos(\cos K) &= \arccos(0.818) \\ m\angle K &= \arccos(0.818)\end{aligned}$$

Finally use your calculator or a table to find the arccosine of 0.818.

$$m\angle K \approx 35^\circ$$

General Trends in Trigonometric Ratios

Now that you know how to find the trigonometric ratios as well as their inverses, it is helpful to look at trends in the different values. Remember that each ratio will have a constant value for a specific angle. In any right triangle, the sine of a 30° angle will always be 0.5—it doesn't matter how long the sides are. You can use that information to find missing lengths in triangles where you know the angles, or to identify the measure of an angle if you know two of the sides.

Examine the table below for trends. It shows the sine, cosine, and tangent values for eight different angle measures.

Table 8.4:

	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.174	0.342	0.5	0.643	0.766	0.866	0.940	0.985
Cosine	0.985	0.940	0.866	0.766	0.643	0.5	0.342	0.174
Tangent	0.176	0.364	0.577	0.839	1.192	1.732	2.747	5.671

Example 7

Using the table above, which value would you expect to be greater: the sine of 25° *or the*

cosine of 25°*?*

You can use the information in the table to solve this problem. The sine of 20° is 0.342 and the sine of 30° is 0.5. So, the sine of 25° will be between the values 0.342 and 0.5. The cosine of 20° is 0.940 and the cosine of 30° is 0.866. So, the cosine of 25° will be between the values of 0.866 and 0.940. Since the range for the cosine is greater, than the range for the sine, it can be assumed that the cosine of 25° will be greater than the sine of 25°.

Notice that as the angle measures approach 90°, sin approaches 1. Similarly, as the value of the angles approach 90°, the cos approaches 0. In other words, as the sin gets greater, the cos gets smaller for the angles in this table.

The tangent, on the other hand, increases rapidly from a small value to a large value (infinity, in fact) as the angle approaches 90°.

Lesson Summary

In this lesson, we explored how to work with different radicals both in theory and in practical situations. Specifically, we have learned:

- How to identify and use the arctangent ratio in a right triangle.
- How to identify and use the arcsine ratio in a right triangle.
- How to identify and use the arccosine ratio in a right triangle.
- How to understand the general trends of trigonometric ratios.

These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to find relationships between sides and angles in triangles.

Points to Consider

To this point, all of the trigonometric ratios you have studied have dealt exclusively with right triangles. Can you think of a way to use trigonometry on triangles that are acute or obtuse?

Review Questions

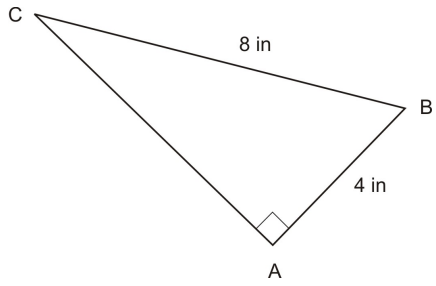
1. Solve for $m\angle G$.

$$\cos G = 0.53$$

2. Solve for $m\angle V$.

$$\tan V = 2.25$$

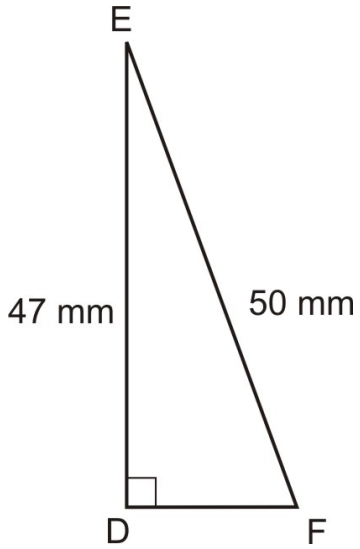
3. What is the measure of $\angle B$ in the triangle below?



4. Solve for $m\angle M$.

$$\sin M = 0.978$$

5. What is the measure of $\angle F$ in the triangle below?



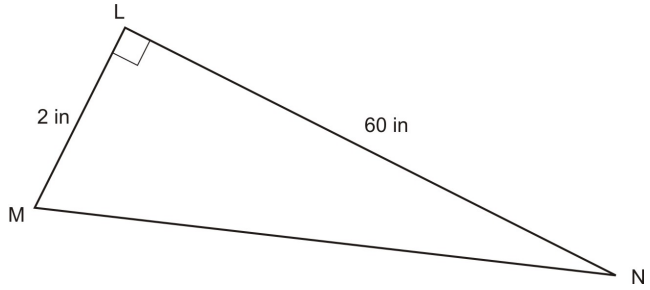
6. Solve for $m\angle L$.

$$\tan L = 1.04$$

7. Solve for $m\angle D$.

$$\cos D = 0.07$$

8. What is the measure of $\angle M$ in the triangle below?



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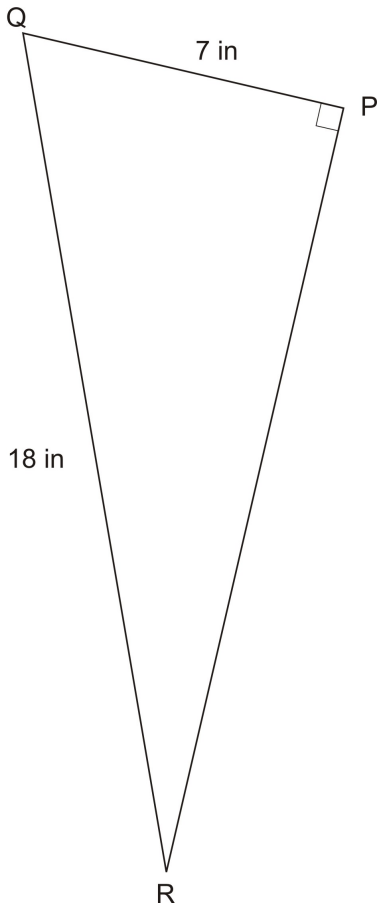
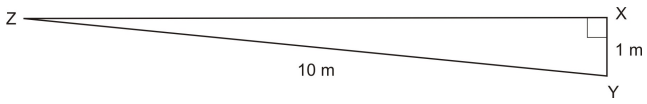
9. What is the measure of $\angle Q$ in the triangle below?



10. What is the measure of $\angle Z$ in the triangle below?



Review Answers

1. 58°
2. 66°
3. 60°
4. 78°
5. 70°
6. 46°
7. 86°
8. 89°
9. 67°
10. 5.7°

8.8 Acute and Obtuse Triangles

Learning Objectives

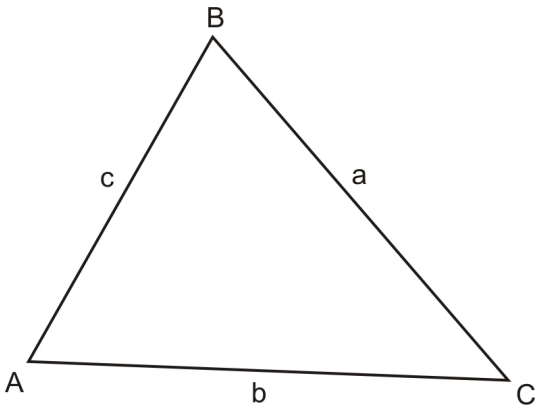
- Identify and use the Law of Sines.
- Identify and use the Law of Cosines.

Introduction

Trigonometry is most commonly learned on right triangles, but the ratios can have uses for other types of triangles, too. This lesson focuses on how you can apply sine and cosine ratios to angles in acute or obtuse triangles. Remember that in an acute triangle, all angles measure less than 90°. In an obtuse triangle, there will be one angle that has a measure that is greater than 90°.

The Law of Sines

The **Law of Sines** states that in any triangle, the ratio of the length of a side to the sine of the angle opposite it will be constant. That is, the ratio is the same for all three angles and their opposite sides. Thus, if you find the ratio, you can use it to find missing angle measure and side lengths.

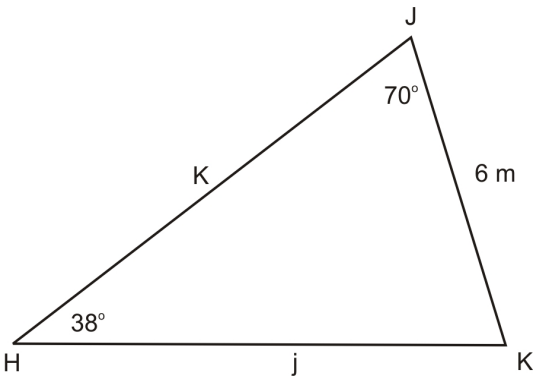


$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note the convention that A denotes $\angle A$ and a is the length of the side opposite $\angle A$.

Example 1

Examine the triangle in the following diagram.



What is the length of the side labeled j?

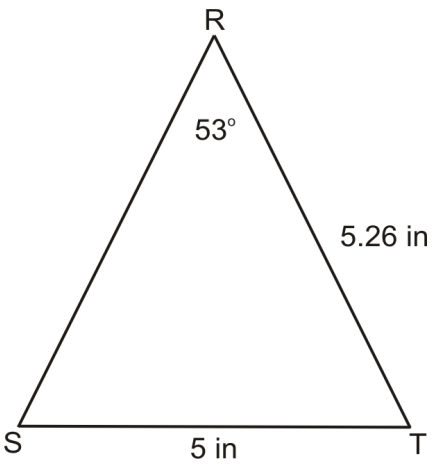
You can use the law of sines to solve this problem. Because you have one side and the angle opposite, you can find the constant that applies to the entire triangle. This ratio will be equal to the proportion of side j and $\angle J$. You can use your calculator to find the value of the sines.

$$\begin{aligned}
\frac{h}{\sin H} &= \frac{j}{\sin J} \\
\frac{6}{\sin 38} &= \frac{7}{\sin 70} \\
\frac{6}{0.616} &= \frac{j}{0.940} \\
9.74 &= \frac{j}{0.940} \\
9.2 &\approx j
\end{aligned}$$

So, using the law of sines, the length of j is about 9.2 meters.

Example 2

Examine the triangle in the following diagram.



What is the measure of $\angle S$?

You can use the law of sines to solve this problem. Because you have one side and the angle opposite, you can find the constant that applies to the entire triangle. This ratio will be equal to the proportion of side r and angle R. You can use your calculator to find the value of the sines.

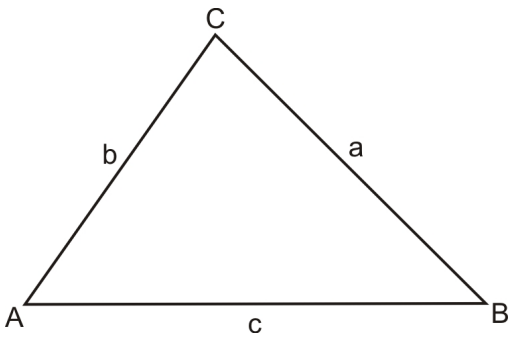
$$\begin{aligned}
\frac{r}{\sin R} &= \frac{s}{\sin S} \\
\frac{5}{\sin 53} &= \frac{5.26}{\sin S} \\
\frac{5}{0.788} &= \frac{5.26}{\sin S} \\
6.345 &= \frac{5.26}{\sin S} \\
(6.345) \cdot \sin S &= 5.26 \\
\sin S &= \frac{5.26}{6.345} \\
\sin S &= 0.829 \\
\arcsin(\sin S) &= \arcsin 0.829 \\
m\angle S &\approx 56^\circ
\end{aligned}$$

So, using the law of sines, the angle labeled S must measure about 56°.

The Law of Cosines

There is another law that works on acute and obtuse triangles in addition to right triangles. The **Law of Cosines** uses the cosine ratio to identify either lengths of sides or missing

angles. To use the law of cosines, you must have either the measures of all three sides, or the measure of two sides and the measure of the included angle.

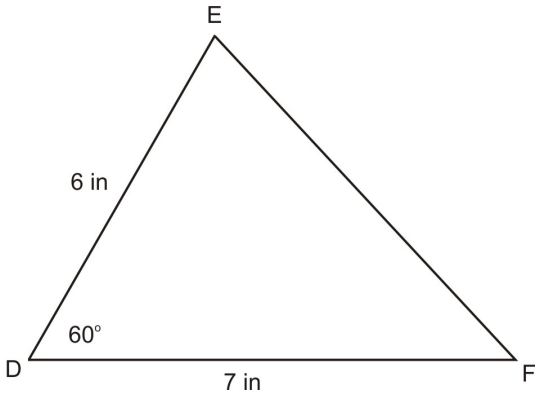


$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

It doesn't matter how you assign the variables to the three sides of the triangle, but the angle C must be opposite side c.

Example 3

Examine the triangle in the following diagram.



What is the measure of side $\overline{EF}$?

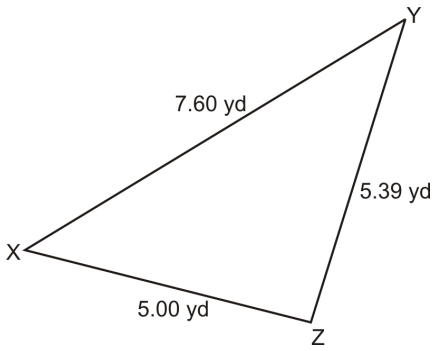
Use the Law of Cosines to find EF. Since $\overline{EF}$ is opposite $\angle D$, we will call the length of $\overline{EF}$ by the letter d.

$$
\begin{aligned}
d^2 &= e^2 + f^2 - 2ab(\cos D) \\
d^2 &= (6)^2 + (7)^2 - 2(6)(7)(\cos 60) \\
d^2 &= 36 + 49 - 84(\cos 60) \\
d^2 &= 85 - 84(0.5) \\
d^2 &= 85 - 42 \\
d^2 &= 43 \\
d &= \sqrt{43} \\
d &\approx 6.56
\end{aligned}
$$

So, EF is about 6.56 inches.

Example 4

Examine the triangle in the following diagram.



What is the measure of $\angle X$?

Use the Law of Cosines to find the measure of $\angle X$.

$$
\begin{aligned}
x^2 &= y^2 + z^2 - 2yz(\cos X) \\
(5.39)^2 &= (5)^2 + (7.6)^2 - 2(7.6)(5)(\cos X) \\
29.05 &= 25 + 57.76 - 76(\cos X) \\
29.05 &= 82.76 - 76(\cos X) \\
-53.71 &= -76(\cos X) \\
0.707 &= (\cos X) \\
\arccos(0.707) &= \arccos(cosX) \\
45^\circ &\approx m\angle X
\end{aligned}
$$

So, $m\angle X$ is about 45°.

Lesson Summary

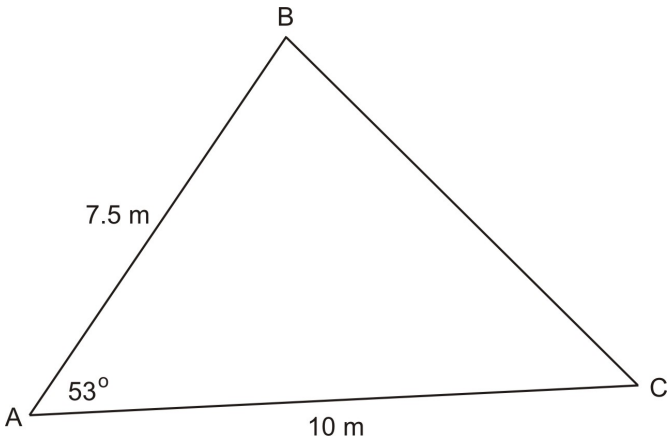
In this lesson, we explored how to work with different radical expressions both in theory and in practical situations. Specifically, we have learned:

- how to identify and use the law of sines.
- how to identify and use the law of cosines.

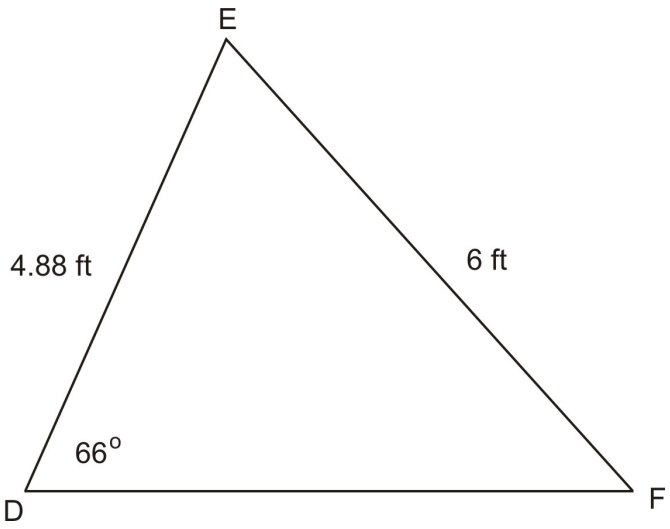
These skills will help you solve many different types of problems. Always be on the lookout for new and interesting ways to find relationships between sides and angles in triangles.

Review Questions

Exercises 1 and 2 use the triangle in the following diagram.

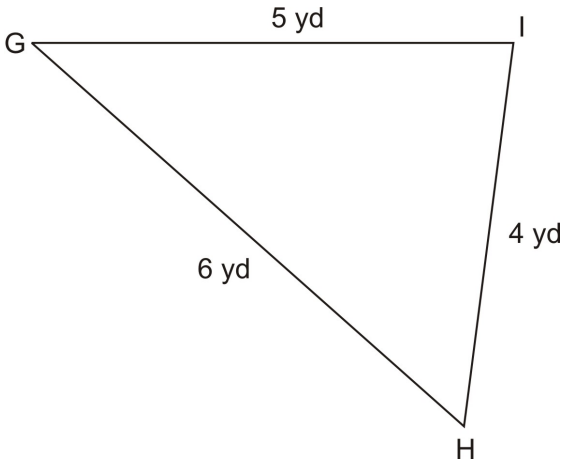


1. What is the length of side BC?
2. What is $m\angle C$?
3. Examine the triangle in the following diagram.



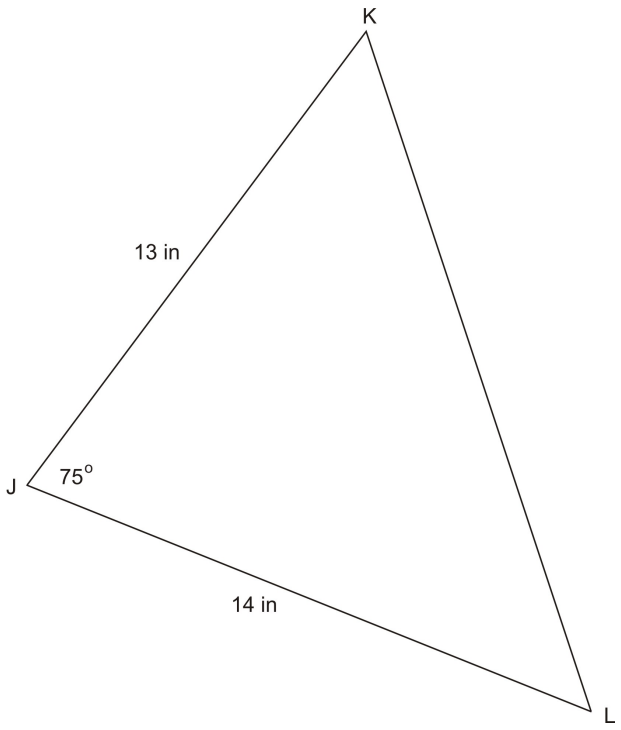
What is the measure of $\angle F$?

4. Examine the triangle in the following diagram.



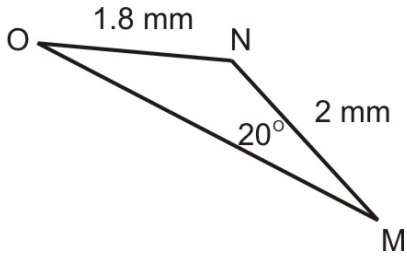
What is the measure of $\angle I$?

5. Examine the triangle in the following diagram.



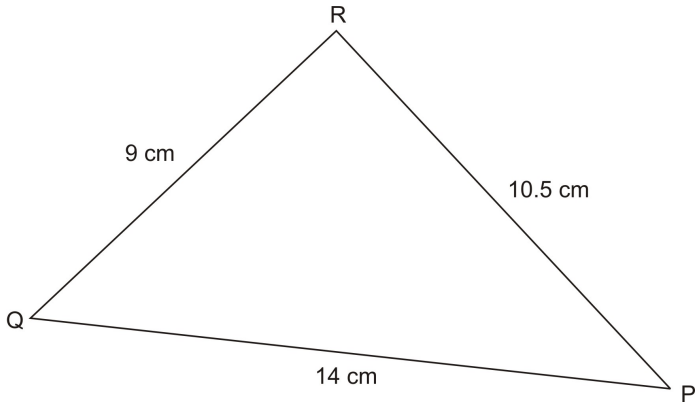
What is the measure of side KL?

6. Examine the triangle in the following diagram.



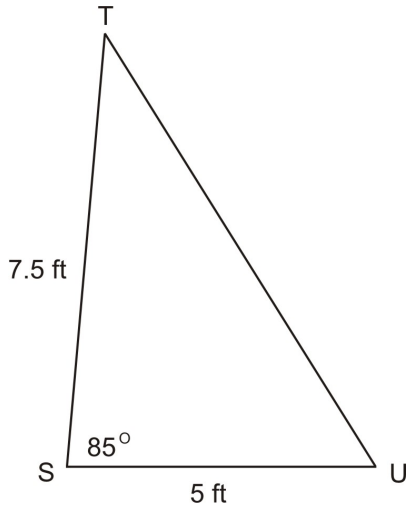
What is the measure of $\angle O$?

Use the triangle in the following diagram for exercises 7 and 8.



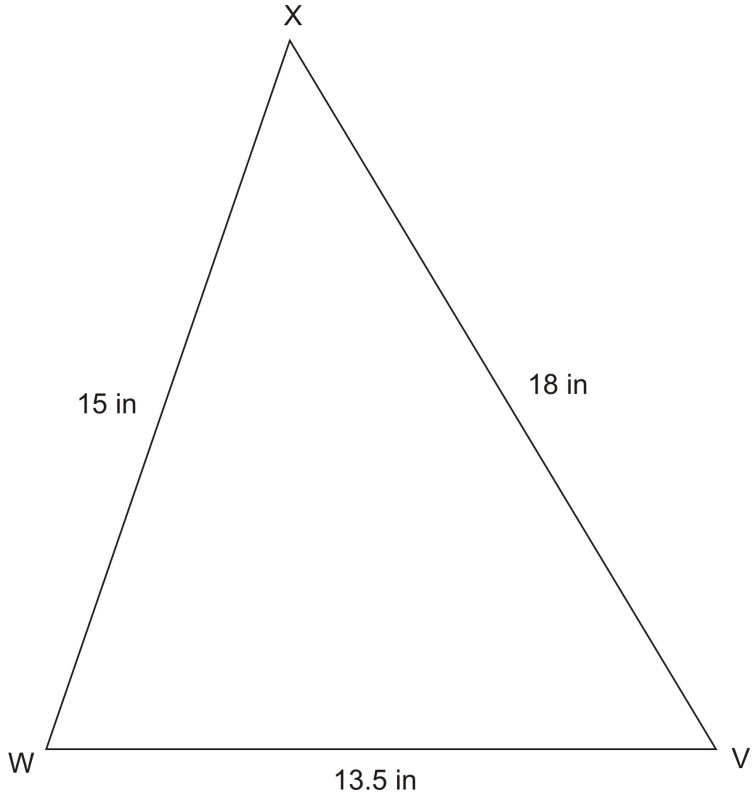
7. What is the measure of $\angle P$?

8. What is the measure of $\angle Q$?
9. Examine the triangle in the following diagram.



What is the measure of $\angle T$?

10. Examine the triangle in the following diagram.



What is the measure of $\angle W$?

Review Answers

1. 8 inches
2. 47.6°
3. 48°
4. 83°
5. 16.5 inches
6. 22.3°
7. 40°
8. 48.6°
9. 35°
10. 78°

Chapter 9

Circles

9.1 About Circles

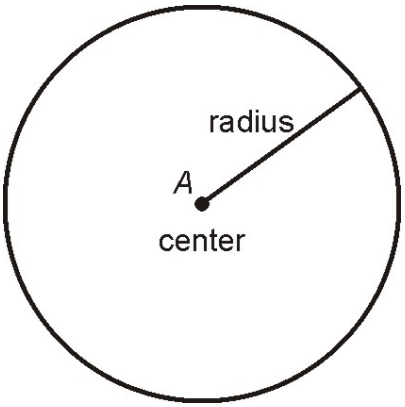
Learning Objectives

- Distinguish between radius, diameter, chord, tangent, and secant of a circle.
- Find relationships between congruent and similar circles.
- Examine inscribed and circumscribed polygons.
- Write the equation of a circle.

Circle, Center, Radius

A **circle** is defined as the set of all points that are the same distance away from a specific point called the **center** of the circle. Note that the circle consists of only the curve but not of the area inside the curve. The distance from the center to the circle is called the **radius** of the circle.

We often label the center with a capital letter and we refer to the circle by that letter. For example, the circle below is called circle A or $\odot A$.

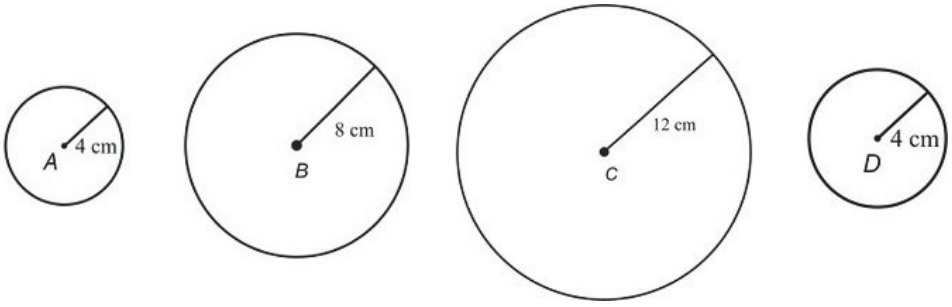


Congruent Circles

Two circles are **congruent** if they have the same radius, regardless of where their centers are located. For example, all circles of radius of 2 centimeters are congruent to each other. Similarly, all circles with a radius of 254 miles are congruent to each other. If circles are not congruent, then they are **similar** with the similarity ratio given by the ratio of their radii.

Example 1

Determine which circles are congruent and which circles are similar. For similar circles find the similarity ratio.



$\odot A$ and $\odot D$ are congruent since they both have a radius of 4 cm.

$\odot A$ and $\odot B$ are similar with similarity ratio of 1 : 2.

$\odot A$ and $\odot C$ are similar with similarity ratio of 1 : 3.

$\odot B$ and $\odot C$ are similar with similarity ratio of 2 : 3.

$\odot B$ and $\odot D$ are similar with similarity ratio of 2 : 1.

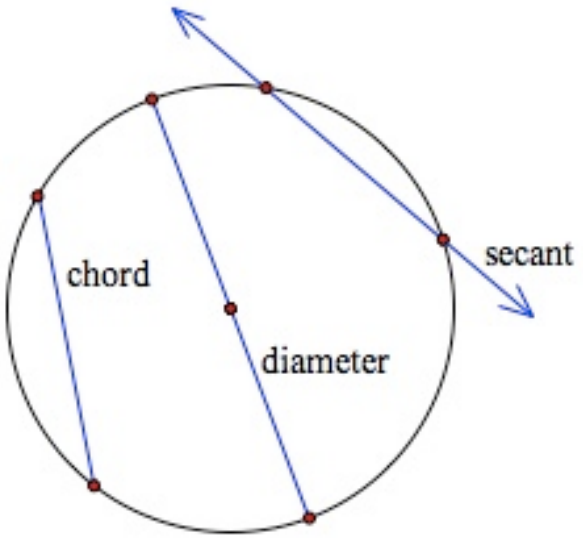
$\odot C$ and $\odot D$ are similar with similarity ratio of 3 : 1.

Chord, Diameter, Secant

A **chord** is defined as a line segment starting at one point on the circle and ending at another point on the circle.

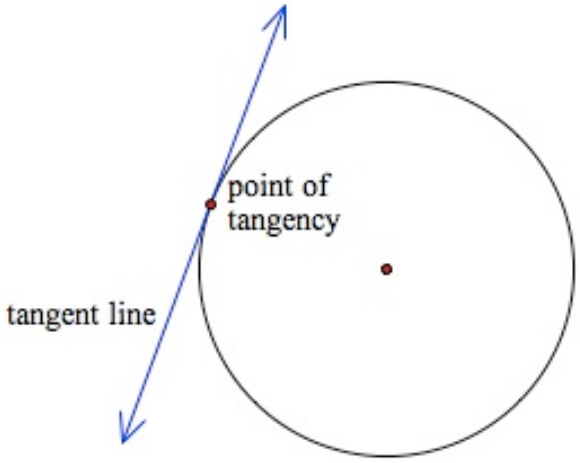
A chord that goes through the center of the circle is called the **diameter** of the circle. Notice that the diameter is twice as long as the radius of the circle.

A **secant** is a line that cuts through the circle and continues infinitely in both directions.



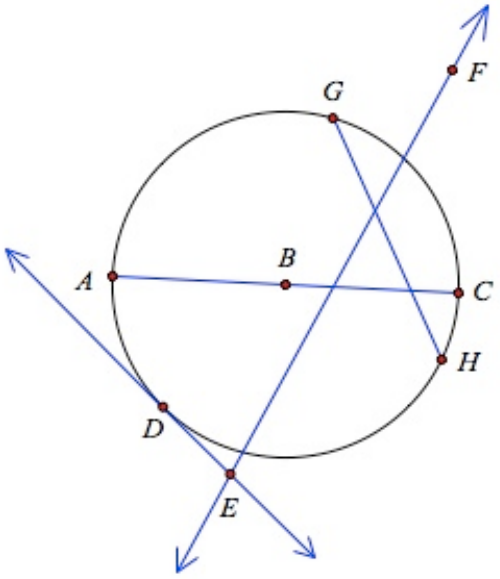
Point of Tangency and Tangent

A **tangent line** is defined as a line that touches the circle at exactly one point. This point is called the **point of tangency.**



Example 2

Identify the following as a secant, chord, diameter, radius, or tangent:



A. $\overline{AC}$

B. $\overline{AB}$

C. $\overline{GH}$

D. $\overline{DE}$

E. $\overline{EF}$

F. $\overline{BC}$

A. $\overline{AC}$ is a diameter of the circle.

B. $\overline{AB}$ is a radius of the circle.

C. $\overline{GH}$ is a chord of the circle.

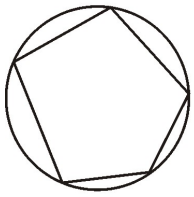
D. $\overline{DE}$ is a tangent of the circle.

E. $\overline{EF}$ is a secant of the circle.

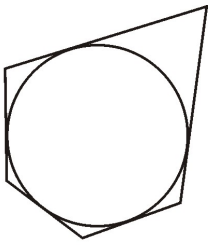
F. $\overline{BC}$ is a radius of the circle.

Inscribed and Circumscribed Polygons

A convex polygon whose vertices all touch a circle is said to be an **inscribed polygon.** A convex polygon whose sides all touch a circle is said to be a **circumscribed polygon.** The figures below show examples of inscribed and circumscribed polygons.



inscribed polygon

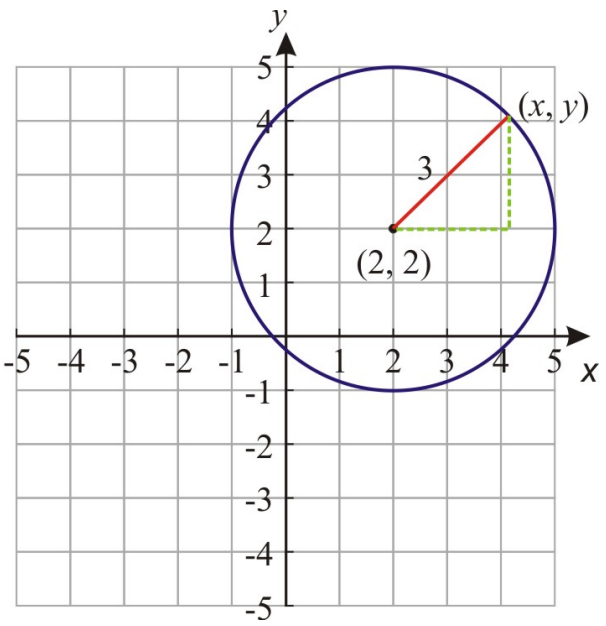


circumscribed polygon

Equations and Graphs of Circles

A circle is defined as the set of all points that are the same distance from a single point called the *center*. This definition can be used to find an equation of a circle in the coordinate plane.

Let's consider the circle shown below. As you can see, this circle has its center at point $(2, 2)$ and it has a radius of 3.



All the points x, y on the circle are a distance of 3 units away from the center of the circle.

We can express this information as an equation with the help of the Pythagorean Theorem. The right triangle shown in the figure has legs of length $x - 2$ and $y - 2$ and hypotenuse of length 3. We write:

$$(x-2)^2 + (y-2)^2 = 9$$

We can generalize this equation for a circle with center at point (x_0, y_0) and radius r.

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

Example 3

Find the center and radius of the following circles:

A. $(x-4)^2 + (y-1)^2 = 25$

B. $(x+1)^2 + (y-2)^2 = 4$

A. We rewrite the equation as: $(x-4)^2 + (y-1)^2 = 5^2$. The center of the circle is at point $(4, 1)$ and the radius is 5.

B. We rewrite the equation as: $(x-(-1))^2 + (y-2)^2 = 2^2$. The center of the circle is at point $(-1, 2)$ and the radius is 2.

Example 4

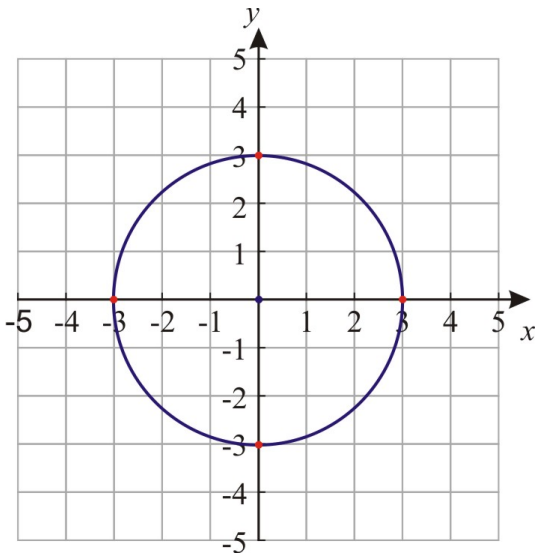
Graph the following circles:

A. $x^2 + y^2 = 9$

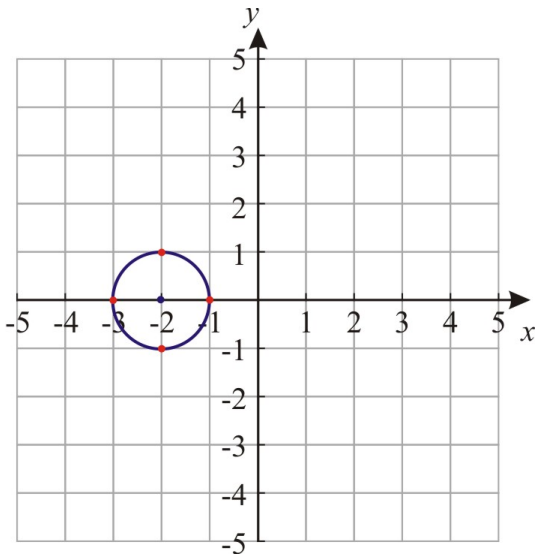
B. $(x + 2)^2 + y^2 = 1$

In order to graph a circle, we first graph the center point and then draw points that are the length of the radius away from the center.

A. We rewrite the equation as: $(x - 0)^2 + (y - 0)^2 = 3^2$. The center of the circle is point at $(0, 0)$ and the radius is 3.

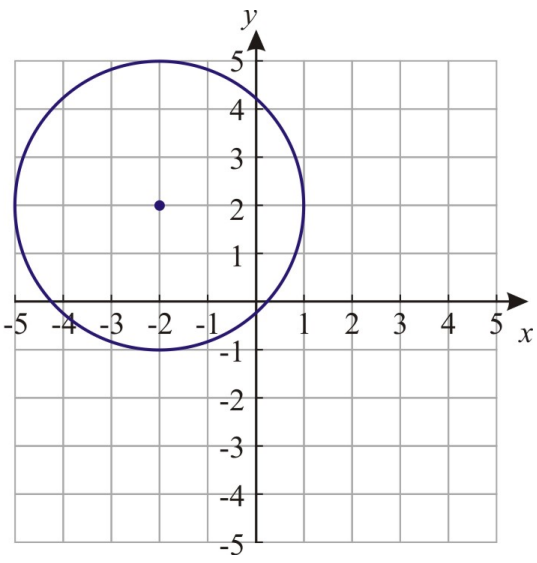


B. We rewrite the equation as: $(x - (-2))^2 + (y - 0)^2 = 1^2$. The center of the circle is point at $(-2, 0)$ and the radius is 1.



Example 5

Write the equation of the circle in the graph.



From the graph we can see that the center of the circle is at point $(-2, 2)$ and the radius is 3 units long.

Thus the equation is:

$$(x+2)^2 + (y-2)^2 = 9$$

Example 6

Determine if the point $(1, 3)$ *is on the circle given by the equation:* $(x-1)^2 + (y+1)^2 = 16$.

In order to find the answer, we simply plug the point $(1, 3)$ into the equation of the circle.

$$\begin{aligned}(1-1)^2 + (3+1)^2 &= 16\\ 0^2 + 4^2 &= 16\end{aligned}$$

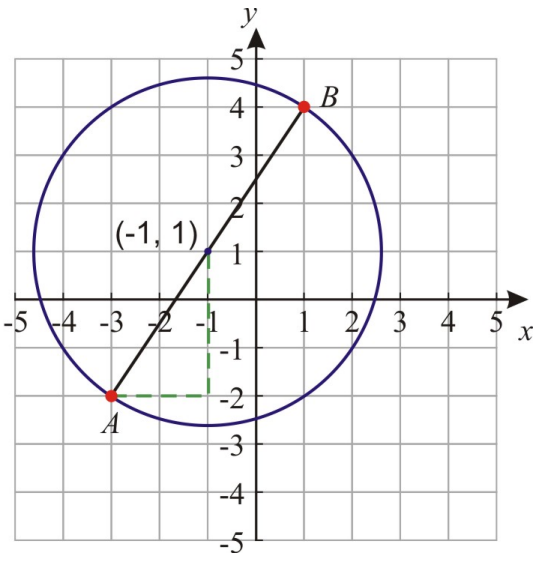
The point $(1, 3)$ satisfies the equation of the circle.

Example 7

Find the equation of the circle whose diameter extends from point $A = (-3, -2)$ *to* $B = (1, 4)$.

The general equation of a circle is: $(x - x_0)^2 + (y - y_0)^2 = r^2$

In order to write the equation of the circle in this example, we need to find the center of the circle and the radius of the circle.



Let's graph the two points on the coordinate plane.

We see that the center of the circle must be in the middle of the diameter.

In other words, the center point is midway between the two points A and B. To get from point A to point B, we must travel 4 units to the right and 6 units up. To get halfway from point A to point B, we must travel 2 units to the right and 3 units up. This means the center of the circle is at point $(-3+2, -2+3)$ or $(-1, 1)$.

We find the length of the radius using the Pythagorean Theorem:

$$r^2 = 2^2 + 3^2 \Rightarrow r^2 = 13 \Rightarrow r = \sqrt{13}$$

Thus, the equation of the circle is: $(x+1)^2 + (y-1)^2 = 13$.

Completing the Square:

You saw that the equation of a circle with center at point (x_0, y_0) and radius r is given by:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

This is called the **standard form** of the circle equation. The standard form is very useful because it tells us right away what the center and the radius of the circle is.

If the equation of the circle is not in standard form, we use the method of **completing the square** to rewrite the equation in the standard form.

Example 8

Find the center and radius of the following circle and sketch a graph of the circle.

$$x^2 - 4x + y^2 - 6y + 9 = 0$$

To find the center and radius of the circle we need to rewrite the equation in standard form. The standard equation has two perfect square factors one for the x terms and one for the y terms. We need to complete the square for the x terms and the y terms separately.

$$x^2 - 4x + __ + y^2 - 6y + __ + 9 = __ + __$$

To complete the squares we need to find which constants allow us to factors each trinomial into a perfect square. To complete the square for the x terms we need to add a constant of 4 on both sides.

$$x^2 - 4x + \underline{\ 4\ } + y^2 - 6y + __ + 9 = \underline{\ 4\ } + __$$

To complete the square for the y terms we need to add a constant of 9 on both sides.

$$x^2 - 4x + \underline{\ 4\ } + y^2 - 6y + \underline{\ 9\ } + 9 = \underline{\ 4\ } + \underline{\ 9\ }$$

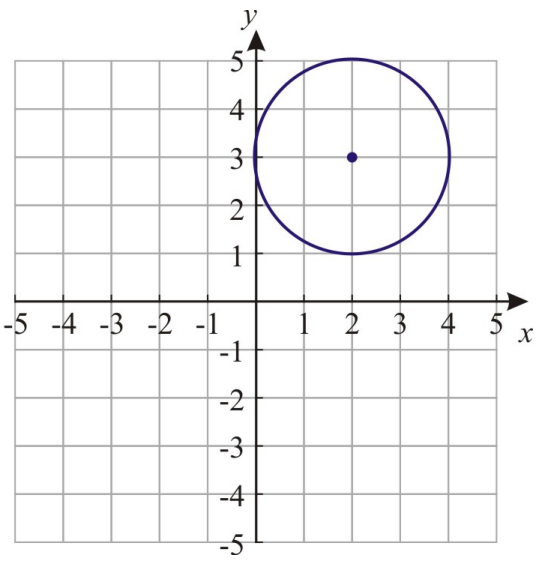
We can factor the separate trinomials and obtain:

$$(x-2)^2 + (y-3)^2 + 9 = 13$$

This simplifies as:

$$(x-2)^2 + (y-3)^2 = 4$$

You can see now that the center of the circle is at point $(2, 3)$ and the radius is 2.

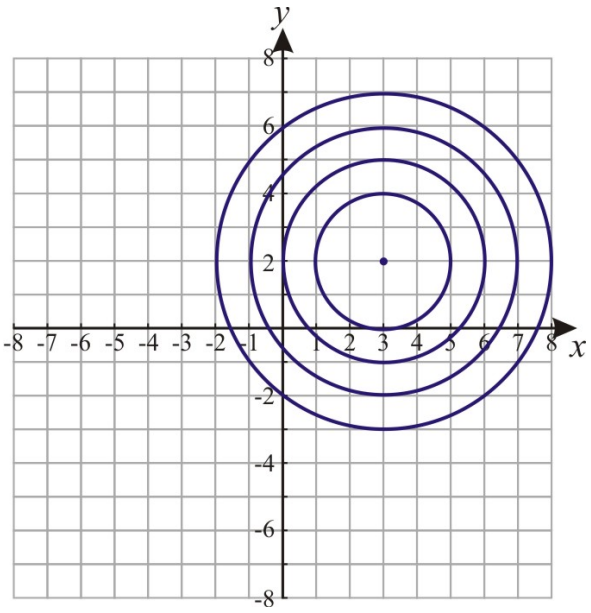


Concentric Circles

Concentric circles are circles of different radii that share the same center point.

Example 9

Write the equations of the concentric circles shown in the graph.



$$(x-3)^2+(y-2)^2=4$$
$$(x-3)^2+(y-2)^2=9$$
$$(x-3)^2+(y-2)^2=16$$
$$(x-3)^2+(y-2)^2=25$$

Example 10

Determine if the circles given by the equations $x^2-10x+y^2-12y+57=0$ *and* $x^2-10x+y^2-12y+36=0$ *are concentric.*

To find the answer to this question, we must rewrite the equations of the circles in standard form and find the center point of each circle.

To rewrite in standard form, we complete the square on the x and y terms separately.

First circle:

$$x^2-10x+__+y^2-12y+__+57=__+__$$
$$x^2-10x+\underline{25}+y^2-12y+\underline{36}+57=\underline{25}+\underline{36}$$
$$(x-5)^2+(y-6)^2+57=61$$
$$(x-5)^2+(y-6)^2=4$$

The center of the first circle is also at point $(5,6)$ so the circles are concentric.

Second circle:

$$x^2-10x+__+y^2-12y+__+36=__+__$$
$$x^2-10x+\underline{25}+y^2-12y+\underline{36}+36=\underline{25}+\underline{36}$$
$$(x-5)^2+(y-6)^2+36=61$$
$$(x-5)^2+(y-6)^2=25$$

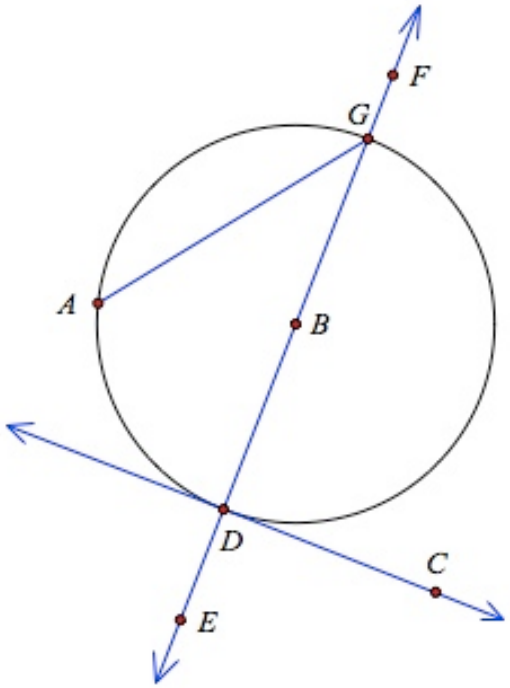
The center of the second circle is at point $(5,6)$.

Lesson Summary

In this section we discussed many terms associated with circles and looked at inscribed and circumscribed polygons. We also covered graphing circles on the coordinate grid and finding the equation of a circle. We found that sometimes we need to use the technique of completing the square to find the equation of a circle.

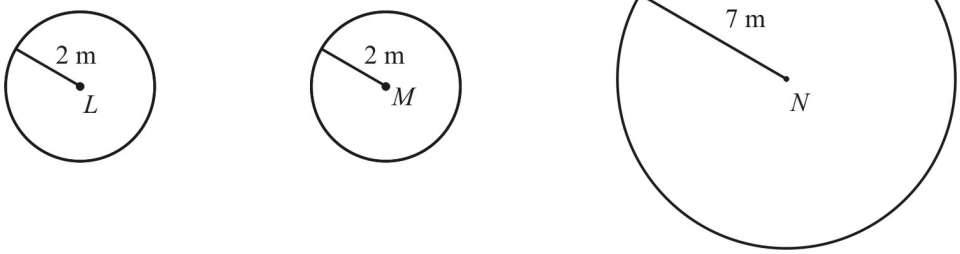
Review Questions

1. Identify each of the following as a diameter, a chord, a radius, a tangent, or a secant line.



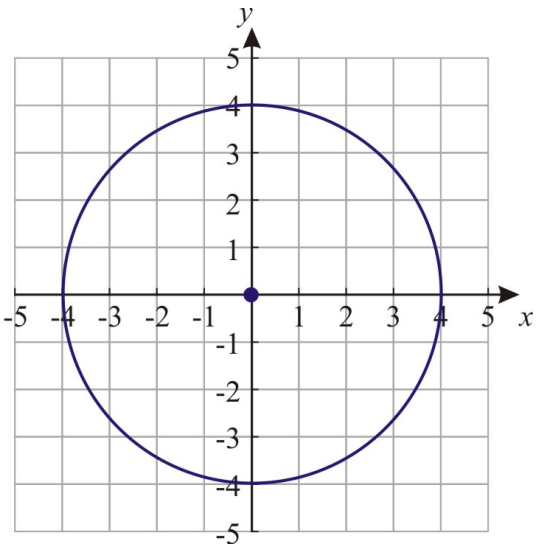
(a) $\overline{BG}$
(b) $\overline{DG}$
(c) $\overline{DC}$
(d) $\overline{FE}$
(e) $\overline{AG}$
(f) $\overline{DB}$

2. Determine which of the following circles are congruent and which are similar. For circles that are similar give the similarity ratio.



For exercises 3-8, find the center and the radius of the circles:

3.



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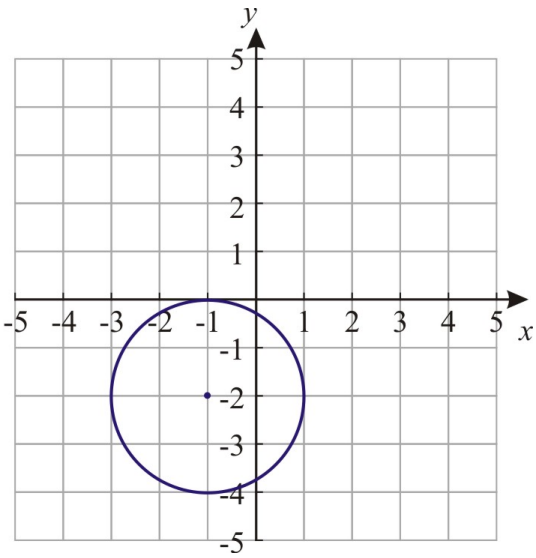
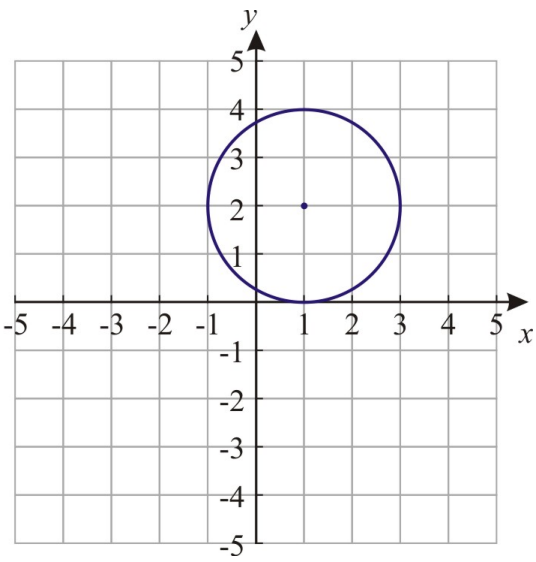


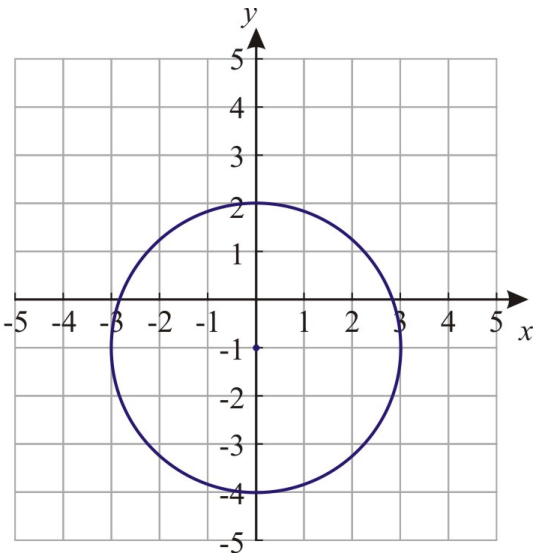
5. $x^2 + y^2 = 1$
6. $(x-3)^2 + (y+5)^2 = 81$
7. $x^2 + (y-2)^2 = 4$
8. $(x+6)^2 + (y+1)^2 = 25$
9. Check that the point $(3, 4)$ is on the circle given by the equation $x^2 + (y-4)^2 = 9$.
10. Check that the point $(-5, 5)$ is on the circle given by the equation $(x+3)^2 + (y+2)^2 = 50$.
11. Write the equation of the circle with center at $(2, 0)$ and radius 4.
12. Write the equation of the circle with center at $(4, 5)$ and radius 9.
13. Write the equation of the circle with center at $(-1, -5)$ and radius 10.

For 14 and 15, write the equation of the circles.

14.



15.



16. In a circle with center $(4, 1)$ one endpoint of a diameter is $(-1, 3)$. Find the other endpoint of the diameter.
17. The endpoints of the diameter of a circle are given by the points $A = (1, 4)$ and $B = (7, 2)$. Find the equation of the circle.
18. A circle has center $(0, 4)$ and contains point $(1, 1)$. Find the equation of the circle.
19. A circle has center $(-2, -2)$ and contains point $(4, 4)$. Find the equation of the circle.
20. Find the center and the radius of the following circle: $x^2 + 8x + y^2 - 2y - 19 = 0$.
21. Find the center and the radius of the following circle: $x^2 - 10x + y^2 + 6y - 15 = 0$.
22. Find the center and the radius of the following circle: $x^2 + 20x + y^2 - 30y + 181 = 0$.
23. Determine if the circles given by the equations are concentric.
    (a) $x^2 - 4x + y^2 + 2y + 4 = 0$ and $x^2 - 4x + y^2 + 2y - 31 = 0$

(b) $x^2 + 6x + y^2 + 8y = 0$ and $x^2 + 6x + y^2 - 8y + 9 = 0$
(c) $x^2 + y^2 + 21 = 10y$ and $x^2 + y^2 - 8y = 33$
(d) $x^2 + y^2 + 26 = 10x + 10y$ and $x^2 + y^2 = 10x + 10y + 14$

Review Answers

1. (a) $\overline{BG}$ is a radius.
   (b) $\overline{DG}$ is a diameter.
   (c) $\overline{DC}$ is a tangent.
   (d) $\overline{FE}$ is a secant.
   (e) $\overline{AG}$ is a chord.
   (f) $\overline{DB}$ is a radius.
2. $\odot L$ is congruent to $\odot M$; $\odot L$ is similar to $\odot N$ with similarity ratio $2 : 7$; $\odot M$ is similar to $\odot N$ with similarity ratio $2 : 7$.
3. The center is located at $(0, 0)$, radius $= 4$.
4. The center is located at $(-1, -2)$, radius $= 2$.
5. The center is located at $(0, 0)$, radius $= 1$.
6. The center is located at $(3, -5)$, radius $= 9$.
7. The center is located at $(0, 2)$, radius $= 2$.
8. The center is located at $(-6, -1)$, radius $= 5$.
9. $3^2 + (4 - 4)^2 = 3^2 = 9$. The point is on the circle.
10. $(-5 + 3)^2 + (5 + 2)^2 = 4 + 49 = 53 \neq 50$. The point is not on the circle.
11. $(x - 2)^2 + y^2 = 16$
12. $(x - 4)^2 + (y - 5)^2 = 81$
13. $(x + 1)^2 + (y + 5)^2 = 100$
14. $(x - 1)^2 + (y - 2)^2 = 4$
15. $x^2 + (y + 1)^2 = 9$
16. $(9, -1)$
17. $(x - 4)^2 + (y - 3)^2 = 10$
18. $x^2 + (y - 4)^2 = 10$
19. $(x + 2)^2 + (y + 2)^2 = 72$
20. The center is located at $(-4, 1)$, radius $= 6$.
21. The center is located at $(5, -3)$, radius $= 7$.
22. The center is located at $(-10, 15)$, radius $= 12$.
23. (a) Yes
    (b) No
    (c) No
    (d) Yes

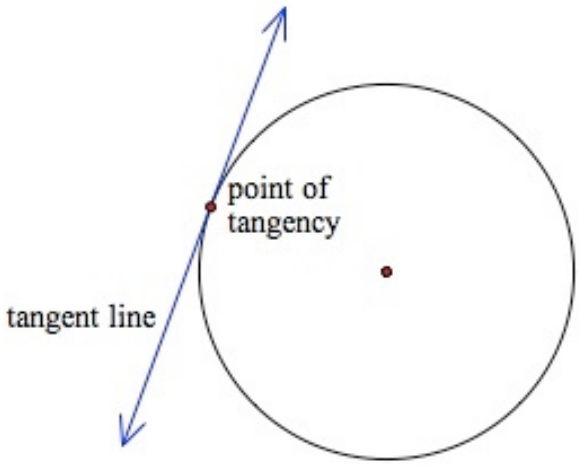
9.2 Tangent Lines

Learning Objectives

- Find the relationship between a radius and a tangent to a circle.
- Find the relationship between two tangents drawn from the same point.
- Circumscribe a circle.
- Find equations of concentric circles.

Introduction

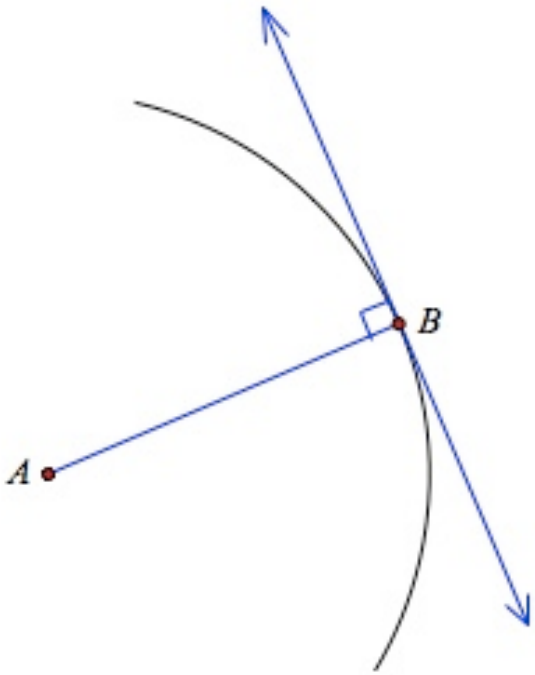
In this section we will discuss several theorems about tangent lines to circles and the applications of these theorems to geometry problems. Remember that a **tangent to a circle** is a line that intersects the circle at exactly one point and that this intersection point is called the **point of tangency.**



Tangent to a Circle

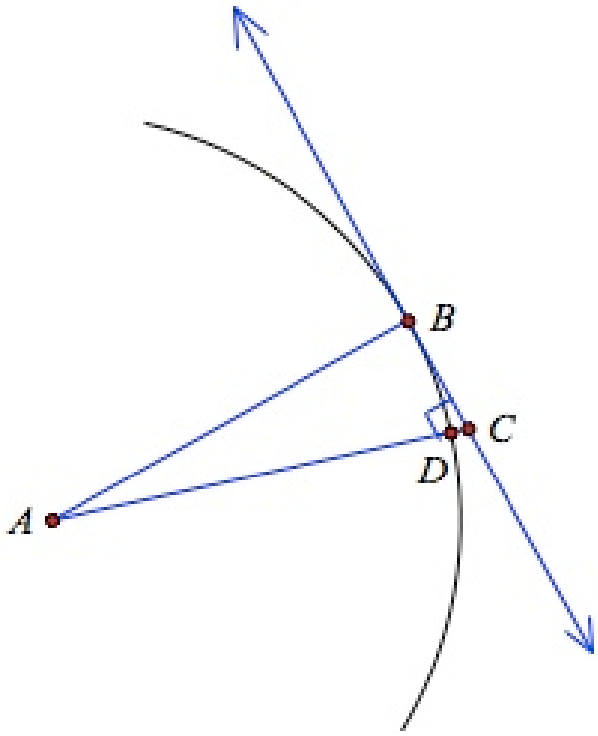
Tangent to a Circle Theorem:

A tangent line is always at right angles to the radius of the circle at the point of tangency.



Proof. We will prove this theorem by contradiction.

We start by making a drawing. $\overline{AB}$ is a radius of the circle. A is the center of the circle and B is the point of intersection between the radius and the tangent line.



Assume that the tangent line is **not** perpendicular to the radius.

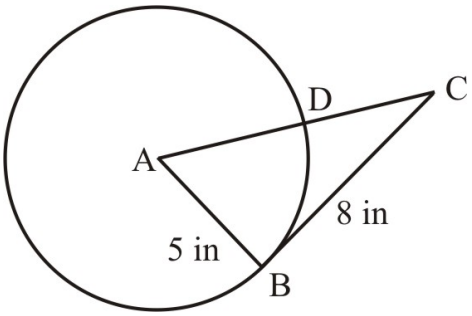
There must be another point C on the tangent line such that $\overline{AC}$ is perpendicular to the tangent line. Therefore, in the right triangle $ACB, \overline{AB}$ is the hypotenuse and $\overline{AC}$ is a leg of the triangle. However, this is not possible because $AC > AB$. (Note that $AC =$ length of the radius $+ DC$).

Since our assumption led us to a contradiction, this means that our assumption was incorrect. Therefore, the tangent line must be perpendicular to the radius of the circle. ♦

Since the tangent to a circle and the radius of the circle make a right angle with each other, we can often use the Pythagorean Theorem in order to find the length of missing line segments.

Example 1

In the figure, $\overline{CB}$ is tangent to the circle. Find CD.



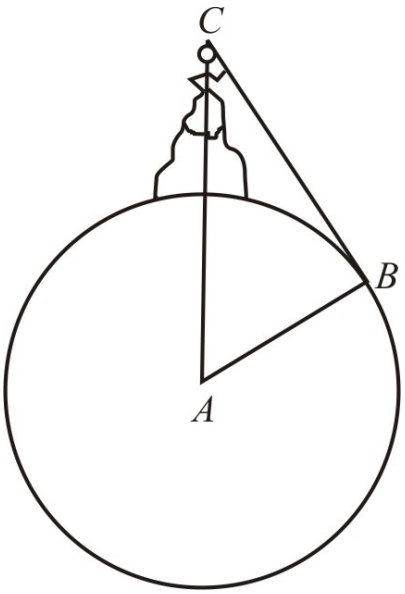
Since $\overline{CB}$ is tangent to the circle, then $\overline{CB} \perp \overline{AB}$.

This means that $\triangle ABC$ is a right triangle and we can apply the Pythagorean Theorem to find the length of $\overline{AC}$.

$$\begin{aligned}(AC)^2 &= (AB)^2 + (BC)^2\\ (AC)^2 &= 25 + 64 = 89\\ AC &= \sqrt{89} \approx 9.43 \text{ in.}\\ CD &= AC - AD \approx 9.43 - 5 \approx 4.43 \text{ in.}\end{aligned}$$

Example 2

Mark is standing at the top of Mt. Whitney, which is 14,500 feet *tall. The radius of the Earth is approximately* 3,960 miles. *(There are* 5,280 feet *in one mile.) How far can Mark see to the horizon?*



We start by drawing the figure above.

The distance to the horizon is given by the line segment CB.

Let us convert the height of the mountain from feet into miles.

$$14500 \text{ feet} \times \frac{1 \text{ mile}}{5280 \text{ feet}} = 2.75 \text{ miles}$$

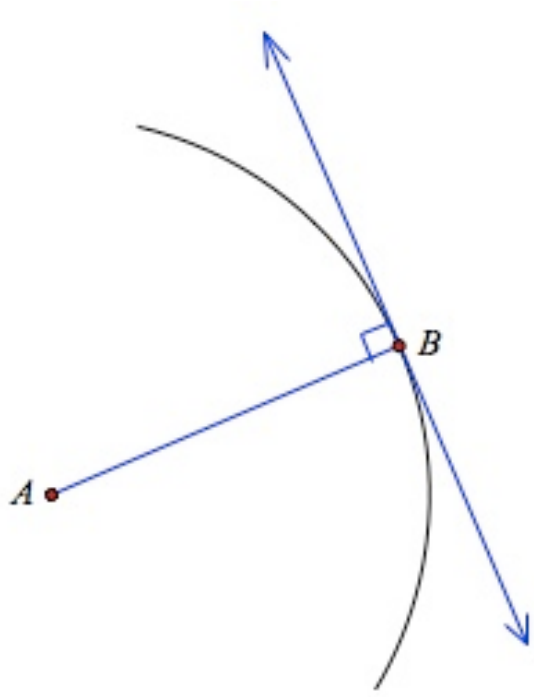
Since $\overline{CB}$ is tangent to the Earth, $\triangle ABC$ is a right triangle and we can use the Pythagorean Theorem.

$$\begin{aligned} (AC)^2 &= (CB)^2 + (AB)^2 \\ (3960 + 2.75)^2 &= (CB)^2 + 3960^2 \\ CB &= \sqrt{3962.75^2 - 3960^2} \approx 147.6 \text{ miles} \end{aligned}$$

Converse of a Tangent to a Circle

Converse of a Tangent to a Circle Theorem

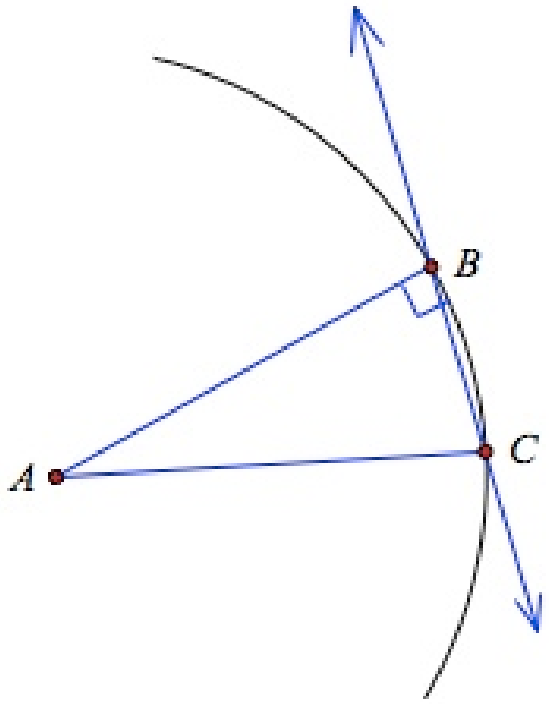
If a line is perpendicular to the radius of a circle at its outer endpoint, then the line is tangent to the circle.



Proof.

We will prove this theorem by contradiction. Since the line is perpendicular to the radius at its outer endpoint it must touch the circle at point B. For this line to be tangent to the circle, it must only touch the circle at this point and no other.

Assume that the line also intersects the circle at point C.



Since both $\overline{AB}$ and $\overline{AC}$ are radii of the circle, $\overline{AB} \cong \overline{AC}$, and $\triangle ABC$ is isosceles.

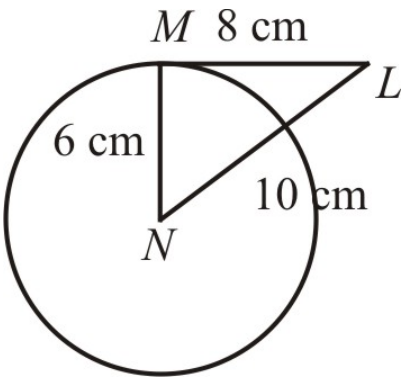
This means that $m\angle ABC = m\angle ACB = 90°$.

It is impossible to have two right angles in the same triangle.

We arrived at a contradiction so our assumption must be incorrect. We conclude that line BC is tangent to the circle at point B. ♦

Example 3

Determine whether $\overline{LM}$ is tangent to the circle.



$\overline{LM}$ is tangent to the circle if $\overline{LM} \perp \overline{NM}$.

To show that $\triangle LMN$ is a right triangle we use the Converse of the Pythagorean Theorem:

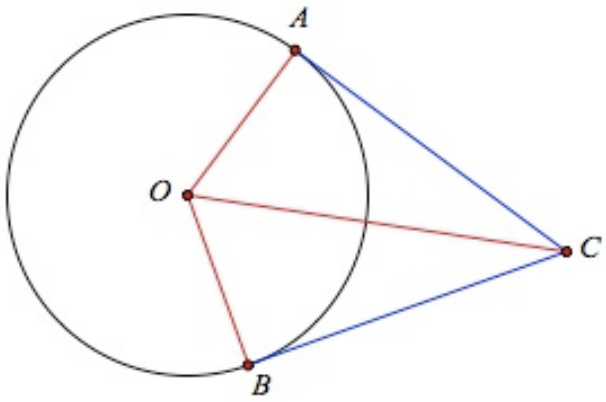
$$(LM)^2 + (MN)^2 = 64 + 36 = 100 = 10^2$$

The lengths of the sides of the triangle satisfy the Pythagorean Theorem, so $\overline{LM}$ is perpendicular to $\overline{MN}$ and is therefore tangent to the circle.

Tangent Segments from a Common External Point

Tangent Segments from a Common External Point Theorem

If two segments from the same exterior point are tangent to the circle, then they are congruent.



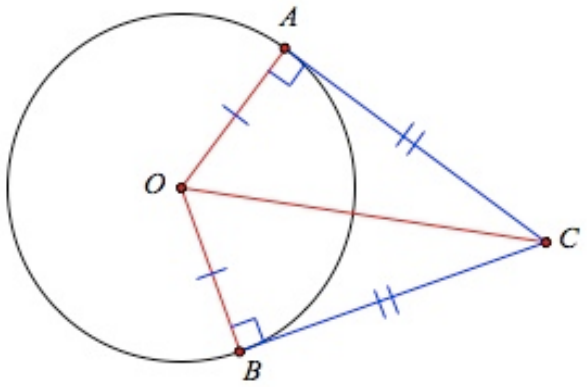
Proof.

The figure above shows a diagram of the situation.

- Given: $\overline{AC}$ is a tangent to the circle and $\overline{BC}$ is a tangent to the circle
- Prove: $\overline{AC} \cong \overline{BC}$

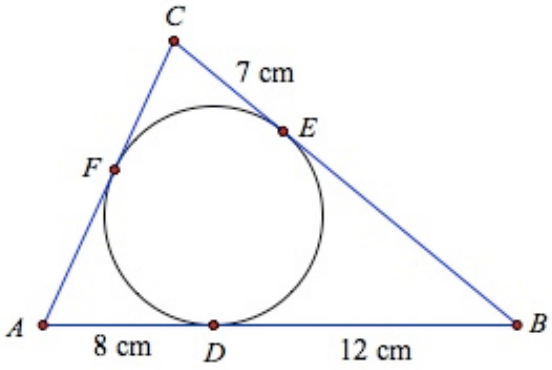
Table 9.1:

Statement	Reason
1. $\overline{AB}$ is tangent to the circle	1. Given
2. $\overline{AC} \perp \overline{OA}$	2. Tangent to a Circle Theorem
3. $\overline{BC}$ is a tangent to the circle	3. Given
4. $\overline{BC} \perp \overline{OB}$	4. Tangent to a Circle Theorem
5. $\overline{OA} \cong \overline{OB}$	5. Radii of same the circle
6. $\overline{OC} \cong \overline{OC}$	6. Same line
7. $\triangle AOC \cong \triangle BOC$	7. Hypotenuse-Leg congruence
8. $\overline{AC} \cong \overline{BC}$	8. Congruent Parts of Congruent Triangles are Congruent ♦



Example 4

Find the perimeter of the triangle.



All sides of the triangle are tangent to the circle.

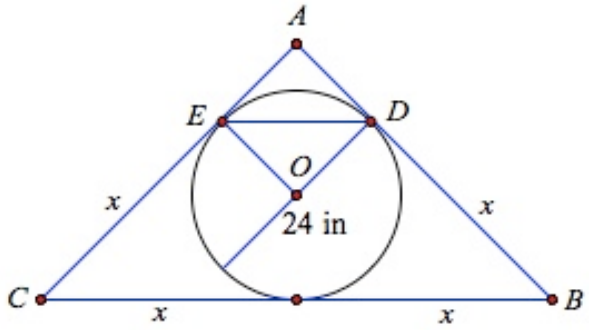
The Tangent Segments from a Common External Point Theorem tells us that:

$CE = FC = 7$ cm
$FA = AD = 8$ cm
$DB = BE = 12$ cm

The perimeter of the triangle $= AF + FC + CE + EB + BD + AD$
$= 8 \text{ cm} + 7 \text{ cm} + 7 \text{ cm} + 12 \text{ cm} + 12 \text{ cm} + 8 \text{ cm} = 54 \text{ cm}$

Example 5

An isosceles right triangle is circumscribed about a circle with diameter of 24 inches. *Find the hypotenuse of the triangle.*



Let's start by making a sketch.

Since $\overline{EO}$ and $\overline{DO}$ are radii of the circle and $\overline{AC}$ and $\overline{AB}$ are tangents to the circle,

$\overline{EO} \perp \overline{AC}$ and $\overline{DO} \perp \overline{AB}$.

Therefore, quadrilateral $ADOE$ is a square.

Therefore, $AE = AD = 12$ in.

We can find the length of side $\overline{ED}$ by using the Pythagorean Theorem.

$$\begin{aligned}(ED)^2 &= (AE)^2 + (AD)^2\\ (ED)^2 &= 144 + 144 = 288\\ (ED) &= 12\sqrt{2} \text{ in.}\end{aligned}$$

$\triangle ADE$ and $\triangle ABC$ are both isosceles right triangles, therefore $\triangle ADE \sim \triangle ABC$ and all the corresponding sides are proportional:

$$\frac{AE}{AC} = \frac{AD}{AB} = \frac{ED}{CB}$$

We can find the length of $\overline{EC}$ by using one of the ratios above:

$$\frac{AE}{AC} = \frac{ED}{CB} \Rightarrow \frac{12}{12+x} = \frac{12\sqrt{2}}{2x}$$

Cross-multiply to obtain:

$$\begin{aligned}24x &= 144\sqrt{2} + 12\sqrt{2}x\\ 24x - 12\sqrt{2}x &= 144\sqrt{2}\\ 12x(2-\sqrt{2}) &= 144\sqrt{2}\\ x &= \frac{144\sqrt{2}}{12(2-\sqrt{2})} = \frac{12\sqrt{2}}{2-\sqrt{2}}\end{aligned}$$

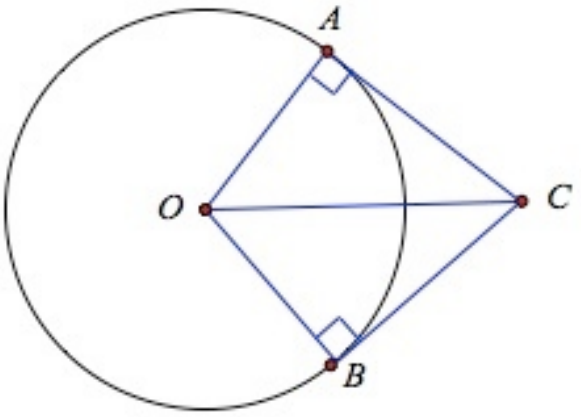
Rationalize the denominators:

$$\begin{aligned}x &= \frac{12\sqrt{2}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{24\sqrt{2}+24}{2}\\ x &= (12\sqrt{2}+12) \text{ in.}\end{aligned}$$

The length of the hypotenuse is $\overline{BC} = 2x = 24\sqrt{2} + 24$ in ≈ 58 in.

Corollary to Tangent Segments Theorem

A line segment from an external point to the center of a circle bisects the angle formed by the tangent segments starting at that same external point.



Proof.

- Given:
  - $\overline{AC}$ is a tangent to the circle
  - $\overline{BC}$ is a tangent to the circle
  - O is the center of the circle

- Prove
  - $\angle ACO \cong \angle BCO$

Proof.

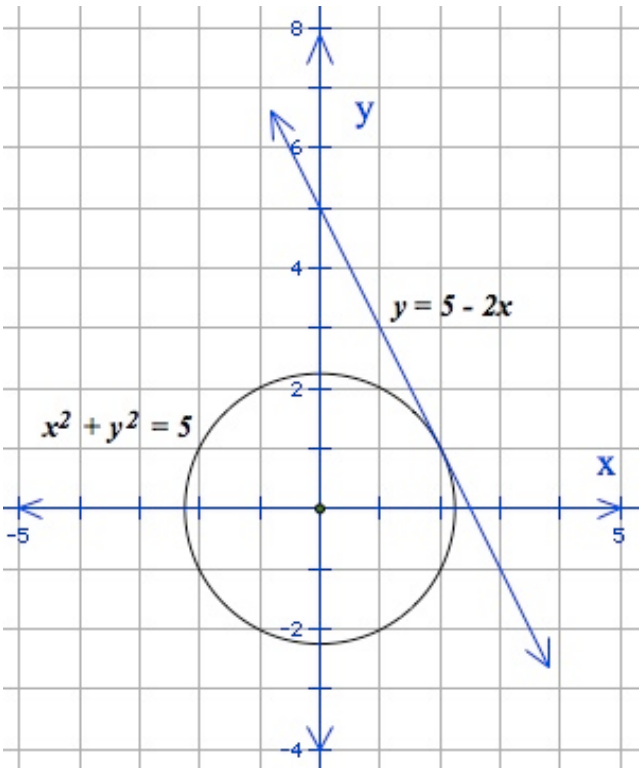
We will use a similar figure to the one we used we used to prove the tangent segments theorem (pictured above).

$\triangle AOC \cong \triangle BOC$ by HL congruence.

Therefore, $\angle ACO \cong \angle BCO$. ♦

Example 6

Show that the line $y = 5 - 2x$ is tangent to the circle $x^2 + y^2 = 5$. Find an equation for the line perpendicular to the tangent line at the point of tangency. Show that this line goes through the center of the circle.



To check that the line is tangent to the circle, substitute the equation of the line into the equation for the circle.

$$\begin{aligned} x^2 + (5 - 2x)^2 &= 5 \\ x^2 + 25 - 20x + 4x^2 &= 5 \\ 5x^2 - 20x + 20 &= 0 \\ x^2 - 4x + 4 &= 0 \\ (x - 2)^2 &= 0 \end{aligned}$$

This has a double root at $x = 2$. This means that the line intersects the circle at only one point $(2, 1)$.

A perpendicular line to the tangent line would have a slope that is the negative reciprocal of the tangent line or $m = \frac{1}{2}$.

The equation of the line can be written: $y = \frac{1}{2}x + b$.

We find the value of b by plugging in the tangency point: $(2, 1)$.

$$1 = \frac{1}{2}(2) + b \Rightarrow b = 0$$

The equation is $y = \frac{1}{2}x$ and we know that it passes through the origin since the $y-$intercept is zero.

This means that the radius of the circle is perpendicular to the tangent to the circle.

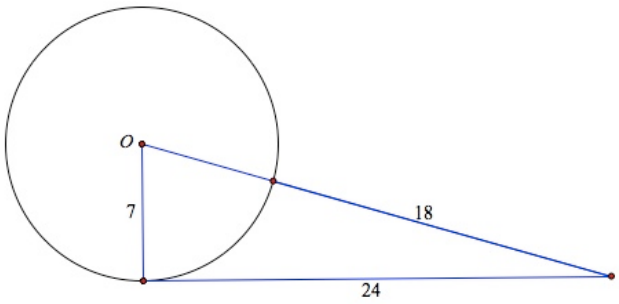
Lesson Summary

In this section we learned about tangents and their relationship to the circle. We found that a tangent line touches the circle at one point, which is the endpoint of a radius of the circle. The radius and tangent line are perpendicular to each other. We found out that if two segments are tangent to a circle, and share a common endpoint outside the circle; the segments are congruent.

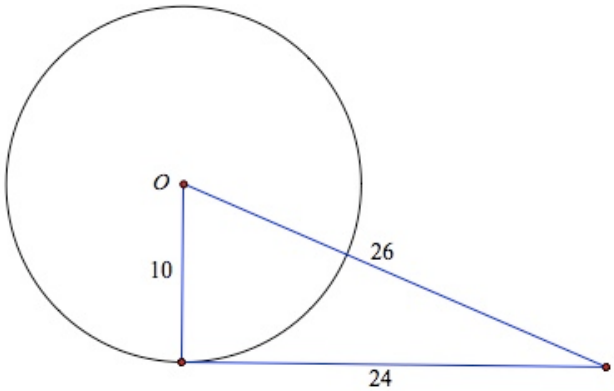
Review Questions

1. Determine whether each segment is tangent to the given circle:

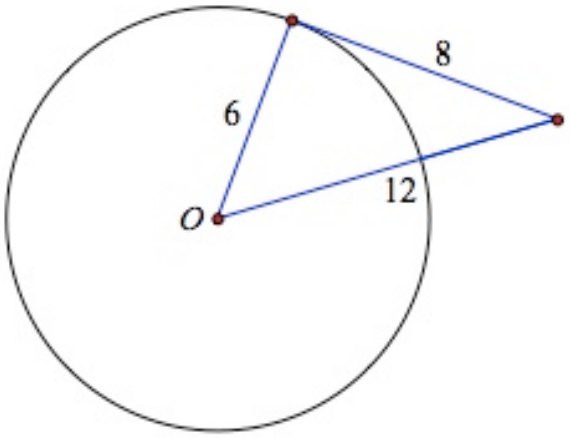
(a)



(b)

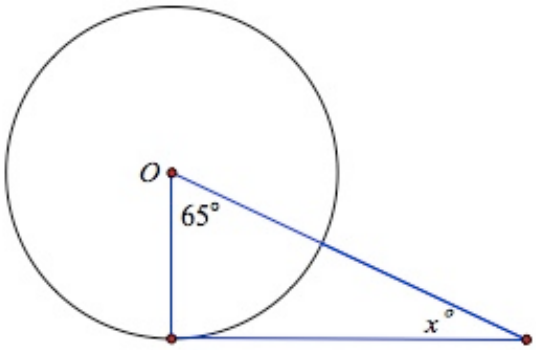


(c)

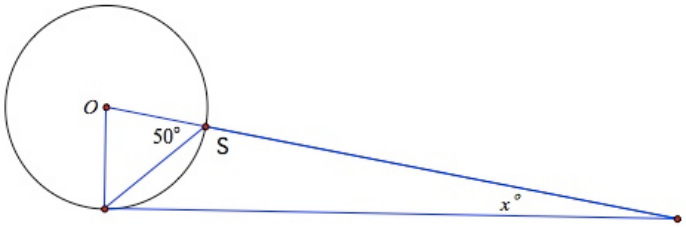


2. Find the measure of angle x.

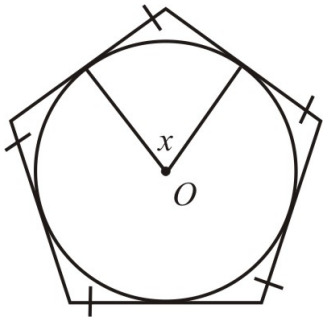
(a)



(b)

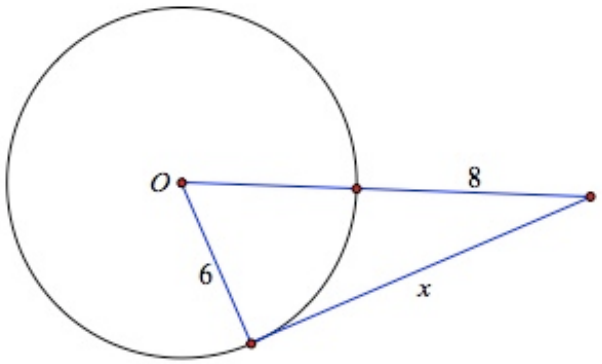


(c)

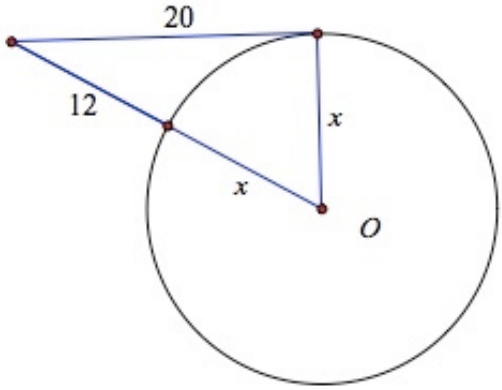


3. Find the missing length:

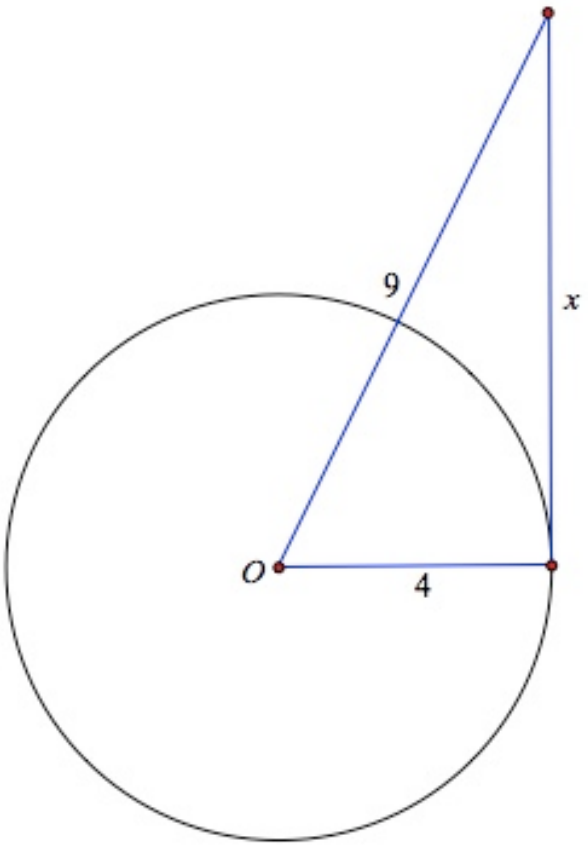
(a)



(b)

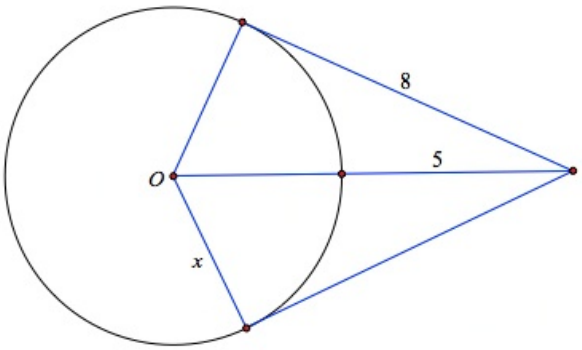


(c)

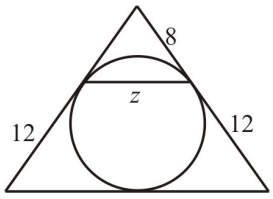


4. Find the values of the missing variables

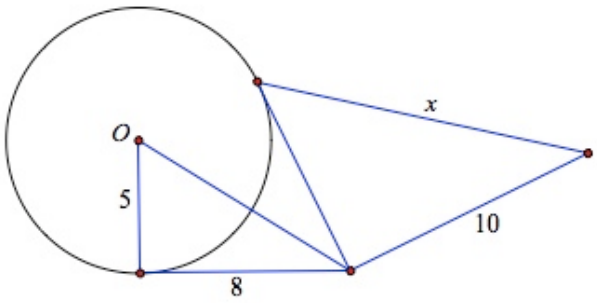
(a)



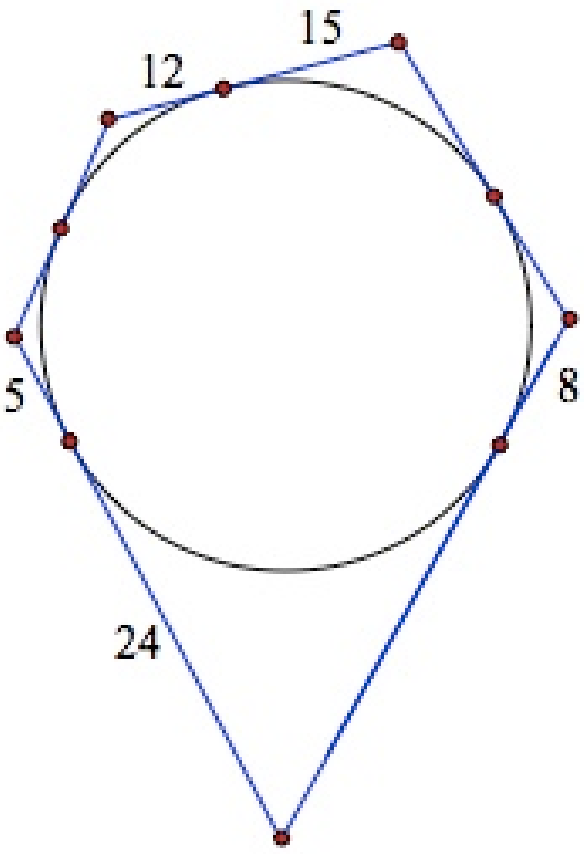
(b)



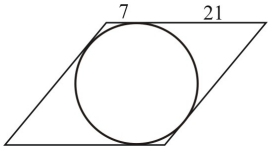
(c)



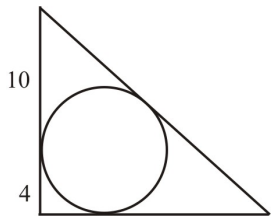
5. Find the perimeter of the pentagon:



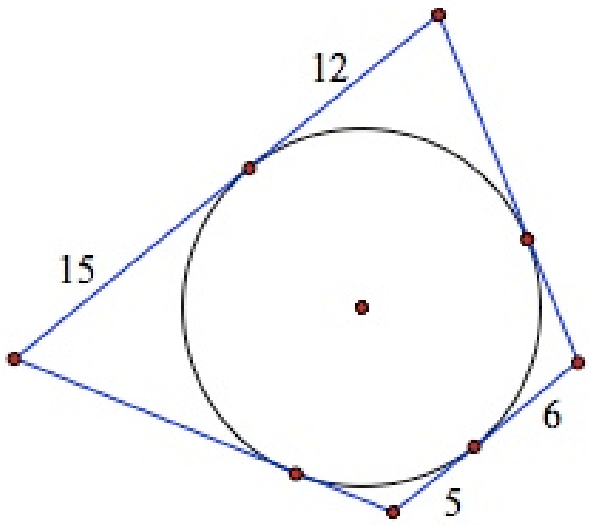
6. Find the perimeter of the parallelogram:



7. Find the perimeter of the right triangle:



8. Find the perimeter of the polygon:



9. Draw the line $y = 3x + 10$ and the circle $x^2 + y^2 = 10$. Show that these graphs touch at only one point.

Find the slope of the segment that joins this point to the center of the circle, and compare your answer with the slope of the line $y = 3x + 10$.

Review Answers

1. (a) Yes
   (b) Yes
   (c) No
2. (a) 25°
   (b) 10°
   (c) 72°
3. (a) 12.6
   (b) 10.67
   (c) 8.1
4. (a) 3.9

(b) 9.6
(c) 12.8

5. 128
6. 112
7. 48
8. 76
9. $x^2 + (3x + 10)^2 = 10$ solve for x to obtain double root $(-3, 1)$.

The slope of the line from $(0, 0)$ to $(-3, 1) = -1/3$, which is the negative reciprocal of the slope of the line.

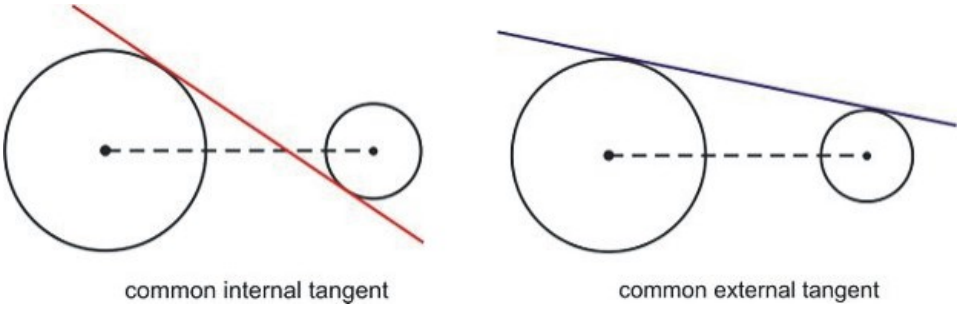
This means that the tangent line and radius are perpendicular.

9.3 Common Tangents and Tangent Circles

Learning Objectives

- Solve problems involving common internal tangents of circles.
- Solve problems involving common external tangents of circles.
- Solve problems involving externally tangent circles.
- Solve problems involving internally tangent circles.

Common tangents to two circles may be internal or external. A **common internal tangent** intersects the line segment connecting the centers of the two circles whereas a **common external tangent** does not.

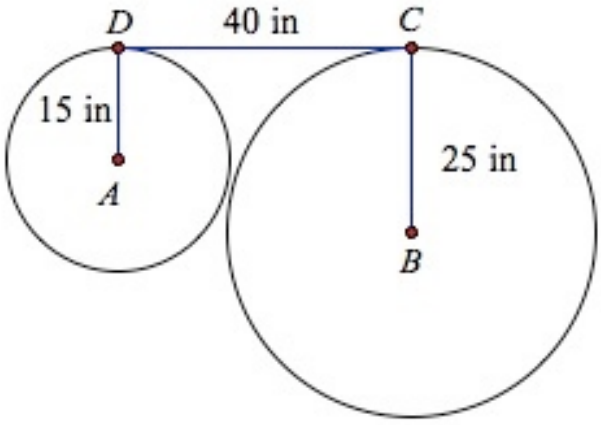


Common External Tangents

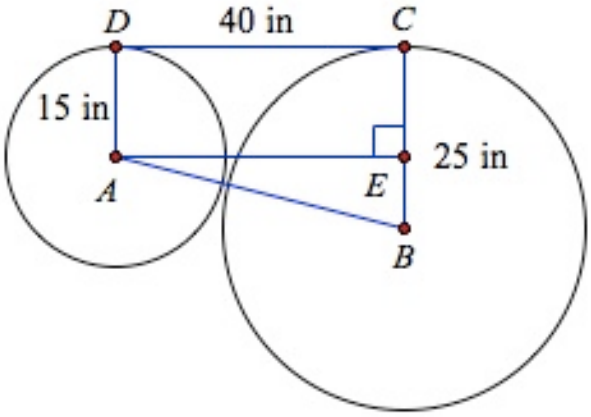
Here is an example in which you might encounter the use of common external tangents.

Example 1

Find the distance between the centers of the circles in the figure.



Let's label the diagram and draw a line segment that joins the centers of the two circles. Also draw the segment $\overline{AE}$ perpendicular the radius $\overline{BC}$.



Since $\overline{DC}$ is tangent to both circles, $\overline{DC}$ is perpendicular to both radii: $\overline{AD}$ and $\overline{BC}$.

We can see that $AECD$ is a rectangle, therefore $EC = AD = 15$ in.

This means that $BE = 25 \text{ in} - 15 \text{ in} = 10$ in.

$\triangle ABE$ is a right triangle with $AE = 40$ in and $BE = 10$ in. We can apply the Pythagorean Theorem to find the missing side, $\overline{AB}$.

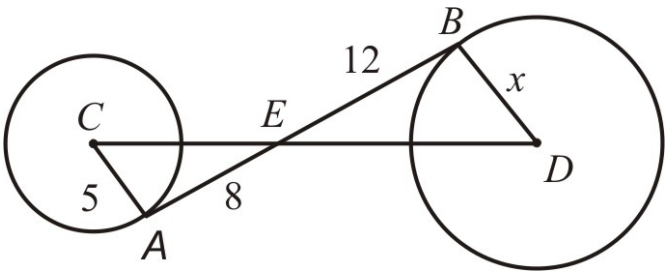
$$\begin{aligned}(AB)^2 &= (AE)^2 + (BE)^2 \\ (AB)^2 &= 40^2 + 10^2 \\ AB &= \sqrt{1700} \approx 41.2 \text{ in}\end{aligned}$$

The distance between the centers is *approximately* 41.2 inches.

Common Internal Tangents

Here is an example in which you might encounter the use of common internal tangents.

Example 2



$\overline{AB}$ is tangent to both circles. Find the value of x and the distance between the centers of the circles.

$\overline{AC} \perp \overline{AB}$	Tangent is perpendicular to the radius
$\overline{BD} \perp \overline{AB}$	Tangent is perpendicular to the radius
$\angle CAE \cong \angle DBE$	Both equal $90°$
$\angle CEA \cong \angle BED$	Vertical angles
$\triangle CEA \sim \triangle BED$	AA similarity postulate

Therefore,

$$\frac{AC}{BD} = \frac{AE}{EB} \Rightarrow \frac{5}{x} = \frac{8}{12} \Rightarrow x = \frac{5 \times 12}{8} \Rightarrow x = 7.5$$

Using the Pythagorean Theorem on $\triangle CEA$:

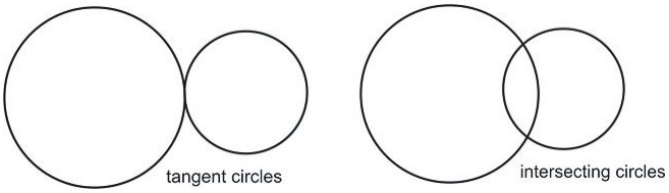
$$(CE)^2 = (AC)^2 + (AE)^2$$
$$(CE)^2 = 5^2 + 8^2 = 89$$
$$CE \approx 9.43$$

Using the Pythagorean Theorem on $\triangle BED$:

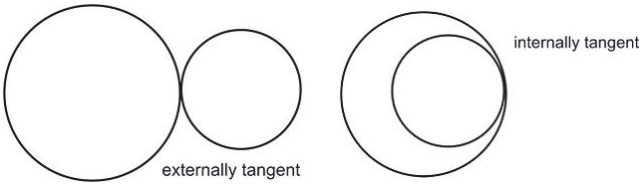
$$(DE)^2 = (BD)^2 + (DE)^2$$
$$(DE)^2 = 7.5^2 + 12^2 = 200.25$$
$$BE \approx 14.15$$

The distance between the centers of the circles is $CE + DE \approx 9.43 + 14.15 \approx 23.58$.

Two circles are **tangent** to each other if they have only one common point. Two circles that have two common points are said to **intersect** each other.



Two circles can be **externally tangent** if the circles are situated outside one another and **internally tangent** if one of them is situated inside the other.

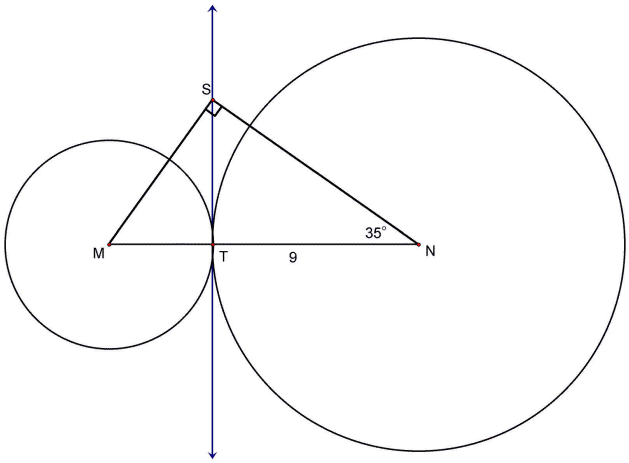


Externally Tangent Circles

Here are some examples involving externally tangent circles.

Example 3

Circles tangent at T are centered at M and N. Line ST is tangent to both circles at T. Find the radius of the smaller circle if $\overline{SN} \perp \overline{SM}$.



$\overline{ST} \perp \overline{TM}$ tangent is perpendicular to the radius.

$\overline{ST} \perp \overline{TN}$ tangent is perpendicular to the radius.

In the right triangle $\triangle STN, \cos 35^\circ = \frac{9}{SN} \Rightarrow SN = \frac{9}{cos35^\circ} \approx 11$.

We are also given that $\overline{SN} \perp \overline{SM}$.

Therefore, $m\angle MSN = 90^\circ \Rightarrow m\angle SMT = 90^\circ - 35^\circ = 55^\circ$.

Also, $m\angle STN = 90^\circ \Rightarrow m\angle TSN = 90^\circ - 35^\circ = 55^\circ$.

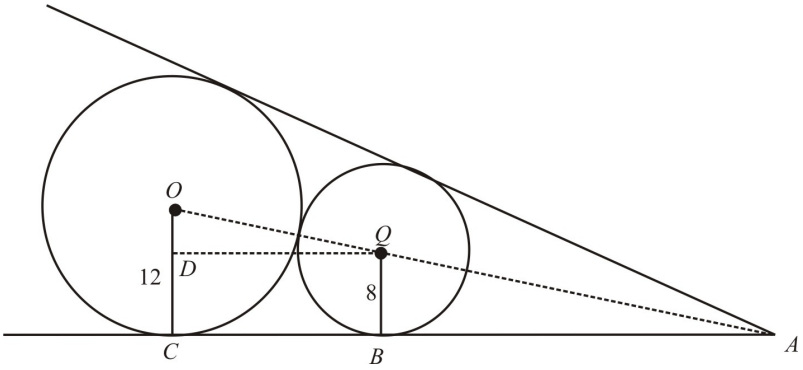
Therefore, $\triangle SNM \sim \triangle SNT$ by the AA similarity postulate.

$$\frac{SN}{MN} = \frac{TN}{SN} \Rightarrow \frac{11}{TM+9} = \frac{9}{11}$$
$$9(TM+9) = 121$$
$$9TM + 81 = 121$$
$$9TM = 40$$
$$TM \approx 4.44$$

The radius of the smaller circle is approximately 4.44.

Example 4

Two circles that are externally tangent have radii of 12 inches *and* 8 inches *respectively. Find the length of tangent* $\overline{AB}$.



Label the figure as shown.

In $\triangle DOQ, OD = 4$ and $OQ = 20$.

Therefore,

$$(DQ)^2 = (OQ)^2 - (OD)^2$$
$$(DQ)^2 = 20^2 - 4^2 = 384$$
$$DQ \approx 19.6 \Rightarrow CB \approx 19.6.$$

$\overline{AC} \perp \overline{QB}$ tangent is perpendicular to the radius.

$\overline{AC} \perp \overline{OC}$ tangent is perpendicular to the radius.

Therefore,

$\angle OCA \cong \angle QBA$	both equal 90°
$\angle OAC \cong \angle QAB$	same angle.

$\triangle AOC \sim \triangle AQB$ by the AA similarity postulate.

Therefore,

$$\frac{QB}{OC} = \frac{AB}{AC} \Rightarrow \frac{8}{12} = \frac{AB}{AB + 19.6} \Rightarrow 8(AB + 19.6) = 12AB$$

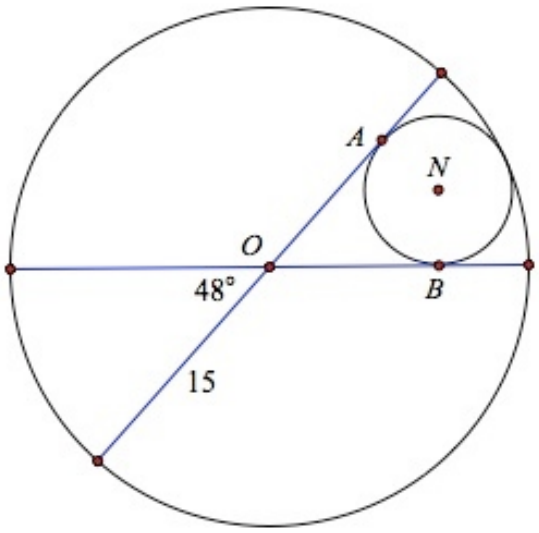
$$\begin{aligned} 8AB + 156.8 &\approx 12AB \\ 4AB &\approx 156.8 \\ AB &\approx 39.2. \end{aligned}$$

Internally Tangent Circles

Here is an example involving internally tangent circles.

Example 5

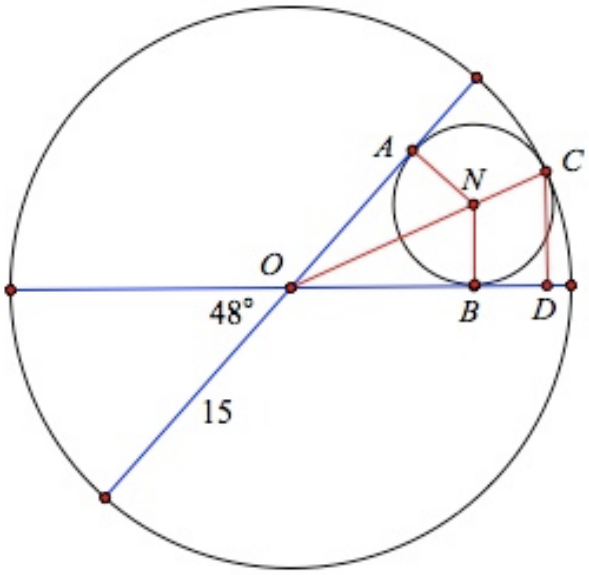
Two diameters of a circle of radius 15 inches are drawn to make a central angle of 48°. A smaller circle is placed inside the bigger circle so that it is tangent to the bigger circle and to both diameters. What is the radius of the smaller circle?



$\overline{OA}$ and $\overline{OB}$ are two tangents to the smaller circle from a common point so by Theorem 9-3, $\overline{ON}$ bisects $\angle AOB \Rightarrow m\angle NOB = 24°$.

In $\triangle ONB$ we use $\sin 24° = \frac{NB}{ON} \Rightarrow ON = \frac{\text{NB}}{\sin\ 24°} \approx 2.46\ NB$.

Draw $\overline{CD}$ from the points of tangency between the circles perpendicular to $\overline{OD}$.



In $\triangle OCD$ we use $\sin 24° = \frac{CD}{OC} \Rightarrow CD = \sin\ 24°(OC) \approx 0.41(15) \approx 6.1$.

We also have $\overline{OB} \perp \overline{NB}$ because a tangent is perpendicular to the radius.

Therefore,

$\angle OBN \cong \angle ODC$ both equal $90°$

$\angle COD \cong \angle BON$ same angle.

Therefore, $\triangle ONB \sim \triangle OCD$ by the AA similarity postulate.

This gives us the ratio $\frac{ON}{OC} = \frac{NB}{CD}$.

$ON = OC - NC = 15 - NB$ ($NB = NC$, since they are both radii of the small circle).

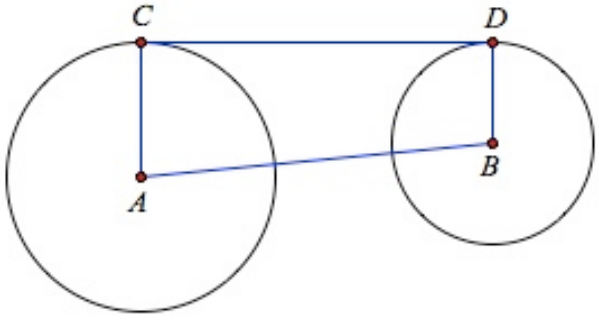
$$\frac{15 - NB}{15} = \frac{NB}{6.1} \Rightarrow 6.1(15 - NB) = 15\ NB$$
$$91.52 - 6.1\ NB = 15\ NB$$
$$91.52 = 21.1\ NB$$
$$NB = 4.34$$

Lesson Summary

In this section we learned about externally and internally tangent circles. We looked at the different cases when two circles are both tangent to the same line, and/or tangent to each other.

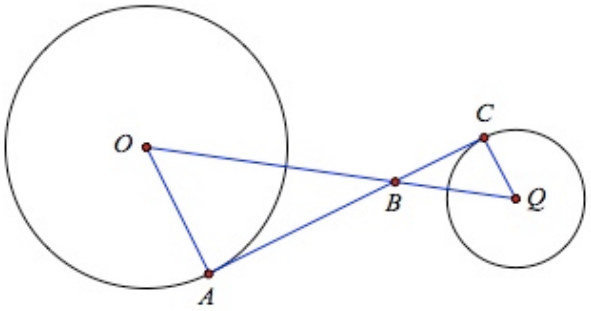
Review Questions

$\overline{CD}$ is tangent to both circles.



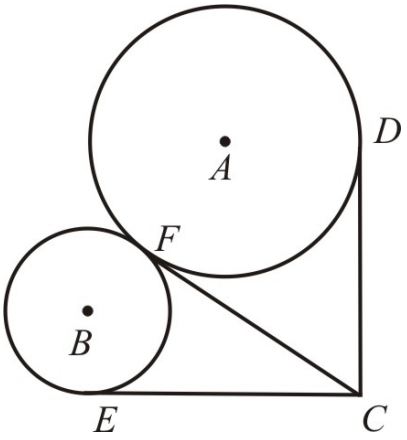
1. $AC = 8, BD = 5, CD = 12$. Find AB.
2. $AB = 20, AC = 15$ and $BD = 10$. Find CD.
3. $AB = 24, AC = 18$ and $CD = 19$. Find BD.
4. $AB = 12, CD = 16$ and $BD = 6$. Find AC.

$\overline{AC}$ is tangent to both circles. Find the measure of angle $\angle CQB$.



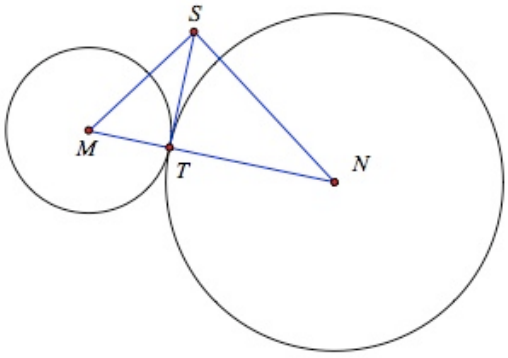
5. $AO = 9$ and $AB = 15$
6. $BQ = 20$ and $BC = 12$
7. $BO = 18, AO = 9$
8. $CB = 7, CQ = 5$

For 9 and 10, find x.

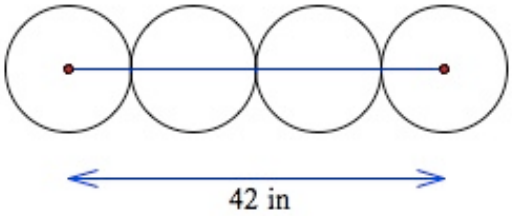


9. $DC = 2x + 3$; $EC = x + 10$
10. $DC = 4x - 9$; $EC = 2x + 21$

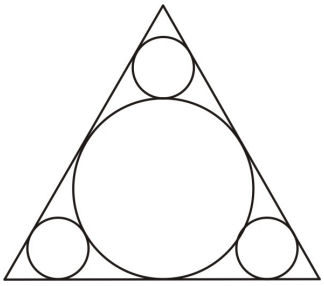
Circles tangent at T are centered at M and N. $\overline{ST}$ is tangent to both circles at T. Find the radius of the smaller circle if $\overline{SN} \perp \overline{SM}$.



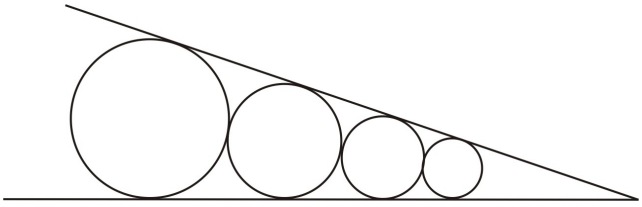
11. $SM = 22, TN = 25, \angle SNT = 40°$
12. $SM = 23, SN = 18, \angle SMT = 25°$
13. Four identical coins are lined up in a row as shown. The distance between the centers of the first and the fourth coin is 42 inches. What is the radius of one of the coins?



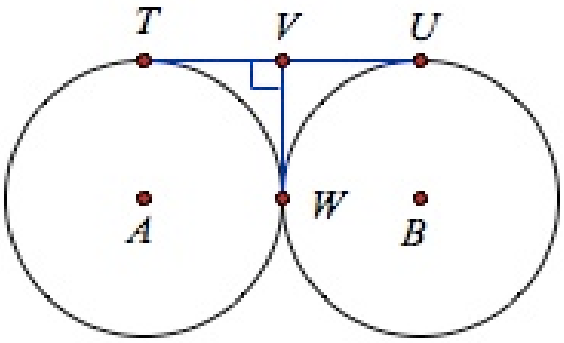
14. Four circles are arranged inside an equilateral triangle as shown. If the triangle has sides equal to 16 cm, what is the radius of the bigger circle? What are the radii of the smaller circles?



15. In the following drawing, each segment is tangent to each circle. The largest circle has a radius of 10 inches. The medium circle has a radius of 8 inches. What is the radius of the smallest circle tangent to the medium circle?



16. Circles centered at A and B are tangent at T. Show that A, B and T are collinear.
17. $\overline{TU}$ is a common external tangent to the two circles. $\overline{VW}$ is tangent to both circles. Prove that $\overline{TV} \cong \overline{VU} \cong \overline{VW}$.



18. A circle with a $5-$ inch radius is centered at A, and a circle with a $12-$ inch radius is centered at B, where A and B are 17 inches apart. The common external tangent touches the small circle at P and the large circle at Q. What kind of quadrilateral is $PABQ$? What are the lengths of its sides?

Review Answers

1. 12.37
2. 19.36
3. 3.34
4. 16.58
5. 59°
6. 36.9°
7. 60°
8. 54.5°
9. 7
10. 15
11. 14.14
12. 7.61
13. 7
14. 4.62; 2.31
15. 6.4
16. Proof
17. Proof
18. Right trapezoid; $AP = 5; BQ = 12; AB = 17; PQ = 15.5$

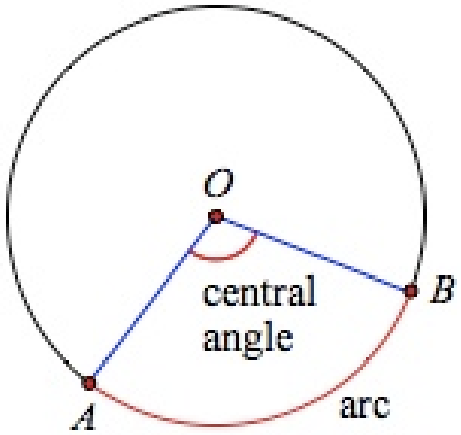
9.4 Arc Measures

Learning Objectives

- Measure central angles and arcs of circles.
- Find relationships between adjacent arcs.
- Find relationships between arcs and chords.

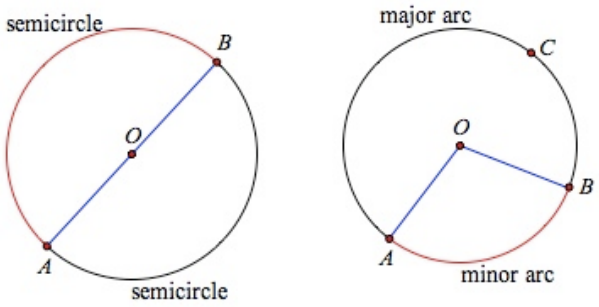
Arc, Central Angle

In a circle, the **central angle** is formed by two radii of the circle with its vertex at the center of the circle. An **arc** is a section of the circle.



Minor and Major Arcs, Semicircle

A **semicircle** is half a circle. A **major arc** is longer than a semicircle and a **minor arc** is shorter than a semicircle.

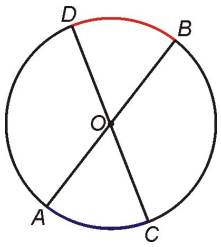


An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this lesson we will concentrate on degree measure. The measure of the minor arc is the same as the measure of the central angle that corresponds to it. The measure of the major arc equals to 360° minus the measure of the minor arc.

Minor arcs are named with two letters—the letters that denote the endpoints of the arc. In the figure above, the minor arc corresponding to the central angle $\angle AOB$ is called $\widehat{AB}$. In order to prevent confusion, major arcs are named with three letters—the letters that denote the endpoints of the arc and any other point on the major arc. In the figure, the major arc corresponding to the central angle $\angle AOB$ is called $\widehat{ACB}$.

Congruent Arcs

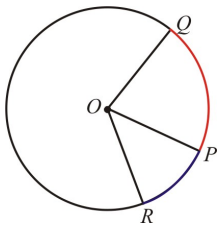
Two arcs that correspond to congruent central angles will also be congruent. In the figure below, $\angle AOC \cong \angle BOD$ because they are vertical angles. This also means that $\widehat{AC} \cong \widehat{DB}$.



Arc Addition Postulate

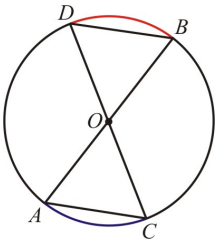
The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.

In other words, $m\widehat{RQ} = m\widehat{RP} + m\widehat{PQ}$.



Congruent Chords Have Congruent Minor Arcs

In the same circle or congruent circles, congruent chords have congruent minor arcs.



Proof. Draw the following diagram, in which the chords $\overline{DB}$ and $\overline{AC}$ are congruent.

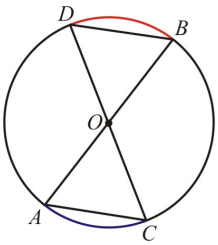
Construct $\triangle DOB$ and $\triangle AOC$ by drawing the radii for the center O to points A, B, C and D respectively.

Then, $\triangle AOC \cong \triangle BOD$ by the SSS postulate.

This means that central angles,$\angle AOC \cong \angle BOD$, which leads to the conclusion that $\widehat{AC} \cong \widehat{DB}$.

Congruent Minor Arcs Have Congruent Chords and Congruent Central Angles

In the same circle or congruent circles, congruent chords have congruent minor arcs.



Proof. Draw the following diagram, in which the $\widehat{AC} \cong \widehat{DB}$. In the diagram $\overline{DO}, \overline{OB}, \overline{AO}$, and $\overline{OC}$ are radii of the circle.

Since $\widehat{AC} \cong \widehat{DB}$, this means that the corresponding central angles are also congruent: $\angle AOC \cong \angle BOD$.

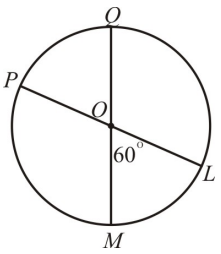
Therefore, $\triangle AOC \cong \triangle BOD$ by the SAS postulate.

We conclude that $\overline{DB} \cong \overline{AC}$.

Here are some examples in which we apply the concepts and theorems we discussed in this section.

Example 1

Find the measure of each arc.



A. $m\widehat{ML}$

B. $m\widehat{PM}$

C. $m\widehat{LMQ}$

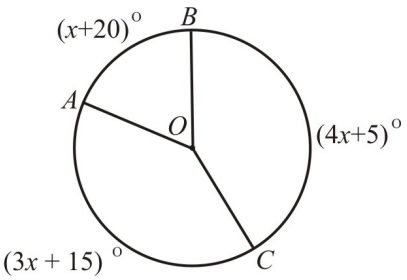
A. $m\widehat{ML} = m\angle LOM = 60°$

B. $m\widehat{PM} = m\angle POM = 180° - \angle LOM = 120°$

C. $m\widehat{LMQ} = m\widehat{ML} + m\widehat{PM} + m\widehat{PQ} = 60° + 120° + 60° = 240°$

Example 2

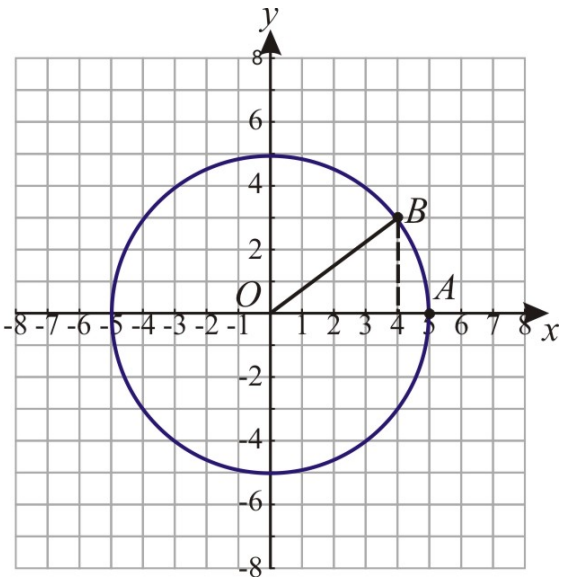
Find $m\widehat{AB}$ in circle O. The measures of all three arcs must add to $360°$.



$$x° + 20° + (4x)° + 5° + (3x)° + 15° = 360°$$
$$(8x)° = 320°$$
$$x = 40$$
$$m\widehat{AB} = 60°$$

Example 3

The circle $x^2 + y^2 = 25$ goes through $A = (5, 0)$ and $B = (4, 3)$. Find $m\widehat{AB}$.



Draw the radii to points A and B.

We know that the measure of the minor arc AB is equal to the measure of the central angle.

$$\tan O = \frac{3}{4} \Rightarrow m\angle AOB = 36.90^\circ$$

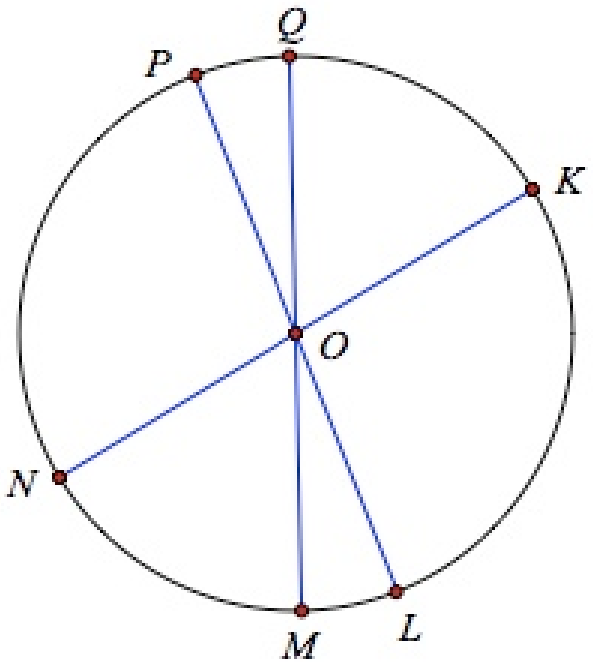
$$m\widehat{AB} \approx 36.9^\circ$$

Lesson Summary

In this section we learned about arcs and chords, and some relationships between them. We found out that there are major and minor arcs. We also learned that if two chords are congruent, so are the arcs they intersect, and vice versa.

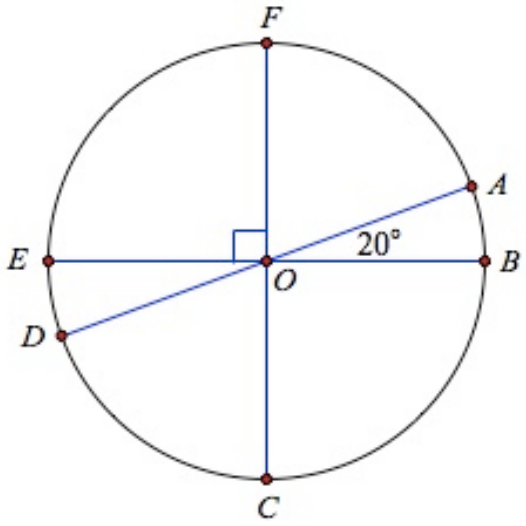
Review Questions

1. In the circle O identify the following:



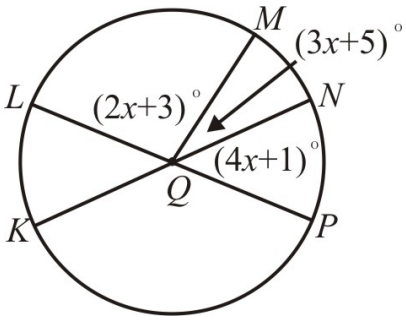
   (a) four radii
   (b) a diameter
   (c) two semicircles
   (d) three minor arcs
   (e) two major arcs

2. Find the measure of each angle in $\odot O$:



(a) $m\angle AOF$
(b) $m\angle AOB$
(c) $m\angle AOC$
(d) $m\angle EOF$
(e) $m\angle AOD$
(f) $m\angle FOD$

3. Find the measure of each angle in $\odot Q$:



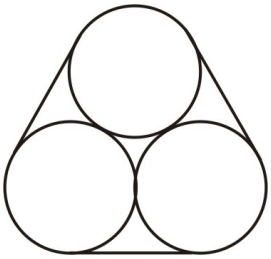
(a) $m\widehat{MN}$
(b) $m\widehat{LK}$
(c) $m\widehat{MP}$
(d) $m\widehat{MK}$
(e) $m\widehat{NPL}$
(f) $m\widehat{LKM}$

4. The students in a geometry class were asked what their favorite pie is. The table below shows the result of the survey. Make a pie chart of this information, showing the measure of the central angle for each slice of the pie.

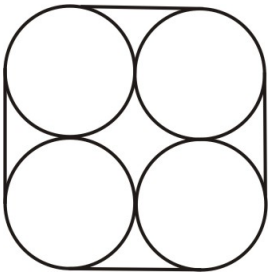
Table 9.2:

Kind of pie	Number of students
apple	6
pumpkin	4
cherry	2
lemon	3
chicken	7
banana	3
total	25

5. Three identical pipes of diameter 14 inches are tied together by a metal band as shown. Find the length of the band surrounding the three pipes.



6. Four identical pipes of diameter 14 inches are tied together by a metal band as shown. Find the length of the band surrounding the four pipes.

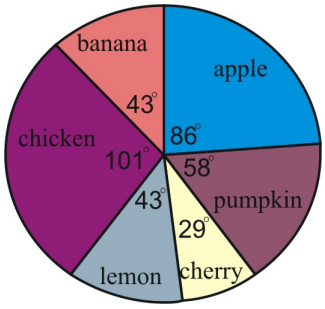


Review Answers

1. (a) $\overline{OQ}, \overline{OK}, \overline{OL}, \overline{OM}, \overline{ON}, \overline{OP}$
   (b) $\overline{KN}, \overline{QM}, \overline{PL}$
   (c) $\widehat{LKP}, \widehat{KPN}, \widehat{QPM}, \widehat{PML}, \widehat{NMK}, \widehat{MLQ}$
   (d) Some possibilities: $\widehat{NM}, \widehat{PK}, \widehat{QL}$.
   (e) Some possibilities: $\widehat{MPK}, \widehat{NQL}$.
2. (a) 70°
   (b) 20°
   (c) 110°

(d) 90°
(e) 180°
(f) 110°

3. (a) 62°
(b) 77°
(c) 139°
(d) 118°
(e) 257°
(f) 319°

4. 

5. $42 + 14\pi \approx$ 86 in.
6. $56 + 14\pi \approx$ 100 in.

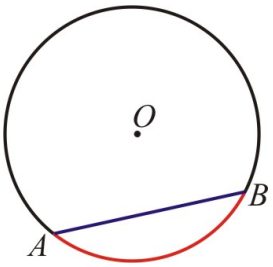
9.5 Chords

Learning Objectives

- Find the lengths of chords in a circle.
- Find the measure of arcs in a circle.

Introduction

Chords are line segments whose endpoints are both on a circle. The figure shows an arc $\overset{\frown}{AB}$ and its related chord $\overline{AB}$.

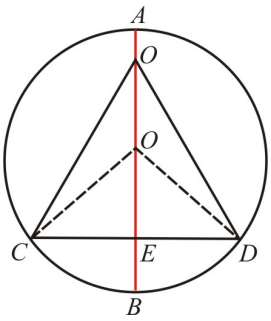


There are several theorems that relate to chords of a circle that we will discuss in the following

sections.

Perpendicular Bisector of a Chord

Theorem The perpendicular bisector of a chord is a diameter.



Proof

We will draw two chords, $\overline{AB}$ and $\overline{CD}$ such that $\overline{AB}$ is the perpendicular bisector of $\overline{CD}$.

We can see that $\triangle COE \cong \triangle DOE$ for any point O on chord $\overline{AB}$.

The congruence of the triangles can be proven by the SAS postulate:

$$\overline{CE} \cong \overline{ED}$$
$$\overline{OE} \cong \overline{OE}$$

$\angle OEC$ and $\angle OED$ are right angles

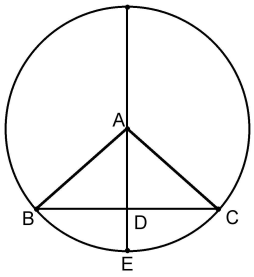
This means that $\overline{CO} \cong \overline{DO}$.

Any point that is equidistant from C and D lies along $\overline{AB}$, by the perpendicular bisector theorem. Since the center of the circle is one such point, it must lie along $\overline{AB}$ so $\overline{AB}$ is a diameter.

If O is the midpoint of $\overline{AB}$ then $\overline{OC}$ and $\overline{OD}$ are radii of the circle and $\overline{AB}$ is a diameter of the circle. ♦

Perpendicular Bisector of a Chord Bisects Intercepted Arc

Theorem The perpendicular bisector of a chord bisects the arc intercepted by the chord.



Proof

We can see that $\triangle CAD \cong \triangle BAD$ because of the SAS postulate.

$$\overline{DA} \cong \overline{DA}$$
$$\overline{BD} \cong \overline{DC}$$

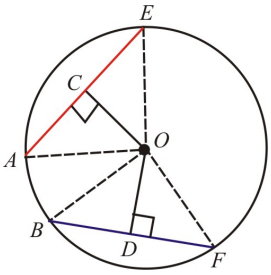
$\angle ADB$ and $\angle ADC$ are right angles.

This means that $\angle DAB \cong \angle DAC \Rightarrow \widehat{BE} \cong \widehat{CE}$.

This completes the proof. ♦

Congruent Chords Equidistant from Center

Theorem Congruent chords in the same circle are equidistant from the center of the circle.



First, recall that the definition of distance from a point to a line is the length of the perpendicular segment drawn to the line from the point. CO and DO are these distances, and we must prove that they are equal.

Proof.

$\triangle AOE \cong \triangle BOF$ by the SSS Postulate.

$$\overline{AE} \cong \overline{BF} \text{ (given)}$$
$$\overline{AO} \cong \overline{BO} \text{ (radii)}$$
$$\overline{EO} \cong \overline{OF} \text{ (radii)}$$

Since the triangles are congruent, their corresponding altitudes $\overline{OC}$ and $\overline{OD}$ must also be congruent: $\overline{CO} \cong \overline{DO}$.

Therefore, $\overline{AE} \cong \overline{BF}$ and are equidistant from the center. ♦

Converse of Congruent Chords Theorem

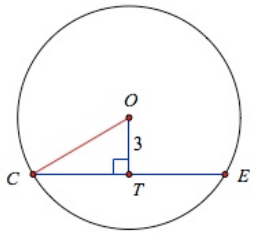
Theorem Two chords equidistant from the center of a circle are congruent.

This proof is left as a homework exercise.

Next, we will solve a few examples that apply the theorems we discussed.

Example 1

$CE = 12$ inches, *and is* 3 in. from the center of circle O.



A. Find the radius of the circle.

B. Find $m\widehat{CE}$

Draw the radius $\overline{OC}$.

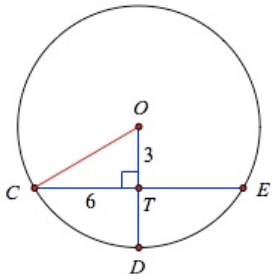
A. $\overline{OC}$ is the hypotenuse of the right triangle $\triangle COT$.

$OT = 3$ in.; $CT = 6$ in.

Apply the Pythagorean Theorem.

$$(OC)^2 = (OT)^2 + (CT)^2$$
$$(OC)^2 = 3^2 + 6^2 = 45$$
$$(OC)^2 = 3\sqrt{5} \approx 6.7 \text{ in.}$$

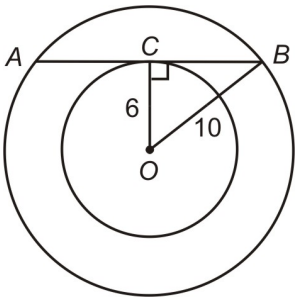
B. Extend $\overline{OT}$ to intersect the circle at point D.



$$\begin{aligned} m\widehat{CE} &= 2m\widehat{CD} \\ m\widehat{CD} &= m\angle COD \\ tan\angle COD &= \frac{6}{3} = 2 \\ m\angle COD &\approx 63.4^\circ \\ m\widehat{CE} &\approx 126.9^\circ \end{aligned}$$

Example 2

Two concentric circles have radii of 6 inches *and* 10 inches. *A segment tangent to the smaller circle is a chord of the larger circle. What is the length of the segment?*



Start by drawing a figure that represents the problem.

$$\begin{aligned} OC &= 6 \text{ in} \\ OB &= 10 \text{ in.} \end{aligned}$$

$\triangle COB$ is a right triangle because the radius $\overline{OC}$ of the smaller circle is perpendicular to the tangent $\overline{AB}$ at point C.

Apply the Pythagorean Theorem.

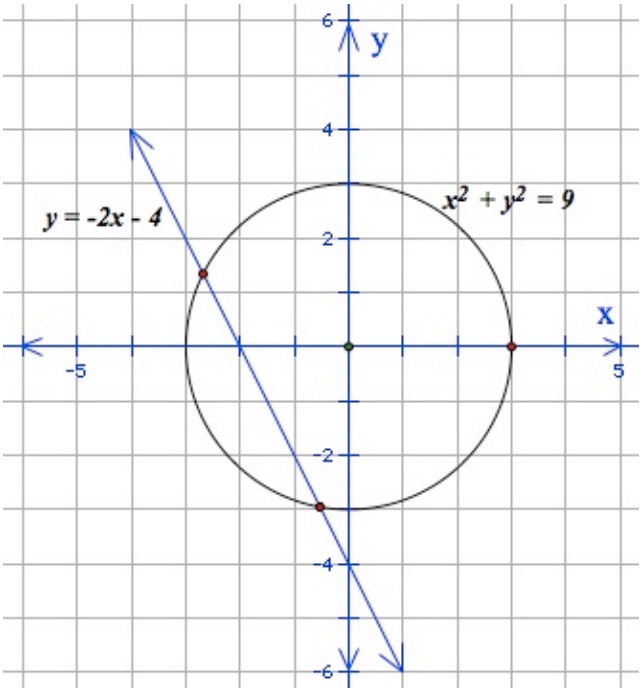
$$
\begin{aligned}
(OB)^2 &= (OC)^2 + (BC)^2 \\
10^2 &= 6^2 + BC^2 \\
BC &= 8 \text{ in} \\
AB &= 2BC \text{ from Theorem 9.6} \\
\text{Therefore,} AB &= 16\text{in.}
\end{aligned}
$$

Example 3

Find the length of the chord of the circle.

$x^2 + y^2 = 9$ that is given by line $y = -2x - 4$.

First draw a graph that represents the problem.



Find the intersection point of the circle and the line by substituting for y in the circle equation.

$$
\begin{aligned}
x^2 + y^2 &= 9 \\
y &= -2x - 4 \\
x^2 + (-2x - 4)^2 &= 9 \\
x^2 + 4x^2 + 16x + 16 &= 9 \\
5x^2 + 16x + 7 &= 0
\end{aligned}
$$

Solve using the quadratic formula.

$x = -0.52$ or -2.68

The corresponding values of y are

$y = -2.96$ or 1.36

Thus, the intersection points are *approximately* $(-0.52, -2.96)$ and $(-2.68, 1.36)$.

We can find the length of the chord by applying the distance formula:

$$d = \sqrt{(-0.52 + 2.68)^2 + (-2.96 - 1.36)^2} \approx 4.83 \text{ units.}$$

Example 4

Let A and B be the positive $x-$intercept and the positive $y-$intercept, respectively, of the circle $x^2 + y^2 = 32$. Let P and Q be the positive $x-$intercept and the positive $y-$intercept, respectively, of the circle $x^2 + y^2 = 64$.

Verify that the ratio of chords AB : PQ is the same as the ratio of the corresponding diameters.

For the circle $x^2 + y^2 = 32$, the $x-$intercept is found by setting $y = 0$. So $A = (4\sqrt{2}, 0)$.

The $y-$intercept is found by setting $x = 0$. So, $B = (0, 4\sqrt{2})$.

AB can be found using the distance formula: $AB = \sqrt{(4\sqrt{2}^2) + (4\sqrt{2})^2} = \sqrt{32 + 32} = 8$.

For the circle $x^2 + y^2 = 64$, $P = (8, 0)$ and $Q = (0, 8)$.

$$PQ = \sqrt{(8)^2 + (8)^2} = \sqrt{64 + 64} = 8\sqrt{2}.$$

The ratio of the $AB : PQ = 8 : 8\sqrt{2} = 1 : \sqrt{2}$.

Diameter of circle $x^2 + y^2 = 32$ is $8\sqrt{2}$.

Diameter of circle $x^2 + y^2 = 64$ is 16.

The ratio of the diameters is $8\sqrt{2} : 16 = 1 : \sqrt{2}$

The ratio of the chords and the ratio of the diameters are the same.

Lesson Summary

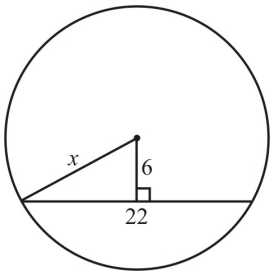
In this section we gained more tools to find the length of chords and the measure of arcs. We learned that the perpendicular bisector of a chord is a diameter and that the perpendicular

bisector of a chord also bisects the corresponding arc. We found that congruent chords are equidistant from the center, and chords equidistant from the center are congruent.

Review Questions

1. Find the value of x:

(a)



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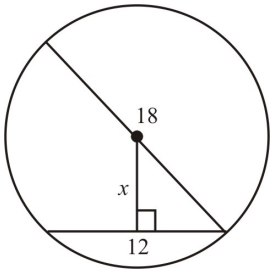

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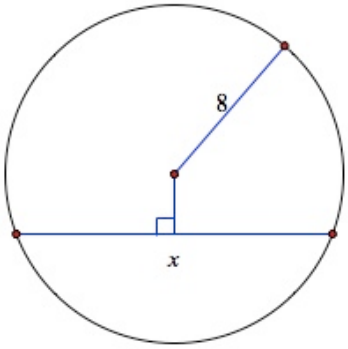
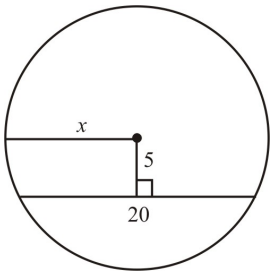
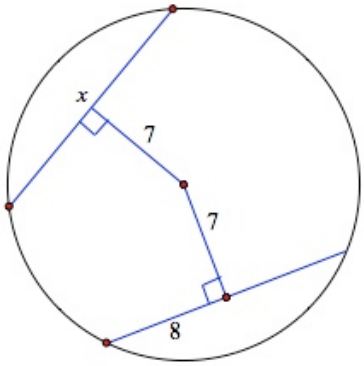


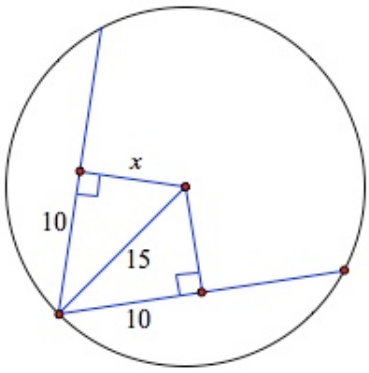
(d)



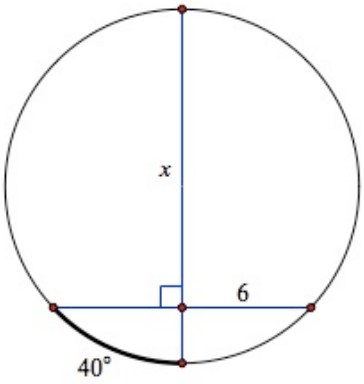
(e)



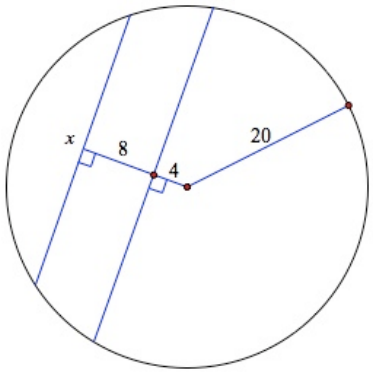
(f)



(g)

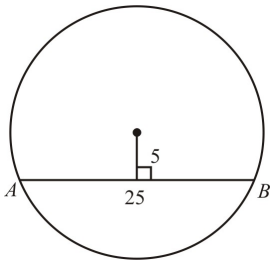


(h)



2. Find the measure of $\widehat{AB}$.

(a)



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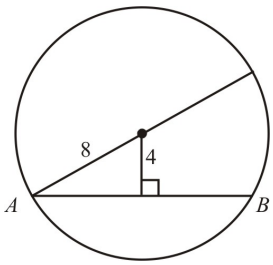

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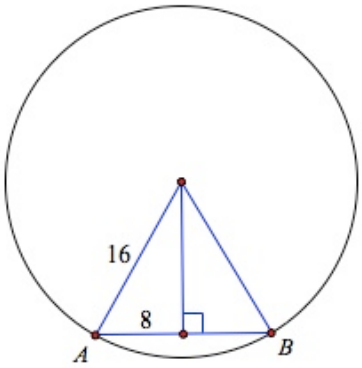

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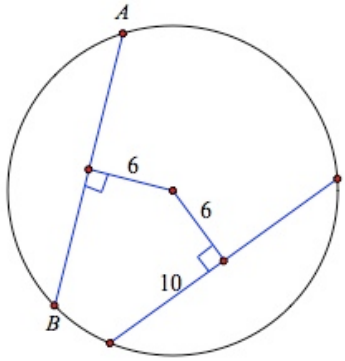

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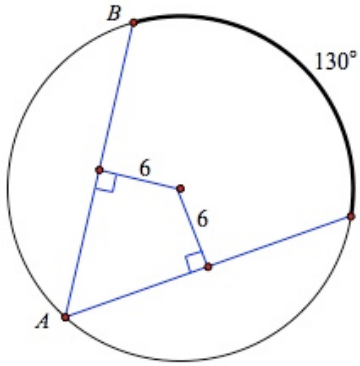

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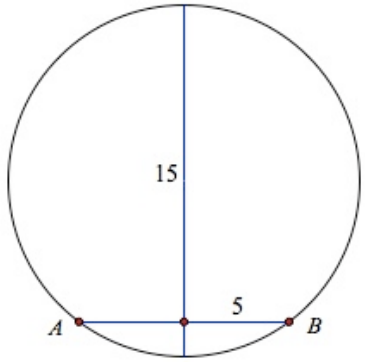
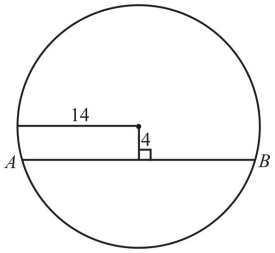
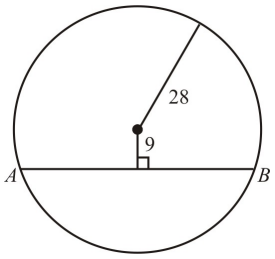


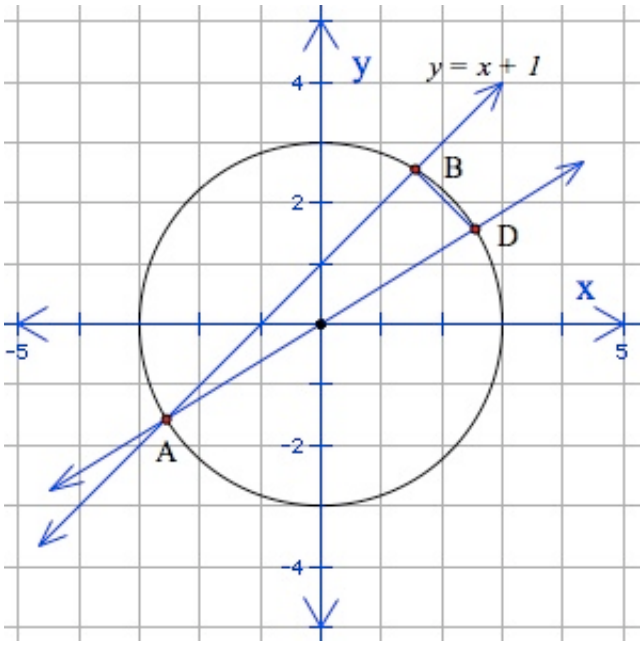
(g)



(h)



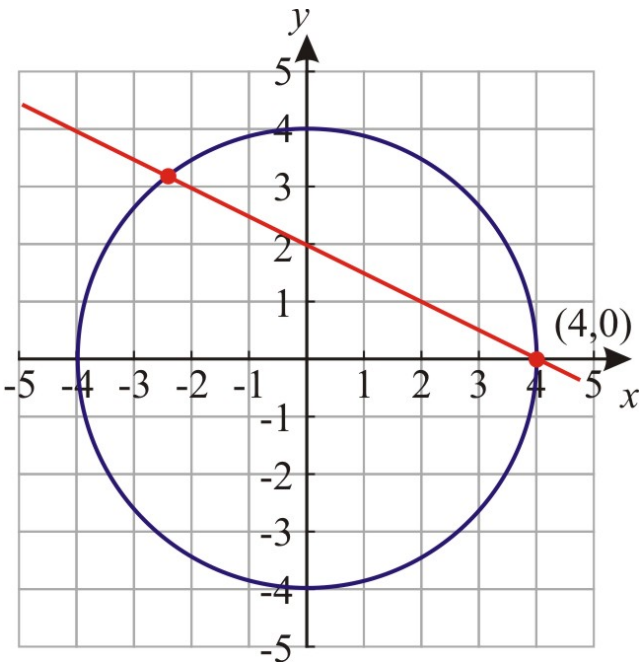
3. Two concentric circles have radii of 3 inches and 8 inches. A segment tangent to the smaller circle is a chord of the larger circle. What is the length of the segment?
4. Two congruent circles intersect at points A and B. $\overline{AB}$ is a chord to both circles. If the segment connecting the centers of the two circles measures 12 in and $\overline{AB} = 8$ in, how long is the radius?
5. Find the length of the chord of the circle $x^2 + y^2 = 16$ that is given by line $y = x + 1$.
6. Prove Theorem 9-9.
7. Sketch the circle whose equation is $x^2 + y^2 = 16$. Using the same system of coordinate axes, graph the line $x + 2y = 4$, which should intersect the circle twice—at $A = (4, 0)$ and at another point B in the second quadrant. Find the coordinates of B.
8. Also find the coordinates for a point C on the circle above, such that $\overline{AB} \cong \overline{AC}$.
9. The line $y = x + 1$ intersects the circle $x^2 + y^2 = 9$ in two points. Call the third quadrant point A and the first quadrant point B, and find their coordinates. Let D be the point where the line through A and the center of the circle intersects the circle again. Show that $\triangle BAD$ is a right triangle.



10. A circular playing field 100 meters in diameter has a straight path cutting across it. It is 25 meters from the center of the field to the closest point on this path. How long is the path?

Review Answers

1. (a) 12.53
   (b) 6.70
   (c) 14.83
   (d) 11.18
   (e) 16
   (f) 11.18
   (g) 16.48
   (h) 32
2. (a) 136.4°
   (b) 120°
   (c) 60°
   (d) 118.07°
   (e) 115°
   (f) 73.74°
   (g) 146.8°
   (h) 142.5°
3. 14.83
4. 7.21
5. 7.88
6. proof
7. $(-12/5, 16/5)$



8. $(-12/5, -16/5)$
9. $B(1.56, 2.56)$; $A(-2.56, -1.56)$; $D(2.56, 1.56)(AB)^2 = 34$; $(AD)^2 = 36$; $(BD)^2 = 2$;

$(AD)^2 = (AB)^2 + (BD)^2$

10. 86.6 meters

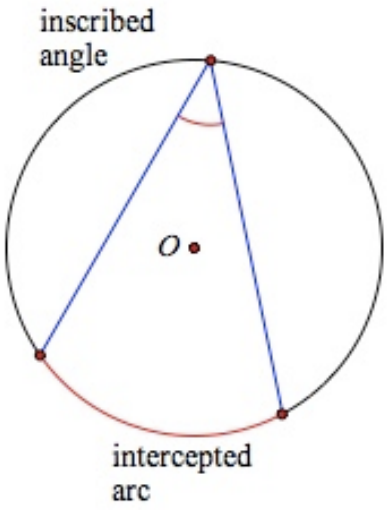
9.6 Inscribed Angles

Learning Objective

- Find the measure of inscribed angles and the arcs they intercept

Inscribed Angle, Intercepted Arc

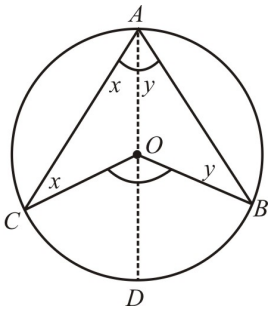
An **inscribed angle** is an angle whose vertex is on the circle and whose sides contain chords of the circle. An inscribed angle is said to **intercept** an arc of the circle. We will prove shortly that the measure of an inscribed angle is half of the measure of the arc it intercepts.



Notice that the vertex of the inscribed angle can be anywhere on the circumference of the circle–it does not need to be diametrically opposite the intercepted arc.

Measure of Inscribed Angle

The measure of a central angle is twice the measure of the inscribed angle that intercepts the same arc.



Proof.

$\angle COB$ and $\angle CAB$ both intercept $\widehat{CB}$. $\angle COB$ is a central angle and angle $\angle CAB$ is an inscribed angle.

We draw the diameter of the circle through points A and O, and let $m\angle CAO = x^\circ$ and $m\angle BAO = y^\circ$.

We see that $\triangle AOC$ is isosceles because $\overline{AO}$ and $\overline{AC}$ are radii of the circle and are therefore congruent.

From this we can conclude that $m\angle ACO = x^\circ$.

Similarly, we can conclude that $m\angle ABO = y^\circ$.

We use the property that the sum of angles inside a triangle equals 180° to find that:

$m\angle AOC = 180^\circ - 2x$ and $m\angle AOB = 180^\circ - 2y$.

Then,

$m\angle COD = 180^\circ - m\angle AOC = 180^\circ - (180^\circ - 2x) = 2x$ and $m\angle BOD = 180^\circ - m\angle AOB = 180^\circ - (180^\circ - 2y) = 2y$.

Therefore

$$m\angle COB = 2x + 2y = 2(x + y) = 2(m\angle CAB).\blacklozenge$$

Inscribed Angle Corollaries a-d

The theorem above has several corollaries, which will be left to the student to prove.

a. Inscribed angles intercepting the same arc are congruent

b. Opposite angles of an inscribed quadrilateral are supplementary

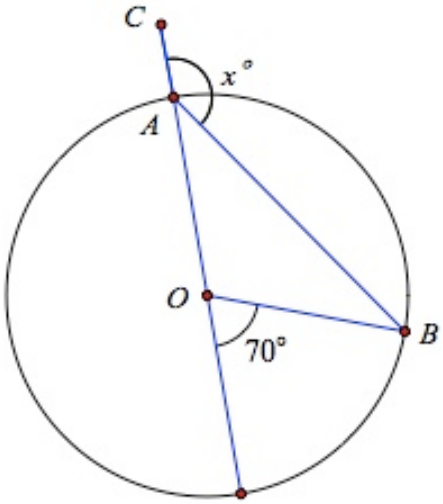
c. An angle inscribed in a semicircle is a right angle

d. An inscribed right angle intercepts a semicircle

Here are some examples the make use of the theorems presented in this section.

Example 1

Find the angle marked x in the circle.



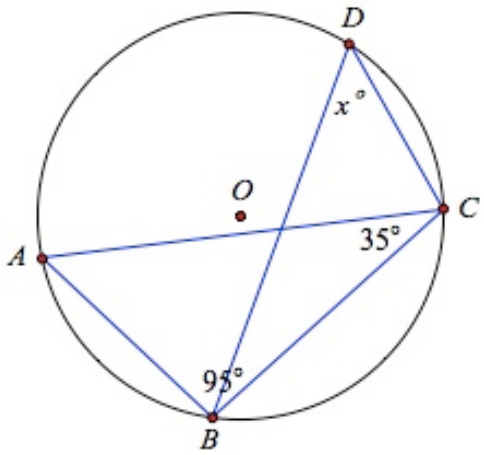
The $m\angle AOB$ is twice the measure of the angle at the circumference because it is a central angle.

Therefore, $m\angle OAB = 35°$.

This means that $x = 180° - 35° = 145°$.

Example 2

Find the angles marked x in the circle.

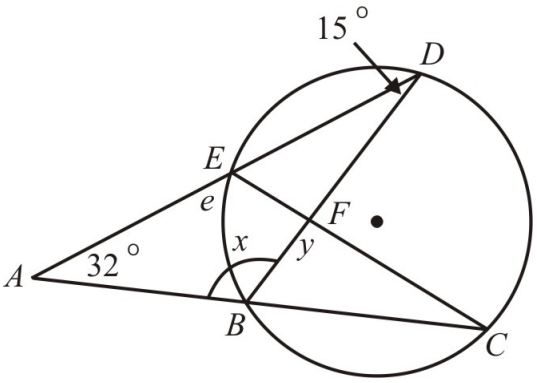


$$m\widehat{ADC} = 2 \times 95^\circ = 190^\circ \Rightarrow m\widehat{ABC} = 360^\circ - 190^\circ = 170^\circ$$
$$m\widehat{AB} = 2 \times 35^\circ = 70^\circ \Rightarrow m\widehat{BC} = 170^\circ - 70^\circ = 100^\circ$$

So, $x = 50^\circ$.

Example 3

Find the angles marked x and y in the circle.



First we use $\triangle ABD$ to find the measure of angle x.

$$x + 15^\circ + 32^\circ = 180^\circ \Rightarrow x = 133^\circ$$

Therefore, $m\angle CBD = 180^\circ - 133^\circ = 47^\circ$.

$m\angle BCE = m\angle BDE$ because they are inscribed angles and intercept the same arc $\Rightarrow$ $m\angle BCE = 15^\circ$.

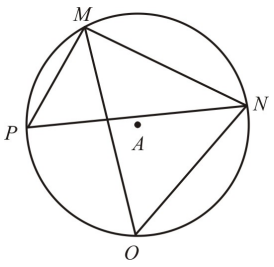
In $\triangle BFC, y + 47^\circ + 15^\circ = 180^\circ \Rightarrow y = 118^\circ$.

Lesson Summary

In this section we learned about inscribed angles. We found that an inscribed angle is half the measure of the arc it intercepts. We also learned some corollaries related to inscribed angles and found that if two inscribed angles intercept the same arc, they are congruent.

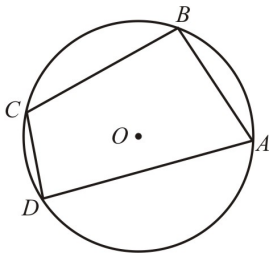
Review Questions

1. In $\odot A, m\widehat{PO} = 90^\circ, m\widehat{ON} = 95^\circ$ and $m\angle MON = 60^\circ$. Find the measure of each angle:



(a) $m\angle PMO$
(b) $m\angle PNO$
(c) $m\angle MPN$
(d) $m\angle PMN$
(e) $m\angle MPO$
(f) $m\angle MNO$

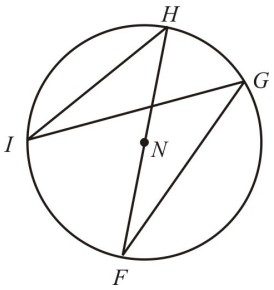
2. Quadrilateral $ABCD$ is inscribed in $\odot O$ such that $m\widehat{AB} = 70°, m\widehat{BC} = 85°, m\widehat{AD} = 130°$.



Find the measure of each of the following angles:

(a) $m\angle A$
(b) $m\angle B$
(c) $m\angle C$
(d) $m\angle D$

3. In the following figure, $m\widehat{IF} = 5x + 60°, m\angle IGF = 3x + 25°$ and $m\widehat{HG} = 2x + 10°$.



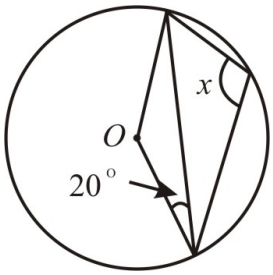
Find the following measures:

(a) $m\angle I$
(b) $m\angle F$

(c) $m\angle H$
(d) $m\widehat{IH}$

4. Prove the inscribed angle theorem corollary a.
5. Prove the inscribed angle theorem corollary b.
6. Prove the inscribed angle theorem corollary c.
7. Prove the inscribed angle theorem corollary d.
8. Find the measure of angle x.

(a)



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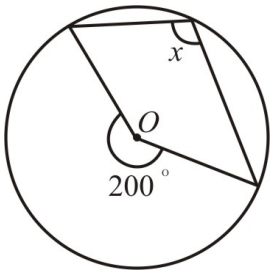

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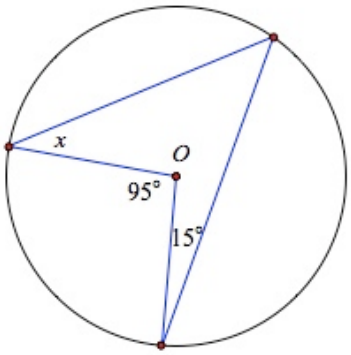

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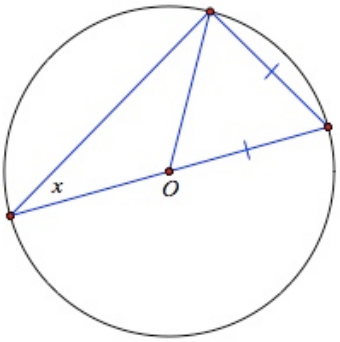
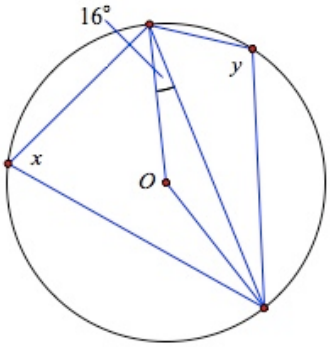


9. Find the measure of the angles x and y.

(a)



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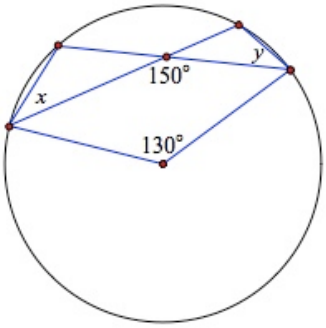


10. Suppose that $\overline{AB}$ is a diameter of a circle centered at O, and C is any other point on the circle. Draw the line through O that is parallel to $\overline{AC}$, and let D be the point where it meets $\widehat{BC}$. Prove that D is the midpoint of $\widehat{BC}$.

Review Answers

1. (a) 45°
   (b) 45°
   (c) 60°

(d) 92.5°
(e) 107.5°
(f) 72.5°

2. (a) 80°
   (b) 102.5°
   (c) 100°
   (d) 77.5°
3. (a) 15°
   (b) 15°
   (c) 55°
   (d) 70°
4. Proof
5. Proof
6. Proof
7. Proof
8. (a) 110°
   (b) 100°
   (c) 32.5°
   (d) 30°
9. (a) $x = 74°, y = 106°$
   (b) $x = 35°, y = 35°$
10. Hint: $\overline{AC} \parallel \overline{OD}$, so $\angle CAB \cong \angle DOB$.

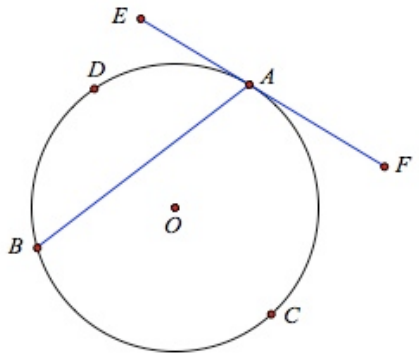
9.7 Angles of Chords, Secants, and Tangents

Learning Objective

- find the measures of angles formed by chords, secants, and tangents

Measure of Tangent-Chord Angle

Theorem The measure of an angle formed by a chord and a tangent that intersect on the circle equals half the measure of the intercepted arc.



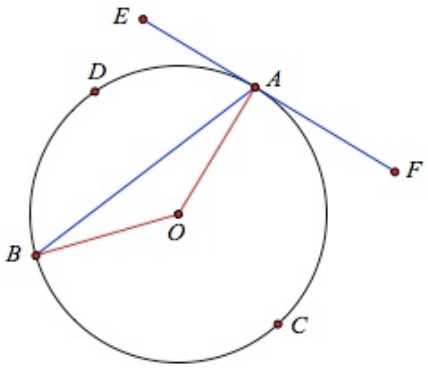
In other words:

$m\angle FAB = \frac{1}{2}m\widehat{ACB}$ and

$m\angle EAB = \frac{1}{2}m\widehat{ADB}$

Proof

Draw the radii of the circle to points A and B.



$\triangle AOB$ is isosceles, therefore

$$m\angle BAO = m\angle ABO = \frac{1}{2}(180^\circ - m\angle AOB) = 90^\circ - \frac{1}{2}m\angle AOB.$$

We also know that, $m\angle BAO + m\angle FAB = 90^\circ$ because FE is tangent to the circle.

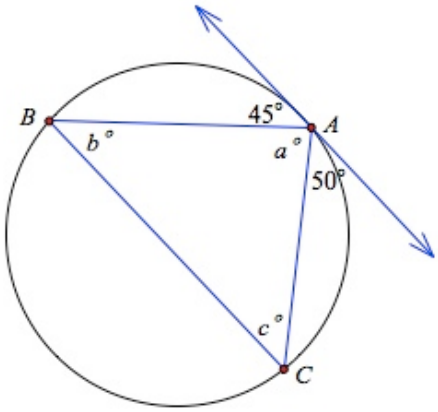
We obtain $90^\circ - \frac{1}{2}m\angle AOB + m\angle FAB = 90^\circ \Rightarrow m\angle FAB = \frac{1}{2}m\angle AOB$.

Since $\angle AOB$ is a central angle that corresponds to $\widehat{ADB}$ then, $m\angle FAB = \frac{1}{2}m\widehat{ADB}$.

This completes the proof. ♦

Example 1

Find the values of a, b and c.



First we find angle a : $50° + 45° + \angle a = 180° \Rightarrow m\angle a = 85°$.

Using the Measure of the Tangent Chord Theorem we conclude that:

$$m\widehat{AB} = 2(45°) = 90°$$

and

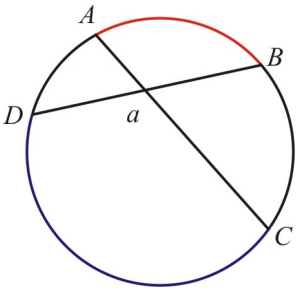
$$m\widehat{AC} = 2(50°) = 100°$$

Therefore,

$$m\angle b = \frac{1}{2}10° = 50°$$
$$m\angle c = \frac{1}{2}90° = 45°$$

Angles Inside a Circle

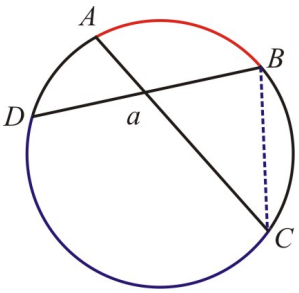
Theorem The measure of the angle formed by two chords that intersect inside a circle is equal to half the sum of the measure of their intercepted arcs. In other words, the measure of the angle is the average (mean) of the measures of the intercepted arcs.



In this figure, $m\angle a = \frac{1}{2}(m\widehat{AB} + m\widehat{DC})$.

Proof

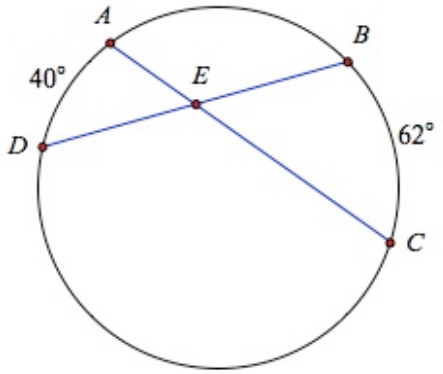
Draw a segment to connect points B and C.



$m\angle DBC = \frac{1}{2}m\widehat{DC}$	Inscribed angle
$m\angle ACB = \frac{1}{2}m\widehat{AB}$	Inscribed angle
$m\angle a = m\angle ACB + m\angle DBC$	The measure of an exterior angle in a triangle is equal to the sum of the measures of the remote interior angles.
$m\angle a = \frac{1}{2}m\widehat{DC} + \frac{1}{2}m\widehat{AB}$	Substitution
$m\angle a = \frac{1}{2}\left(m\widehat{DC} + m\widehat{AB}\right)$ ♦	

Example 2

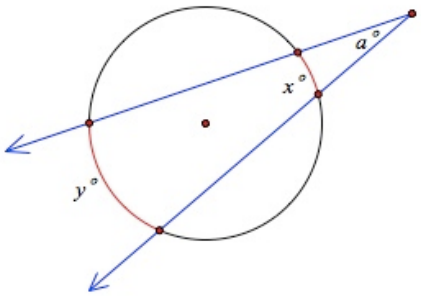
Find $m\angle DEC$.



$$m\angle AED = \frac{1}{2}\left(m\widehat{AD} + m\widehat{BC}\right) = \frac{1}{2}(40^\circ + 62^\circ) = 51^\circ$$
$$m\angle DEC = 180^\circ - m\angle AED$$
$$m\angle DEC = 180^\circ - 51^\circ = 129^\circ$$

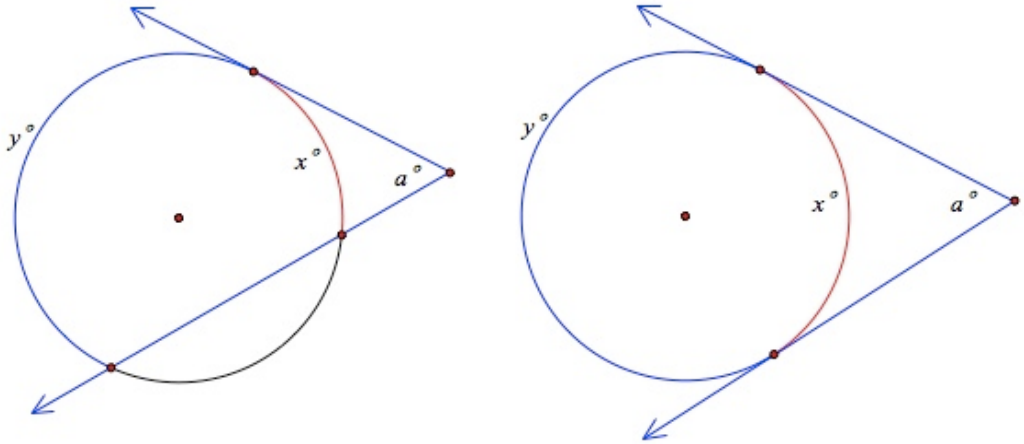
Angles Outside a Circle

Theorem The measure of an angle formed by two secants drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.



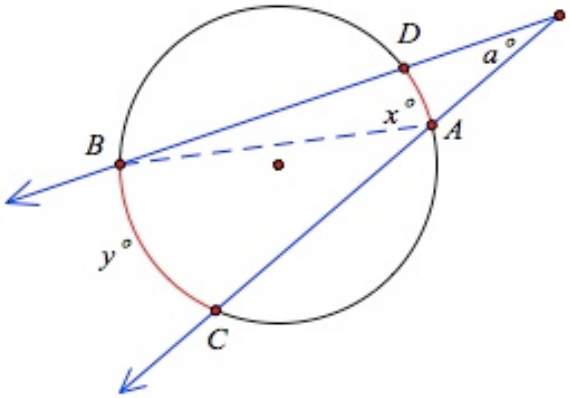
In other words: $m\angle a = \frac{1}{2}(y^\circ - x^\circ)$.

This theorem also applies for an angle formed by two tangents to the circle drawn from a point outside the circle and for an angle formed by a tangent and a secant drawn from a point outside the circle.



Proof

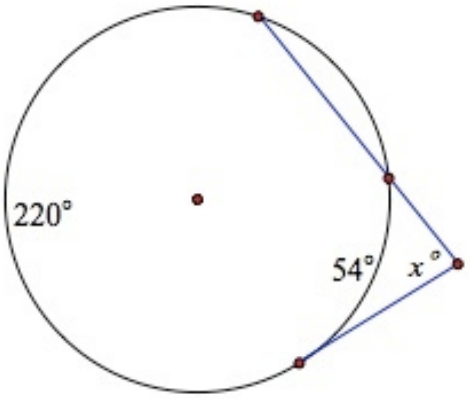
Draw a line to connect points A and B.



$m\angle DBA = \frac{1}{2}x^\circ$	Inscribed angle
$m\angle BAC = \frac{1}{2}y^\circ$	Inscribed angle
$m\angle BAC = m\angle DBA + m\angle a$	The measure of an exterior angle in a triangle is equal to the sum of the measures of the remote interior angles.
$\frac{1}{2}y^\circ = \frac{1}{2}x^\circ + m\angle a$	Substitution
$m\angle a = \frac{1}{2}(y^\circ - x^\circ)$♦	

Example 3

Find the measure of angle x.



$$m\angle x = \frac{1}{2}(220^\circ - 54^\circ) = 83^\circ$$

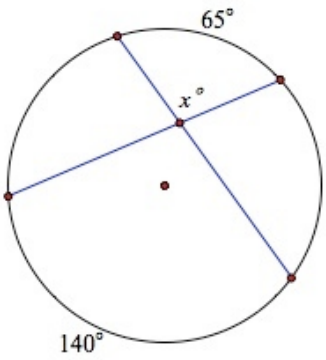
Lesson Summary

In this section we learned about finding the measure of angles formed by chords, secants, and tangents. We looked at the relationship between the arc measure and the angles formed by chords, secants, and tangents.

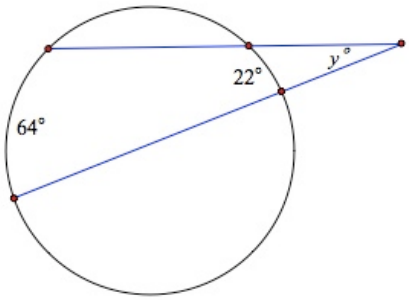
Review Questions

1. Find the value of the variable.

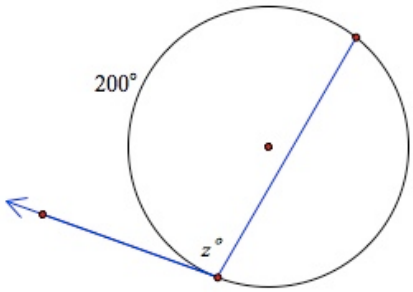
   (a)



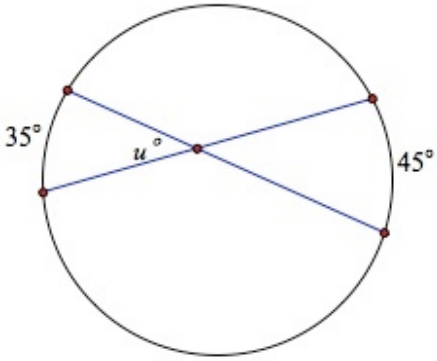
(b)



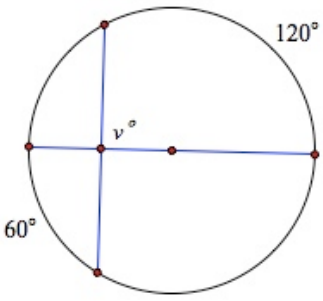
(c)



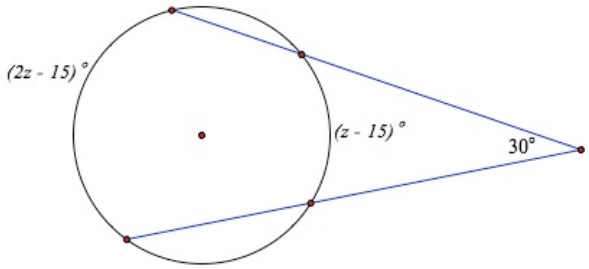
(d)



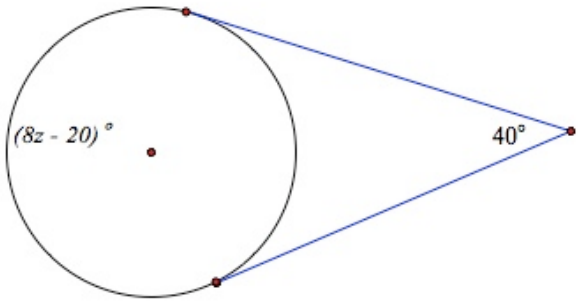
(e)



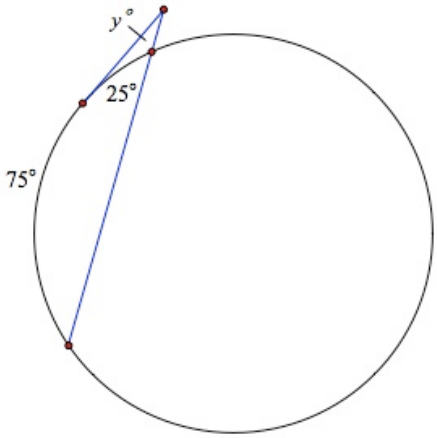
(f)



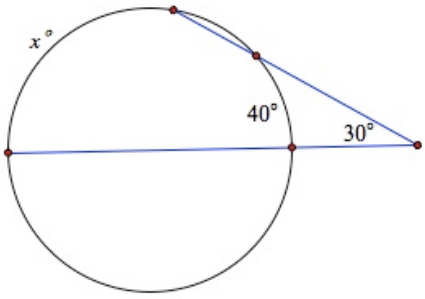
(g)



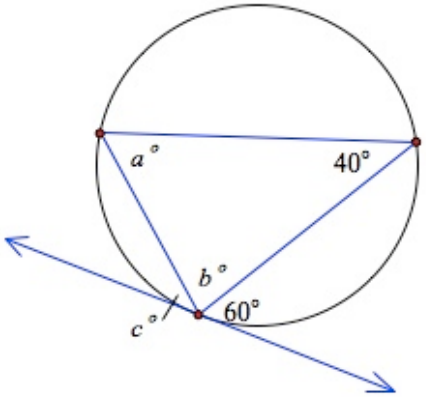
(h)



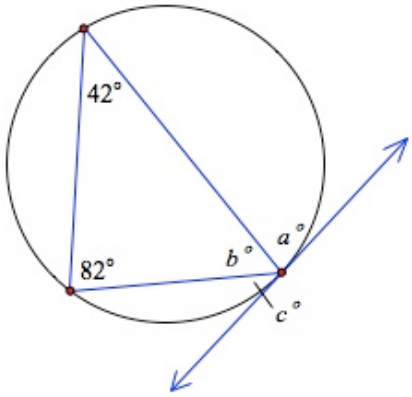
(i)



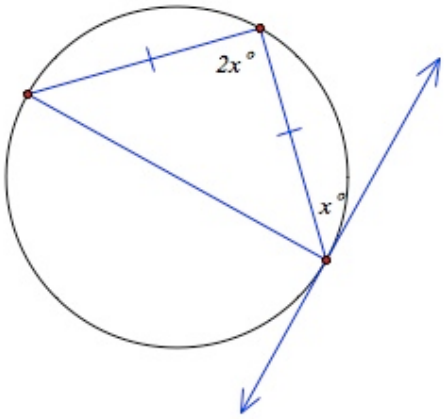
(j)



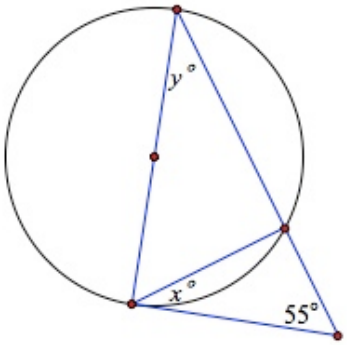
(k)



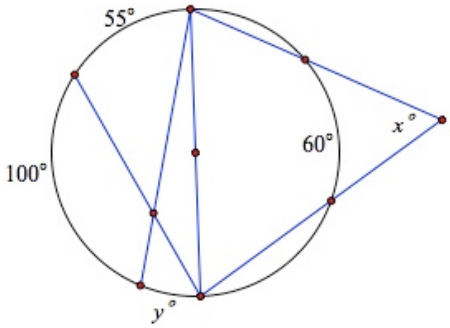
(l)



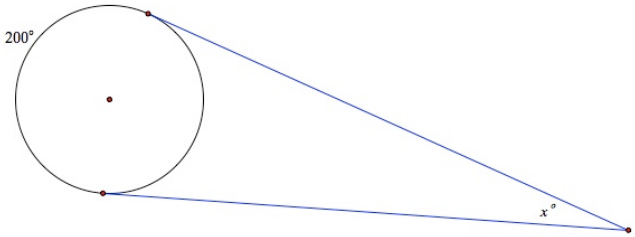
(m)



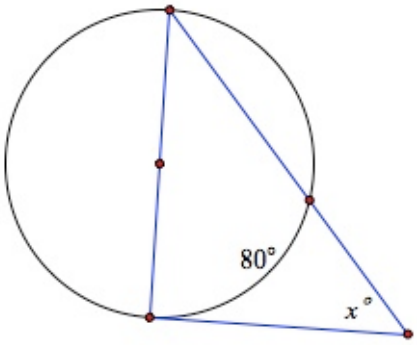
(n)



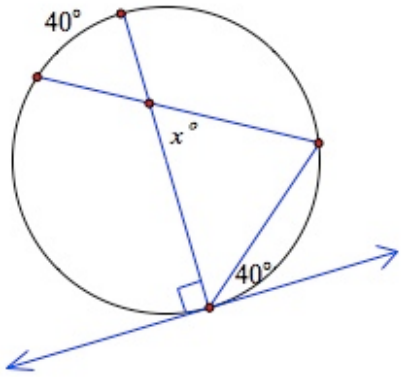
(o)



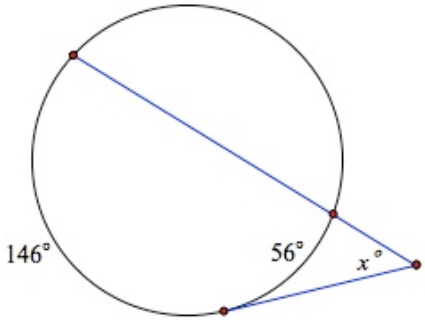
(p)



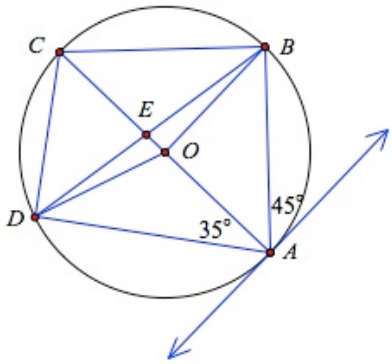
(q)



(r)

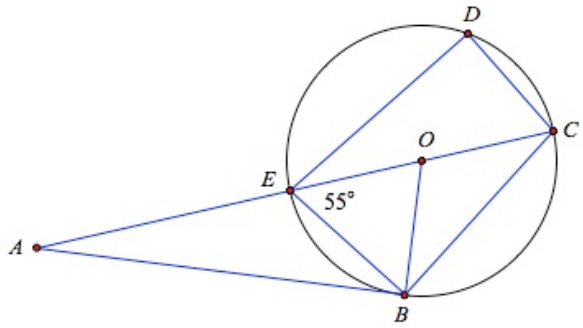


2. Find the measure of the following angles:



(a) $m\angle OAB$
(b) $m\angle COD$
(c) $m\angle CBD$
(d) $m\angle DCO$
(e) $m\angle AOB$
(f) $m\angle DOA$

3. Find the measure of the following angles:



   (a) $m\angle CDE$
   (b) $m\angle BOC$
   (c) $m\angle EBO$
   (d) $m\angle BAC$

4. Four points on a circle divide it into four arcs, whose sizes are $44°, 100°, 106°$, and $110°$, in consecutive order. The four points determine two intersecting chords. Find the sizes of the angles formed by the intersecting chords.

Review Answers

1. (a) $102.5°$
   (b) $21°$
   (c) $100°$
   (d) $40°$
   (e) $90°$
   (f) $60°$
   (g) $30°$
   (h) $25°$
   (i) $100°$
   (j) $a = 60°, b = 80°, c = 40°$
   (k) $a = 82°, b = 56°, c = 42°$
   (l) $45°$
   (m) $x = 35°, y = 35°$
   (n) $x = 60°, y = 25°$
   (o) $20°$
   (p) $50°$
   (q) $60°$
   (r) $45°$
2. (a) $45°$
   (b) $80°$

(c) 40°
(d) 50°
(e) 90°
(f) 110°

3. (a) 90°
(b) 110°
(c) 55°
(d) 20°

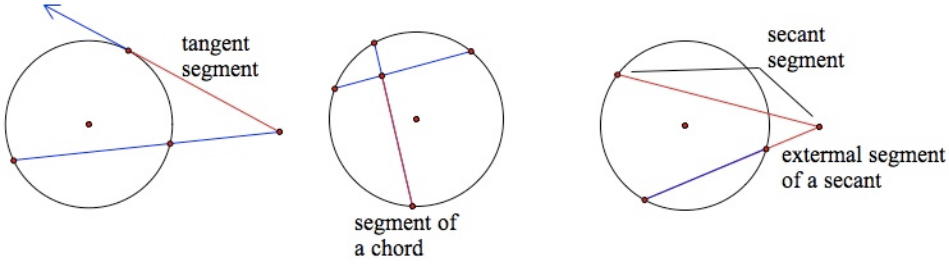
4. 75° and 105°

9.8 Segments of Chords, Secants, and Tangents

Learning Objectives

- Find the lengths of segments associated with circles.

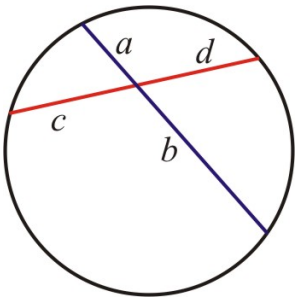
In this section we will discuss segments associated with circles and the angles formed by these segments. The figures below give the names of segments associated with circles.



Segments of Chords

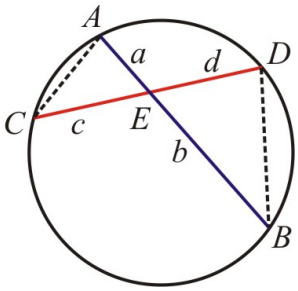
Theorem If two chords intersect inside the circle so that one is divided into segments of length a and b and the other into segments of length b and c then the segments of the chords satisfy the following relationship: $ab = cd$.

This means that the product of the segments of one chord equals the product of segments of the second chord.



Proof

We connect points A and C and points D and B to make $\triangle AEC$ and $\triangle DEB$.



$\angle AEC \cong \angle DEB$	Vertical angles
$\angle CAB \cong \angle BDC$	Inscribed angles intercepting the same arc
$\angle ACD \cong \angle ABD$	Inscribed angles intercepting the same arc

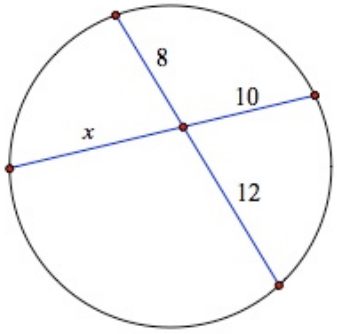
Therefore, $\triangle AEC \sim \triangle DEB$ by the AA similarity postulate.

In similar triangles the ratios of corresponding sides are equal.

$$\frac{c}{b} = \frac{a}{d} \Rightarrow ab = cd \blacklozenge$$

Example 1

Find the value of the variable.

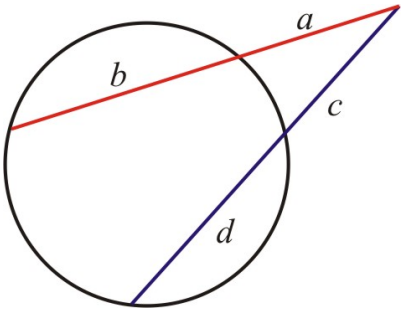


$$10x = 8 \times 12$$
$$10x = 96$$
$$x = 9.6$$

Segments of Secants

Theorem If two secants are drawn from a common point outside a circle and the segments are labeled as below, then the segments of the secants satisfy the following relationship:

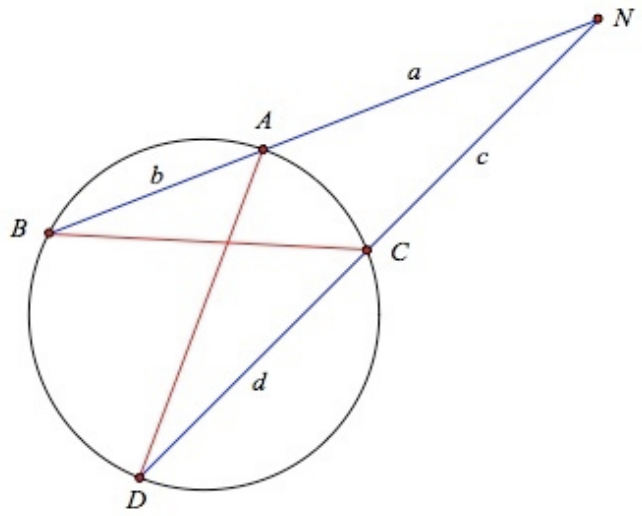
$$a(a+b) = c(c+d)$$



This means that the product of the outside segment of one secant and its whole length equals the product of the outside segment of the other secant and its whole length.

Proof

We connect points A and D and points B and C to make $\triangle BCN$ and $\triangle ADN$.



$\angle BNC \cong \angle DNA$	Same angle
$\angle NBC \cong \angle NDA$	Inscribed angles intercepting the same arc

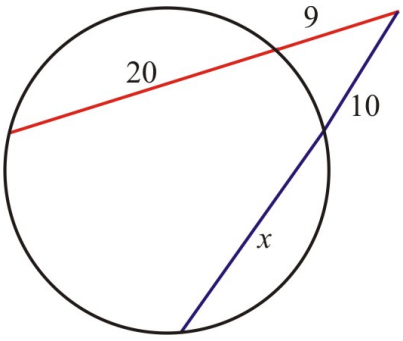
Therefore, $\triangle BCN \sim \triangle DAN$ by the AA similarity postulate.

In similar triangles the ratios of corresponding sides are equal.

$$\frac{a}{c} = \frac{c+d}{a+b} \Rightarrow a(a+b) = c(c+d) \blacklozenge$$

Example 2

Find the value of the variable.

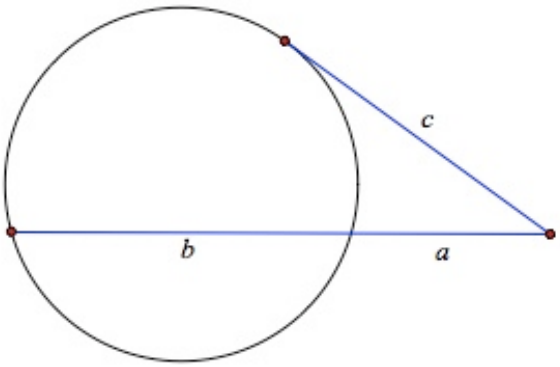


$$\begin{aligned} 10(10+x) &= 9(9+20) \\ 100+10x &= 261 \\ 10x &= 161 \\ x &= 16.1 \end{aligned}$$

Segments of Secants and Tangents

Theorem If a tangent and a secant are drawn from a point outside the circle then the segments of the secant and the tangent satisfy the following relationship

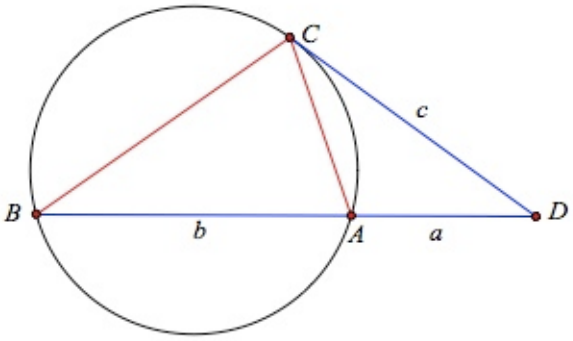
$$a(a+b) = c^2.$$



This means that the product of the outside segment of the secant and its whole length equals the square of the tangent segment.

Proof

We connect points C and A and points B and C to make $\triangle BCD$ and $\triangle CAD$.



$m\angle CDB = m\angle BAC - m\angle DBC$	The measure of an Angle outside a circle is equal to half the difference of the measures of the intercepted arcs
$m\angle BAC = m\angle ACD + m\angle CDB$	The measure of an exterior angle in a triangle equals the sum of the measures of the remote interior angles
$m\angle CDB = m\angle ACD + m\angle CDB - m\angle DBC$	Combining the two steps above
$m\angle DBC = m\angle ACD$	algebra

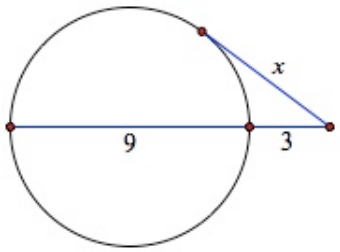
Therefore, $\triangle BCD \sim \triangle CAD$ by the AA similarity postulate.

In similar triangles the ratios of corresponding sides are equal.

$$\frac{c}{a+b} = \frac{a}{c} \Rightarrow a(a+b) = c^2 \blacklozenge$$

Example 3

Find the value of the variable x assuming that it represents the length of a tangent segment.



$$x^2 = 3(9+3)$$
$$x^2 = 36$$
$$x = 6$$

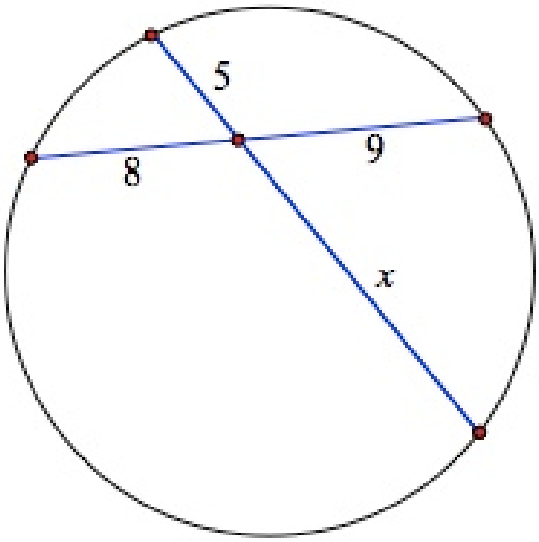
Lesson Summary

In this section, we learned how to find the lengths of different segments associated with circles: chords, secants, and tangents. We looked at cases in which the segments intersect inside the circle, outside the circle, or where one is tangent to the circle. There are different equations to find the segment lengths, relating to different situations.

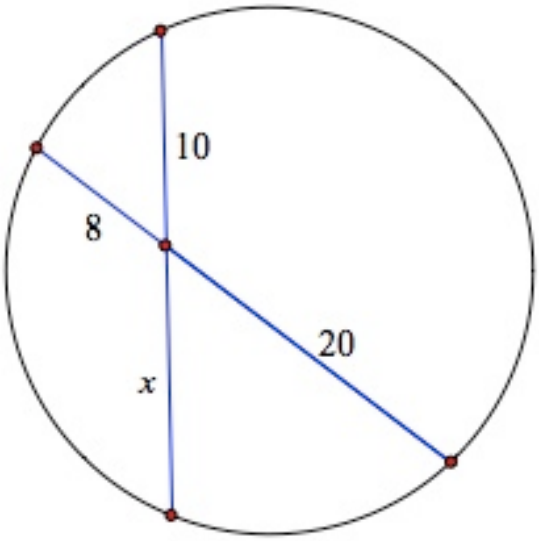
Review Questions

1. Find the value of missing variables in the following figures:

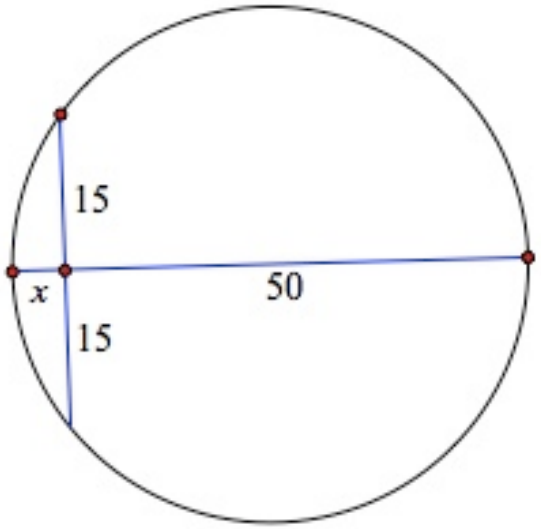
(a)



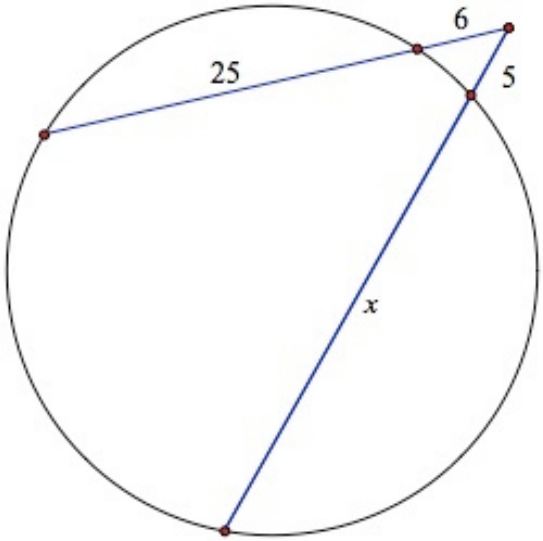
(b)



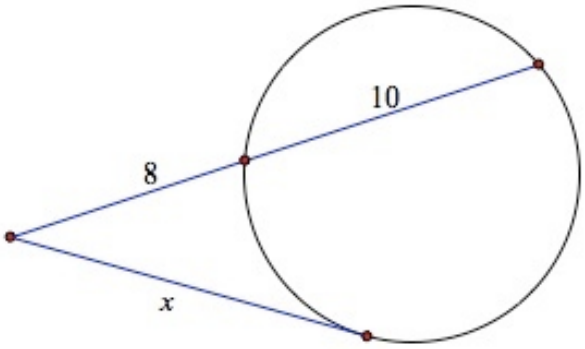
(c)



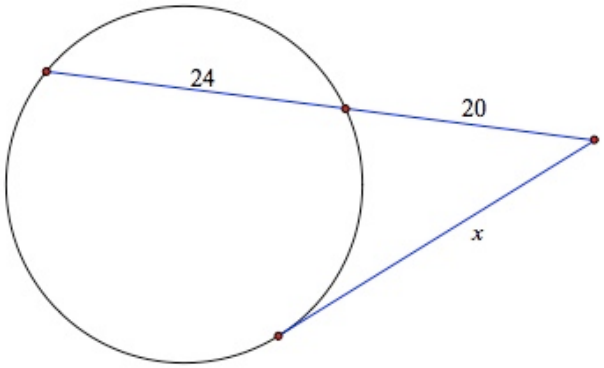
(d)



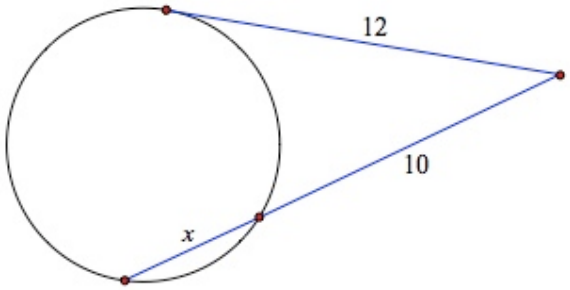
(e)



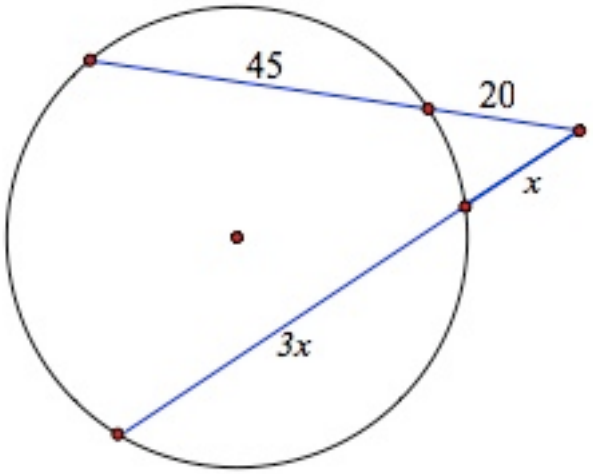
(f)



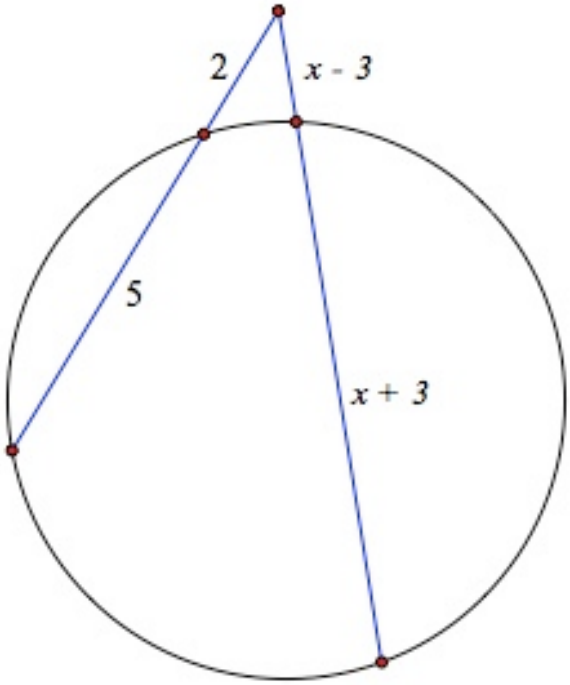
(g)



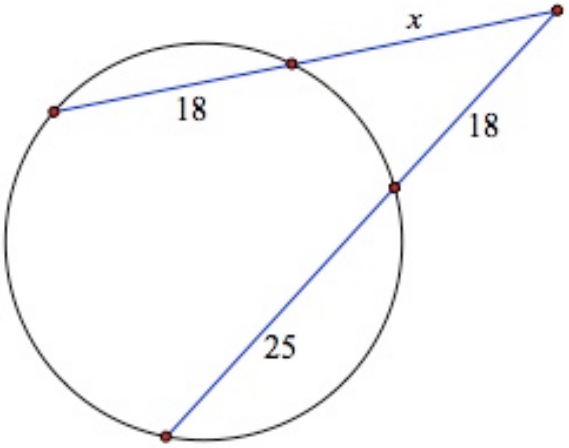
(h)



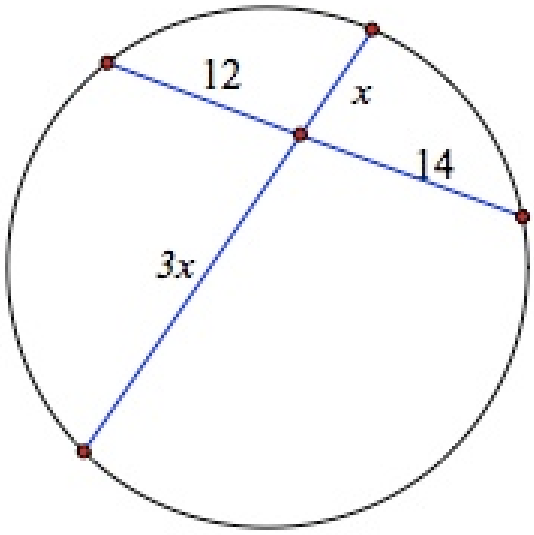
(i)



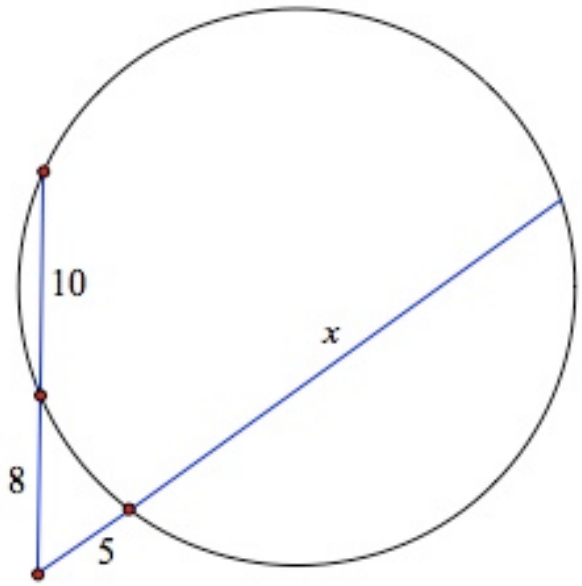
(j)



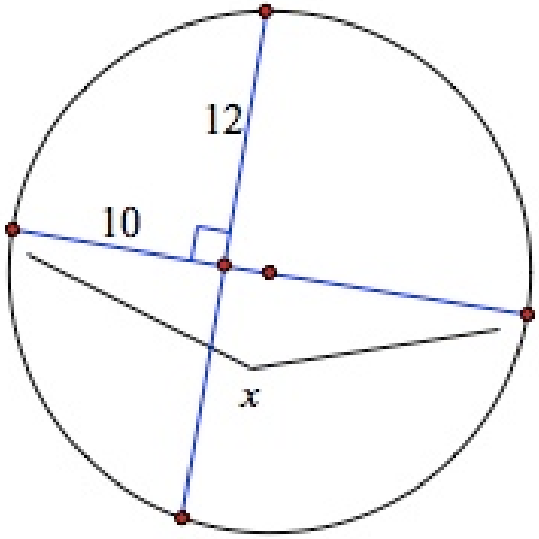
(k)



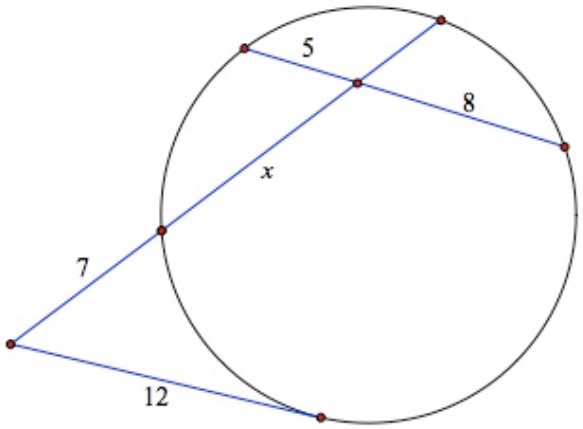
(l)



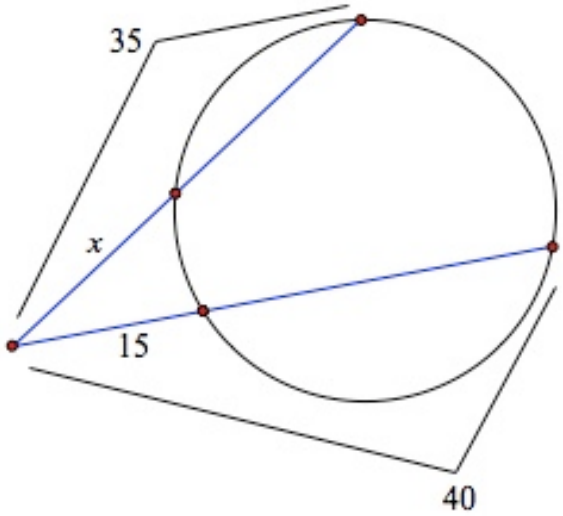
(m)



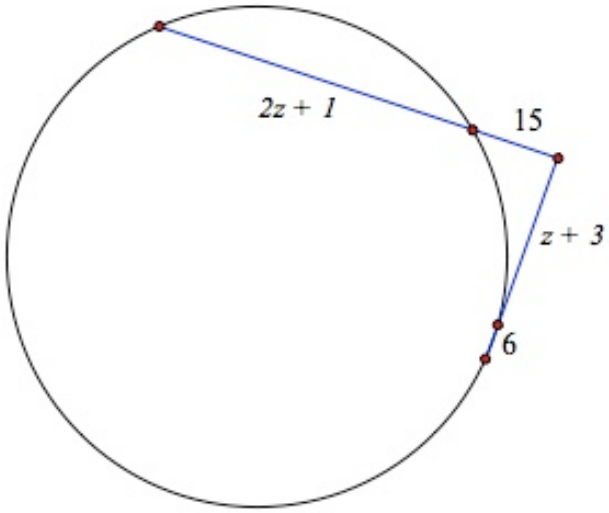
(n)



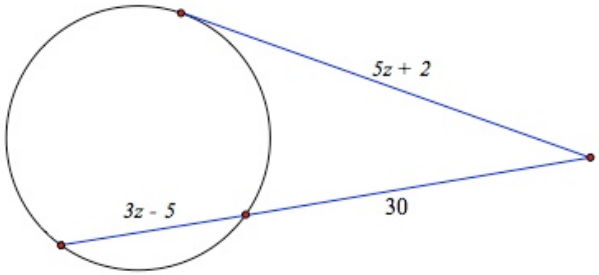
(o)



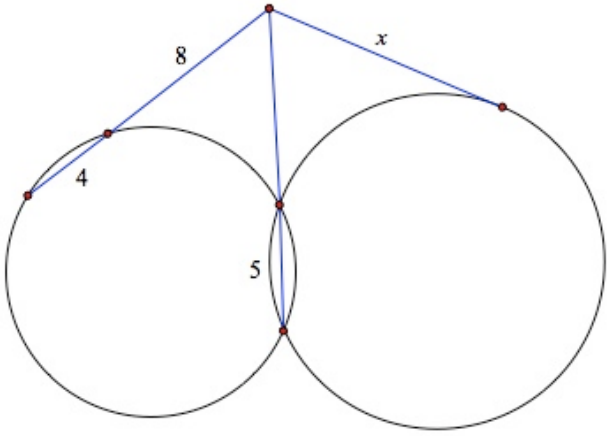
(p)



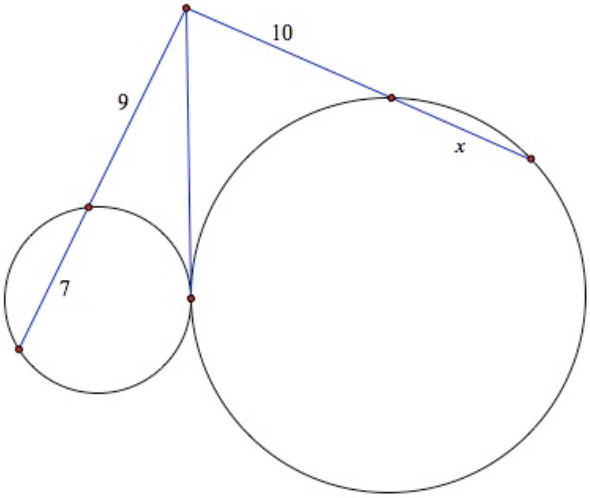
(q)



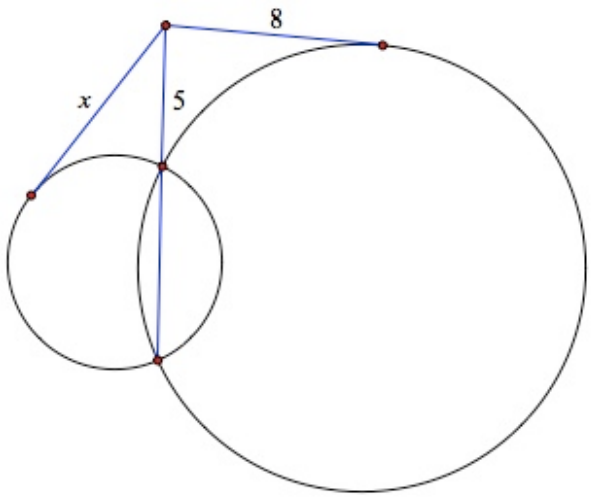
(r)



(s)



(t)



2. A circle goes through the points A, B, C, and D consecutively. The chords $\overline{AC}$ and $\overline{BD}$ intersect at P. Given that $AP = 12, BP = 16$, and $CP = 6$, find DP?
3. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is found to be 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is found to be 1 inch. Find the diameter of the plate.
4. Chords $\overline{AB}$ and $\overline{CD}$ intersect at P. Given $AP = 12, BP = 8$, and $CP = 7$, find DP.

Review Answers

1. (a) 14.4
   (b) 16
   (c) 4.5
   (d) 32.2
   (e) 12
   (f) 29.67
   (g) 4.4
   (h) 18.03
   (i) 4.54
   (j) 20.25
   (k) 7.48
   (l) 23.8
   (m) 24.4
   (n) 9.24 or 4.33
   (o) 17.14
   (p) 26.15
   (q) 7.04

(r) 9.8
(s) 4.4
(t) 8

2. 4.5
3. 10 inches.
4. 13.71

Chapter 10

Perimeter and Area

10.1 Triangles and Parallelograms

Learning Objectives

- Understand basic concepts of the meaning of area.
- Use formulas to find the area of specific types of polygons.

Introduction

Measurement is not a new topic. You have been measuring things nearly all your life. Sometimes you use standard units (pound, centimeter), sometimes nonstandard units (your pace or arm span). Space is measured according to its dimension.

- One-dimensional space: measure the length of a segment on a line.
- Two-dimensional space: measure the area that a figure takes up on a plane (flat surface).
- Three-dimensional space: measure the volume that a solid object takes up in "space."

In this lesson, we will focus on basic ideas about area in two-dimensional space. Once these basic ideas are established we'll look at the area formulas for some of the most familiar two-dimensional figures.

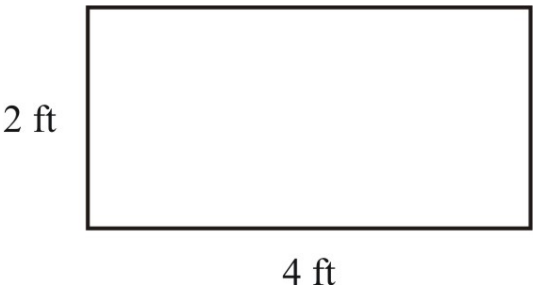
Basic Ideas of Area

Measuring area is just like measuring anything; before we can do it, we need to agree on standard units. People need to say, "These are the basic units of area." This is a matter of

history. Let's re-create some of the thinking that went into decisions about standard units of area.

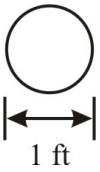
Example 1

What is the area of the rectangle below?

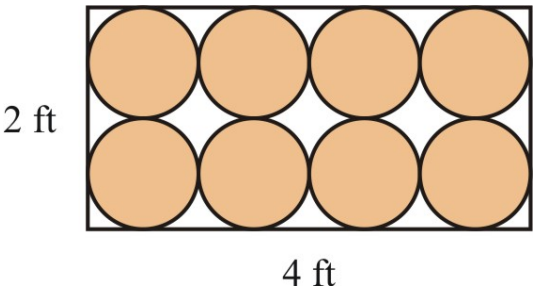


What should we use for a basic unit of area?

As one possibility, suppose we decided to use the space inside this circle as the unit of area.

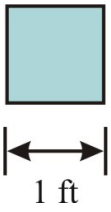


To find the area, you need to count how many of these circles fit into the rectangle, including parts of circles.

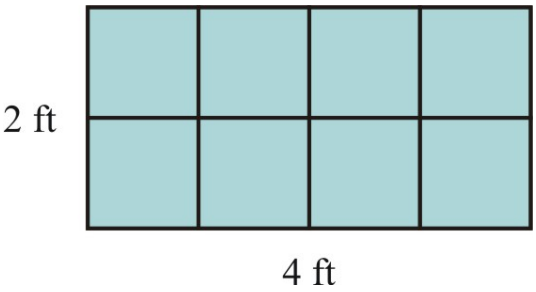


So far you can see that the rectangle's space is made up of 8 whole circles. Determining the fractional parts of circles that would cover the remaining white space inside the rectangle would be no easy job! And this is just for a very simple rectangle. The challenge is even more difficult for more complex shapes.

Instead of filling space with circles, people long ago realized that it is much simpler to use a square shape for a unit of area. Squares fit together nicely and fill space with no gaps. The square below measures 1 foot on each side, and it is called 1 square foot.



Now it's an easy job to find the area of our rectangle.



The area is 8 square feet, because 8 is the number of units of area (square feet) that will exactly fill, or cover, the rectangle.

The principle we used in Example 1 is more general.

The *area* of a two-dimensional figure is the number of square units that will fill, or cover, the figure.

Two Area Postulates

Congruent Areas

If two figures are congruent, they have the same area.

This is obvious because congruent figures have the same amount of space inside them. However, two figures with the same area are not necessarily congruent.

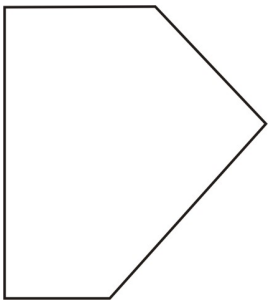
Area of Whole is Sum of Parts

If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

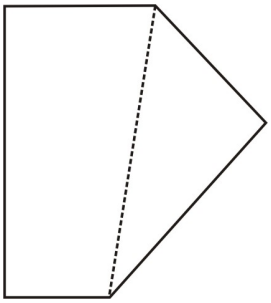
This is the familiar idea that a whole is the sum of its parts. In practical problems you may find it helpful to break a figure down into parts.

Example 2

Find the area of the figure below.

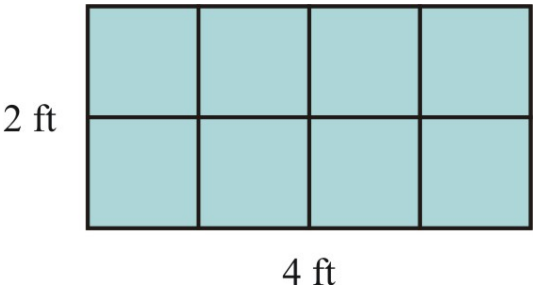


Luckily, you don't have to learn a special formula for an irregular pentagon, which this figure is. Instead, you can break the figure down into a trapezoid and a triangle, and use the area formulas for those figures.



Basic Area Formulas

Look back at Example 1 and the way it was filled with unit area squares.



Notice that the dimensions are:

base (or length) 4 feet

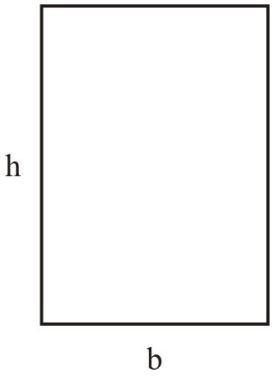
height (or width) 2 feet

But notice, too, that the base is the number of feet in one row of unit squares, and the height is the number of rows. A counting principle tells us that the total number of square feet is the number in one row multiplied by the number of rows.

$$\text{Area} = 8 = 4 \times 2 = \text{base} \times \text{height}$$

Area of a Rectangle

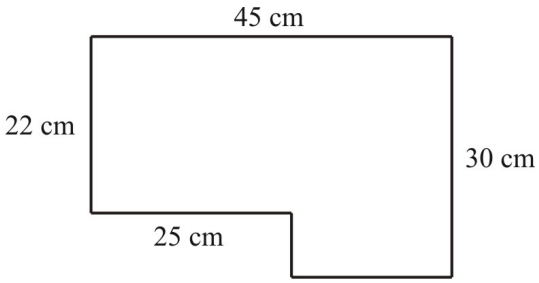
If a rectangle has base b units and height h units, then the area, A, is bh square units.



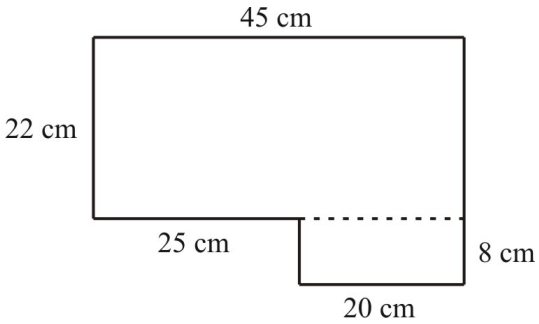
$$A = bh$$

Example 3

What is the area of the figure shown below?



Break the figure down into two rectangles.



$$\text{Area} = 22 \times 45 + 8 \times 20 = 990 + 160 = 1150 \text{ cm}^2$$

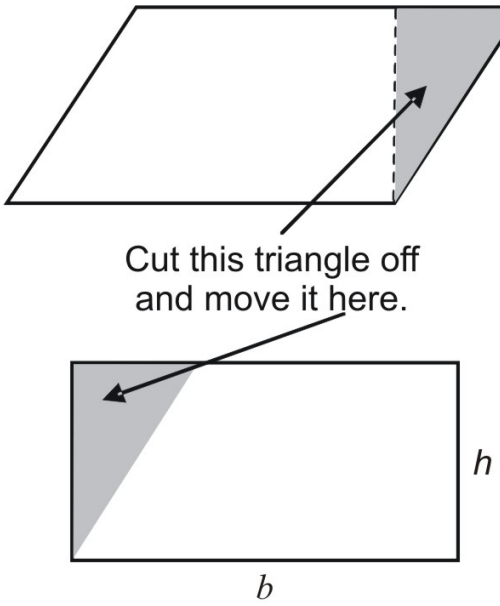
Now we can build on the rectangle formula to find areas of other shapes.

Parallelogram

Example 4

How could we find the area of this parallelogram?

Make it into a rectangle

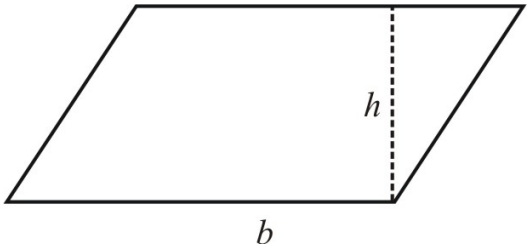


The rectangle is made of the same parts as the parallelogram, so their areas are the same. The area of the rectangle is bh, so the area of the parallelogram is also bh.

Warning: Notice that the height h of the parallelogram is the *perpendicular distance between two parallel sides of the parallelogram*, **not** a side of the parallelogram (unless the parallelogram is also a rectangle, of course).

Area of a Parallelogram

If a parallelogram has base b units and height h units, then the area, A, is bh square units.

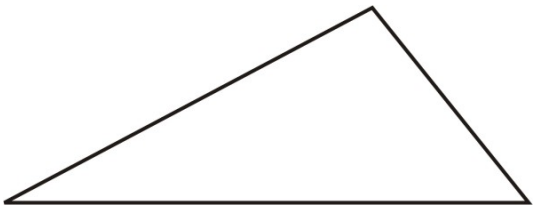


$$A = bh$$

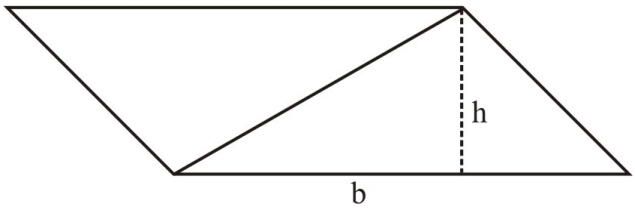
Triangle

Example 5

How could we find the area of this triangle?



Make it into a parallelogram. This can be done by making a copy of the original triangle and putting the copy together with the original.

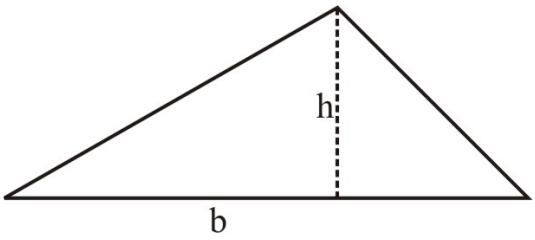


The area of the parallelogram is bh, so the area of the triangle is $\frac{bh}{2}$ or $\frac{1}{2}bh$.

Warning: Notice that the height h (also often called the *altitude*) of the triangle is the *perpendicular distance between a vertex and the opposite side of the triangle.*

Area of a Triangle

If a triangle has base b units and altitude h units, then the area, A, is $\frac{bh}{2}$ or $\frac{1}{2}\ bh$ square units.



$A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$

Lesson Summary

Once we understood the meaning of measures of space in two dimensions—in other words, area—we saw the advantage of using square units. With square units established, the formula for the area of a rectangle is simply a matter of common sense. From that point forward, the formula for the area of each new figure builds on the previous figure. For a parallelogram, convert it to a rectangle. For a triangle, double it to make a parallelogram.

Points to Consider

As we study other figures, we will frequently return to the basics of this lesson—the benefit of square units, and the fundamental formula for the area of a rectangle.

It might be interesting to note that the word geometry is derived from ancient Greek roots that mean *Earth* (geo-) *measure* (-metry). In ancient times geometry was very similar to today's surveying of land. You can see that land surveying became easily possible once knowledge of how to find the area of plane figures was developed.

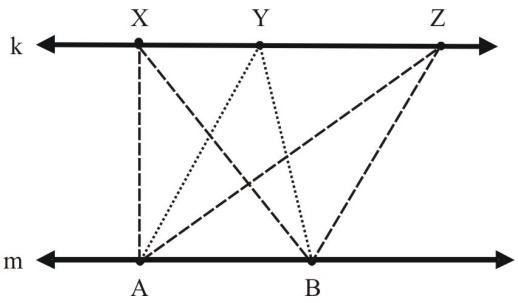
Review Questions

Complete the chart. Base and height are given in units; area is in square units.

Table 10.1:

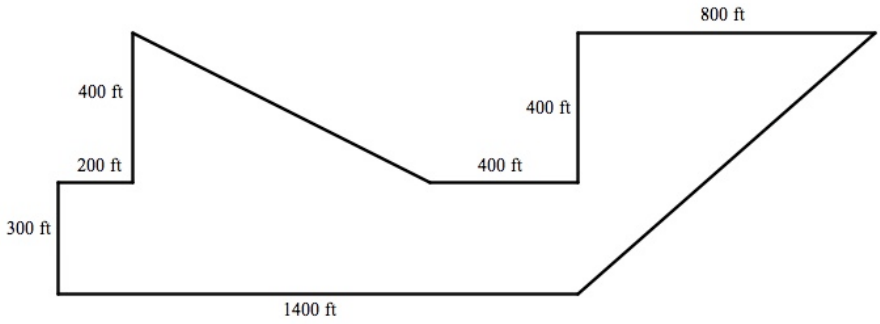
	Base	Height	'Area
1a.	5	8	?
1b.	10	?	40
1c.	1	1	?
1d.	7	?	49
1e.	225	$\frac{1}{3}$	?
1f.	100	?	1

1.
2. The carpet for a $12-$ foot by $20-$ foot room cost \$360. The same kind of carpet cost \$225 for a room with a square floor. What are the dimensions of the room?
3. Explain how an altitude of a triangle can be outside the triangle.
4. Line k and line m are parallel.



Explain how you know that $\triangle ABX, \triangle ABY$, and $\triangle ABZ$ all have the same area.

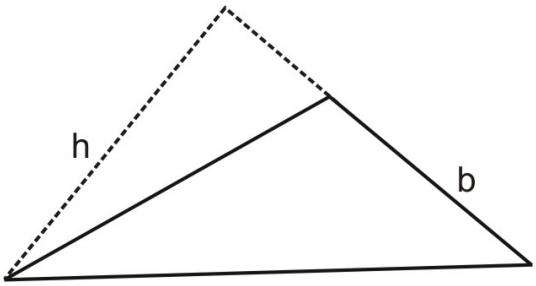
5. Lin bought a tract of land for a new apartment complex. The drawing below shows the measurements of the sides of the tract. Approximately how many acres of land did Lin buy? (1 acre $\approx 40,000$ square feet.)



6. A hexagon is drawn on a coordinate grid. The vertices of the hexagon are $A(1,4), B(3,7), C(8,7), D(6,$ and $F(1,-8)$. What is the area of $ABDCEF$?

Review Answers

1. 1a. 401b. 4
   1c. 1
   1d. 7
   1e. 75
   1f. 0.01
2. 15 feet by 15 feet
3. This happens in a triangle with an obtuse angle. Each altitude to a side of the obtuse angle is outside the triangle.



4. All of the triangles have the same base and altitude, so in each triangle $\frac{bh}{2}$ is the same as in each of the other triangles.
5. $160,000 + 420,000 + 280,000 = 860,000$ sq ft ≈ 21.5 acres
6. 65

10.2 Trapezoids, Rhombi, and Kites

Learning Objectives

- Understand the relationships between the areas of two categories of quadrilaterals: basic quadrilaterals (rectangles and parallelograms), and special quadrilaterals (trapezoids, rhombi, and kites).
- Derive area formulas for trapezoids, rhombi, and kites.
- Apply the area formulas for these special quadrilaterals.

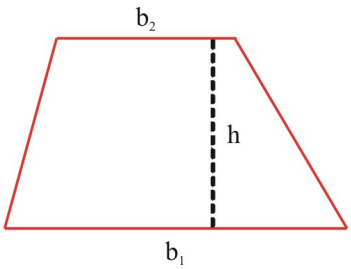
Introduction

We'll use the area formulas for basic shapes to work up to the formulas for special quadrilaterals. It's an easy job to convert a trapezoid to a parallelogram. It's also easy to take apart a rhombus or kite and rebuild it as a rectangle. Once we do this, we can derive new formulas from the old ones.

We'll also need to review basic facts about the trapezoid, rhombus, and kite.

Area of a Trapezoid

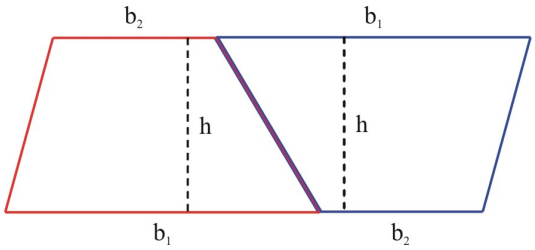
Recall that a trapezoid is a quadrilateral with one pair of parallel sides. The lengths of the parallel sides are the bases. The perpendicular distance between the parallel sides is the height, or altitude, of the trapezoid.



Trapezoid with bases b_1 and b_2 and altitude h

To find the area of the trapezoid, turn the problem into one about a parallelogram. Why? Because you already know how to compute the area of a parallelogram.

- Make a copy of the trapezoid.
- Rotate the copy 180°.
- Put the two trapezoids together to form a parallelogram.



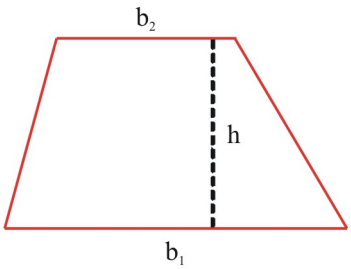
Two things to notice:

1. The parallelogram has a base that is equal to $b_1 + b_2$.
2. The altitude of the parallelogram is the same as the altitude of the trapezoid.

Now to find the area of the trapezoid:

- The area of the parallelogram is base $\times$ altitude $= (b_1 + b_2) \times h$.
- The parallelogram is made up of two congruent trapezoids, so the area of each trapezoid is one-half the area of the parallelogram.
- The area of the trapezoid is one-half of $(b_1 + b_2) \times h$.

Area of Trapezoid with Bases b_1 and b_2 and Altitude h



Trapezoid with bases b_1 and b_2 and altitude h

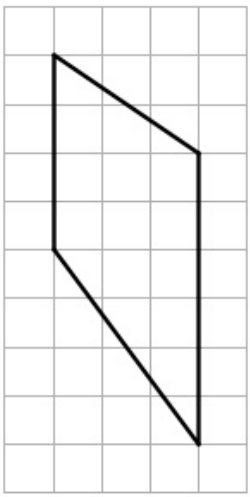
Trapezoid with bases b_1 and b_2 and altitude h

$A = \frac{1}{2}(b_1 + b_2)h$ or $A = \frac{(b_1+b_2)h}{2}$

Notice that the formula for the area of a trapezoid could also be written as the "Average of the bases time the height." This may be a convenient shortcut for memorizing this formula.

Example 1

What is the area of the trapezoid below?



The bases of the trapezoid are 4 and 6. The altitude is 3.

$$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(4 + 6) \times 3 = 15$$

Area of a Rhombus or Kite

First let's start with a review of some of the properties of rhombi and kites.

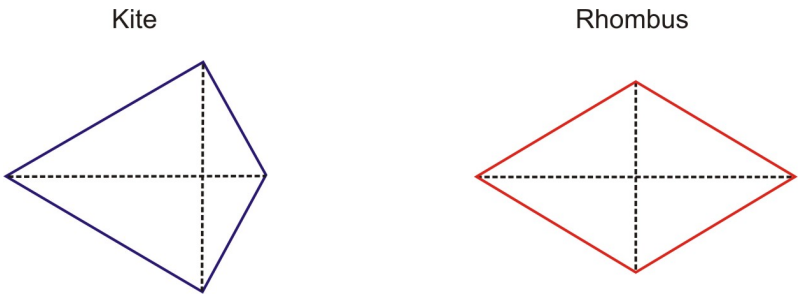
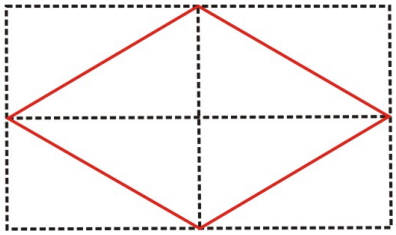


Table 10.2:

	Kite	**Rhombus**
Congruent sides	2 Pairs	All 4
Opposite angles congruent	1 Pair yes. 1 Pair maybe	Both pairs yes
Perpendicular diagonals	Yes	Yes
Diagonals bisected	1 Yes. 1 maybe	Both yes

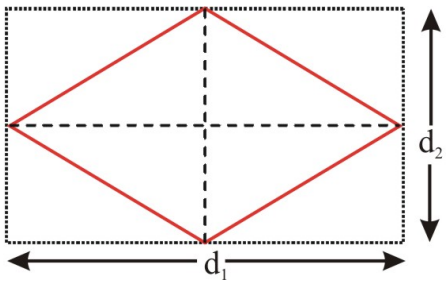
Now you're ready to develop area formulas. We'll follow the command: "Frame it in a rectangle." Here's how you can frame a rhombus in a rectangle.



Notice that:

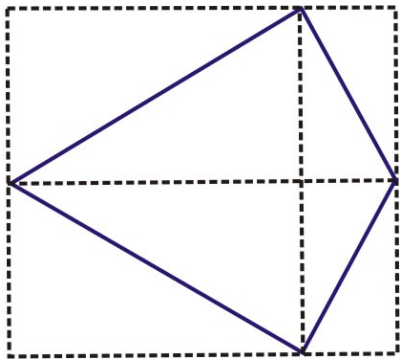
- The *base* and *height* of the **rectangle** are the same as the lengths of the two *diagonals* of the **rhombus**.
- The rectangle is divided into 8 congruent triangles; 4 of the triangles fill the rhombus, so the area of the rhombus is one-half the area of the rectangle.

Area of a Rhombus with Diagonals d_1 and d_2



$$A = \frac{1}{2}d_1 d_2 = \frac{d_1 d_2}{2}$$

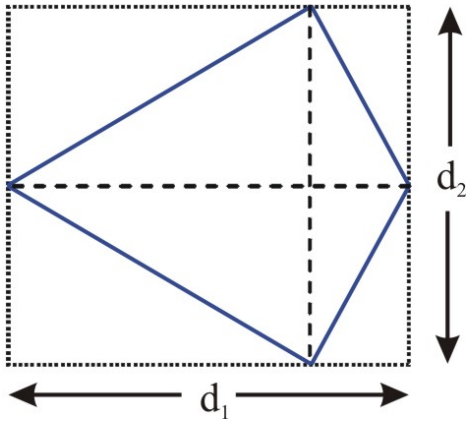
We can go right ahead with the kite. We'll follow the same command again: "Frame it in a rectangle." Here's how you can frame a kite in a rectangle.



Notice that:

- The *base* and *height* of the **rectangle** are the same as the lengths of the two *diagonals* of the **kite**.
- The rectangle is divided into 8 triangles; 4 of the triangles fill the kite. For every triangle inside the kite, there is a congruent triangle outside the kite so the area of the kite is one-half the area of the rectangle.

Area of a Kite with Diagonals d_1 and d_2



$$A = \frac{1}{2}d_1 d_2 = \frac{d_1 d_2}{2}$$

Lesson Summary

We see the principle of "no need to reinvent the wheel" in developing the area formulas in this section. If we wanted to find the area of a trapezoid, we saw how the formula for a parallelogram gave us what we needed. In the same way, the formula for a rectangle was easy to modify to give us a formula for rhombi and kites. One of the striking results is that the same formula works for both rhombi and kites.

Points to Consider

You'll use area concepts and formulas later in this course, as well as in real life.

- Surface area of solid figures: the amount of outside surface.
- Geometric probability: chances of throwing a dart and landing in a given part of a figure.
- Carpet for floors, paint for walls, fertilizer for a lawn, and more: areas needed.

Tech Note - Geometry Software

You saw earlier that the area of a rhombus or kite depends on the lengths of the diagonals.

$$A = \frac{1}{2}d_1d_2 = \frac{d_1d_2}{2}$$

This means that all rhombi and kites with the same diagonal lengths have the same area.

Try using geometry software to experiment as follows.

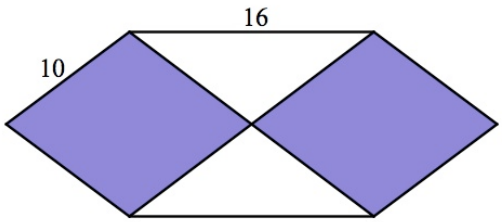
- Construct two perpendicular segments.
- Adjust the segments so that one or both of the segments are bisected.
- Draw a quadrilateral that the segments are the diagonals of. In other words, draw a quadrilateral for which the endpoints of the segments are the vertices.
- Repeat with the same perpendicular, bisected segments, but making a different rhombus or kite. Repeat for several different rhombi and kites.
- Regardless of the specific shape of the rhombus or kite, the areas are all the same.

The same activity can be done on a geoboard. Place two perpendicular rubber bands so that one or both are bisected. Then place another rubber band to form a quadrilateral with its vertices at the endpoints of the two segments. A number of different rhombi and kites can be made with the same fixed diagonals, and therefore the same area.

Review Questions

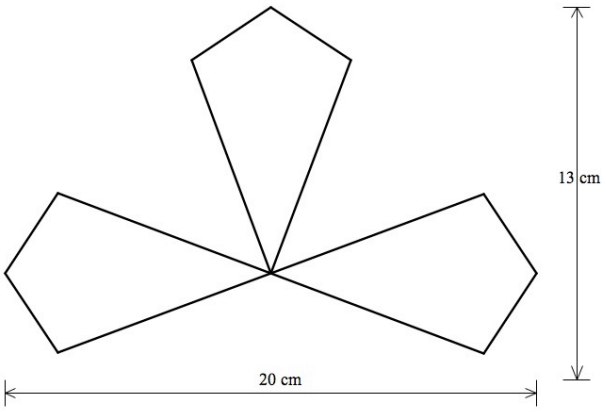
Quadrilateral $ABCD$ has vertices $A(-2, 0), B(0, 2), C(4, 2)$, and $D(0, -2)$ in a coordinate plane.

1. Show that $ABCD$ is a trapezoid.
2. What is the area of $ABCD$?
3. Prove that the area of a trapezoid is equal to the area of a rectangle with height the same as the height of the trapezoid and base equal to the length of the median of the trapezoid.
4. Show that the trapezoid formula can be used to find the area of a parallelogram.
5. Sasha drew this plan for a wood inlay he is making.



10 is the length of the slanted side. 16 is the length of the horizontal line segment. Each shaded section is a rhombus.
The shaded sections are rhombi. Based on the drawing, what is the total area of the shaded sections?

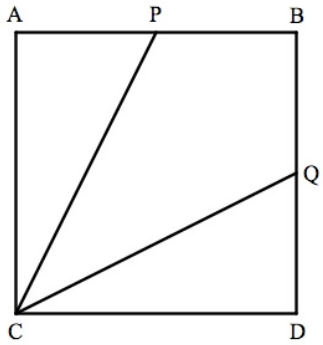
6. Plot 4 points on a coordinate plane.
   - The points are the vertices of a rhombus.
   - The area of the rhombus is 24 square units.
7. Tyra designed the logo for a new company. She used three congruent kites.



What is the area of the entire logo?

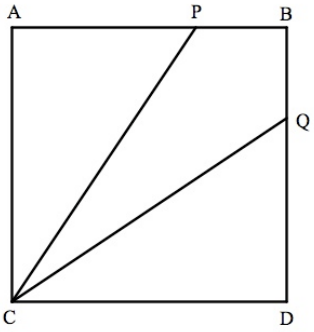
8. In the figure below:

- $ABCD$ is a square
- $AP = PB = BQ$
- $DC = 20$ feet



What is the area of $PBQC$?
In the figure below:

- $ABCD$ is a square
- $AP = 20$ feet
- $PB = BQ = 10$ feet



9. What is the area of $PBQC$?
10. The area of $PBQD$ is what fractional part of the area of $ABCD$?

Review Answers

1. Slope of $\overline{AB} = 1$, slope of $\overline{DC} = 1$

$\overline{AB} \parallel \overline{DC}$ are parallel.

2. $AB = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$
   $DC = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$
   $AD = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

slope of $\overline{AD} = -1$

$\overline{AB}$ and $\overline{DC}$ are the bases, $\overline{AD}$ is an altitude.

$$A = \frac{(b_1 + b_2)h}{2} = \frac{(2\sqrt{2} + 4\sqrt{2})2\sqrt{2}}{2} = \frac{6\sqrt{2}(2\sqrt{2})}{2} = 12$$

3. .
4. For a parallelogram, $b_1 = b_2 = b$ (the "bases" are two of the parallel sides), so by the trapezoid formula the area is
$$\frac{(b_1 + b_2)h}{2} = \frac{(b + b)h}{2} = \frac{2bh}{2} = bh.$$
5. Length of long diagonal of one rhombus is 16. Length of other diagonal is 12 (each rhombus is made of $46 - 8 - 10$ right triangles).

Total area is $2\left[\frac{d_1 d_2}{2}\right] = d_1 d_2 = 16 \times 12 = 192$.

6. Many rhombi work, as long as the product of the lengths of the diagonals is 48.
7. 90 cm^2
8. 200 square feet
9. 300 square feet
10. $\frac{1}{3}$

10.3 Areas of Similar Polygons

Learning Objectives

- Understand the relationship between the scale factor of similar polygons and their areas.
- Apply scale factors to solve problems about areas of similar polygons.
- Use scale models or scale drawings.

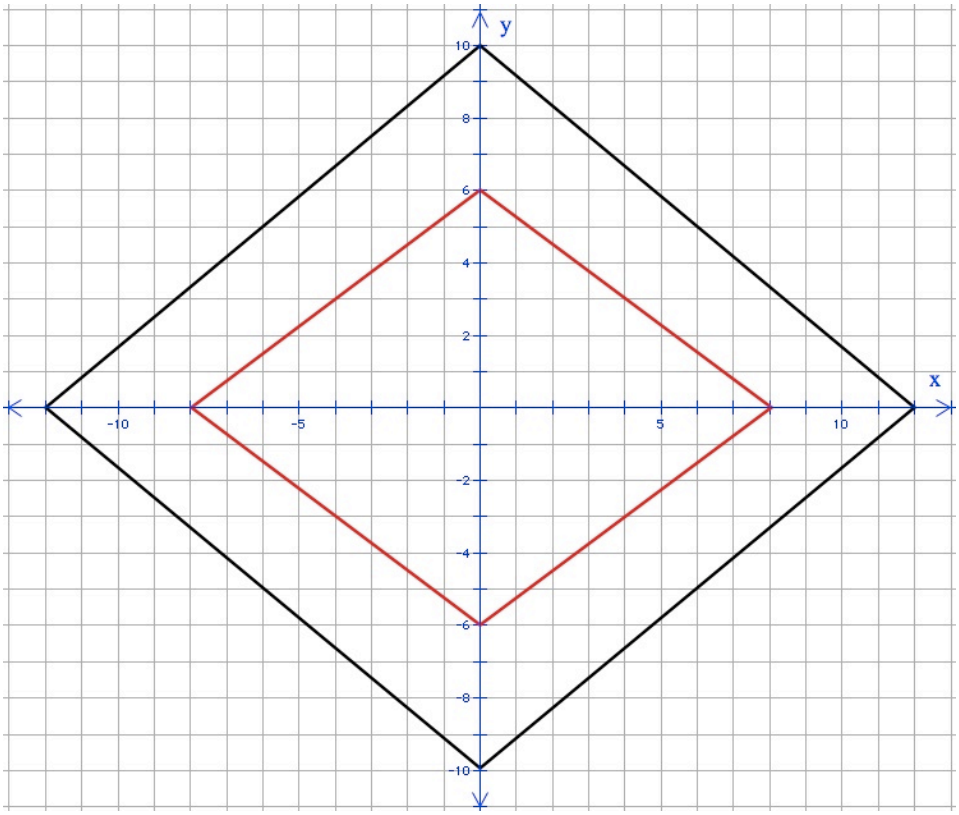
Introduction

We'll begin with a quick review of some important features of similar polygons. You remember that we studied similar figures rather extensively in Chapter 7. There you learned about scale factors and perimeters of similar polygons. In this section we'll take similar figures one step farther. We'll see that the areas of similar figures have a very specific relationship to the scale factor—but it's just a bit tricky! We wrap up the section with some thoughts on why living things are the "right" size, and what geometry has to do with that!

Review - Scale Factors and Perimeter

Example 1

The diagram below shows two rhombi.



a. *Are the rhombi similar? How do you know?*

Yes.

- The sides are parallel, so the corresponding angles are congruent.
- Using the Pythagorean Theorem, we can see that each side of the smaller rhombus has a length of 10, and each side of the larger rhombus has a length of 15.
- So the lengths of the sides are proportional.
- Polygons with congruent corresponding angles and proportional sides are similar.

b. *What is the scale factor relating the rhombi?*

The scale factor relating the smaller rhombus to the larger one is $\frac{15}{10} = \frac{3}{2} = 1.5$.

c. *What is the perimeter of each rhombus?*

Answer

- Perimeter of smaller rhombus $= 4 \times 10 = 40$
- Perimeter of larger rhombus $= 4 \times 15 = 60$

d. *What is the ratio of the perimeters?*

$$\frac{60}{40} = \frac{3}{2} = 1.5$$

e. *What is the area of each rhombus?*

$$\text{Area of smaller rhombus} = \frac{d_1 d_2}{2} = \frac{12 \times 16}{2} = 96$$
$$\text{Area of larger rhombus} = \frac{d_1 d_2}{2} = \frac{18 \times 24}{2} = 216$$

What do you notice in this example? The perimeters have the same ratio as the scale factor.

But what about the areas? The ratio of the areas is certainly *not* the same as the scale factor. If it were, the area of the larger rhombus would be $96 \times 1.5 = 144$, but the area of the larger rhombus is actually 216.

What *IS* the ratio of the areas?

The ratio of the areas is $\frac{216}{96} = \frac{9}{6} = 2.25$. Notice that $\frac{9}{4} = \left(\frac{3}{2}\right)^2$ or in decimal, $2.25 = (1.5)^2$.

So at least in this case we see that the ratio of the areas is the **square** of the scale factor.

Scale Factors and Areas

What happened in Example 1 is no accident. In fact, this is the basic relationship for the areas of similar polygons.

Areas of Similar Polygons

If the scale factor relating the sides of two similar polygons is k, then the area of the larger polygon is k^2 times the area of the smaller polygon. In symbols let the area of the smaller polygon be A_1 and the area of the larger polygon be A_2. Then:

$$A_2 = k^2 A_1$$

Think about the area of a polygon. Imagine that you look at a square with an area of exactly 1 square unit. Of course, the sides of the square are 1 unit of length long. Now think about

another polygon that is similar to the first one with a scale factor of k. Every $1-$by-1 square in the first polygon has a matching $k-$by$-k$ square in the second polygon, and the area of each of these $k-$by$-k$ squares is k^2. Extending this reasoning, every 1 square unit of area in the first polygon has a corresponding k^2 units of area in the second polygon. So the total area of the second polygon is k^2 times the area of the first polygon.

Warning: In solving problems it's easy to forget that you do not always use just the scale factor. Use the scale factor in problems about *lengths*. But use the **square of the scale factor** in problems about *area*!

Example 2

Wu and Tomi are painting murals on rectangular walls. The length and width of Tomi's wall are 3 *times the length and width of Wu's wall.*

a. *The total length of the border of Tomi's wall is* 120 feet. *What is the total length of the border of Wu's wall?*

This is a question about lengths, so you use the scale factor itself. All the sides of Tomi's wall are 3 times the length of the corresponding side of Wu's wall, so the perimeter of Tomi's wall is also 3 times the perimeter of Wu's wall.

The total length of the border (perimeter) of Wu's wall is $\frac{120}{3} = 40$ feet.

b. *Wu can cover his wall with* 6 *quarts of paint. How many quarts of paint will Tomi need to cover her wall?*

This question is about area, since the area determines the amount of paint needed to cover the walls. The ratio of the amounts of paint is the same as the ratio of the areas (which is the square of the scale factor). Let x be the amount of paint that Tomi needs.

$$\frac{x}{6} = k^2 = 3^2 = 9$$
$$x = 6 \times 9 = 54$$

Tomi would need 54 quarts of paint.

Summary of Length and Area Relationships for Similar Polygons

If two similar polygons are related by a scale factor of k, then:

- Length: The lengths of any corresponding parts have the same ratio, k. Note that this applies to sides, *Area: The ratio of the areas is k^2. Note that this applies to areas, and any aspect of an object that

Note: You might be able to make a pretty good guess about the volumes of similar solid $(3-D)$ figures. You'll see more about that in Chapter 11.

Scale Drawings and Scale Models

One important application of similar figures is the use of scale drawings and scale models. These are two-dimensional (scale drawings) or three-dimensional (scale models) representations of real objects. The drawing or model is *similar* to the actual object.

Scale drawings and models are widely used in design, construction, manufacturing, and many other fields. Sometimes a scale is shown, such as "1 inch = 5 miles" on a map. Other times the scale may be calculated, if necessary, from information about the object being modeled.

Example 3

Jake has a map for a bike tour. The scale is 1 inch = 5 miles. *He estimated that two scenic places on the tour were about* $3\frac{1}{2}$ inches *apart on the map. How far apart are these places in reality?*

Each inch on the map represents a distance of 5 miles. The places are about $3\frac{1}{2} \times 5 =$ 17.5 miles apart.

Example 4

Cristy's design team built a model of a spacecraft to be built. Their model has a scale of $1 : 24$. *The actual spacecraft will be* 180 feet *long. How long should the model be?*

Let x be the length of the model.

$$\begin{aligned} \frac{1}{24} &= \frac{x}{180} \\ 24x &= 180 \\ x &= 7.5 \end{aligned}$$

The model should be 7.5 feet long.

Example 5

Tasha is making models of several buildings for her senior project. The models are all made with the same scale. She has started the chart below.

a. *What is the scale of the models?*

$$1250 \div 20 = 62.5$$

The scale is 1 inch = 62.5 feet.

b. *Complete the chart below.*

Table 10.3:

Building	Actual height (feet)	Model height (inches)
Sears Tower (Chicago)	?	23.2
Empire State Building (New York City)	1250	20
Columbia Center (Seattle)	930	?

Sears Tower: $23.2 \times 62.5 = 1450$. It is 1450 feet high.

Columbia Center: Let $x =$ the model height.

$$\begin{aligned} \frac{1250}{20} &= \frac{930}{x} \\ 1250x &= 20 \times 930 \\ x &= \frac{20 \times 930}{1250} \approx 14.9 \end{aligned}$$

The model should be about 14.9 inches high.

Why There Are No 12-Foot-Tall Giants

Why are there no $12-$ foot$-$tall giants? One explanation for this is a matter of similar figures.

Let's suppose that there is a $12-$ foot$-$tall human. Compare this giant (?) to a $6-$ foot$-$tall person. Now let's apply some facts about similar figures.

The scale factor relating these two hypothetical people is $\frac{12}{6} = 2$. Here are some consequences of this scale factor.

- All **linear** dimensions of the giant would be 2 times the corresponding dimensions of the real person. This includes height, bone length, etc.
- All **area** measures of the giant would be $2^2 = 4$ times the corresponding area measures of the real person. This includes respiration (breathing) and metabolism (converting nutrients to usable materials and energy) rates, because these processes take place

along *surfaces* in the lungs, intestines, etc. This also includes the strength of bones, which depends on the cross-section area of the bone.
- All **volume** measures of the giant would be $2^3 = 8$ times the corresponding volume measures of the real person. (You'll learn why in Chapter 11.) The volume of an organism generally determines its weight and mass.

What kinds of problems do we see for our giant? Here are two severe ones.

1. The giant would have bones that are 4 times as strong, but those bones have to carry a body weight that is 8 times as much. The bones would not be up to the task. In fact it appears that the giant's own weight would be able to break its bones.
2. The giant would have 8 times the weight, number of cells, etc. of the real person, but only 4 times as much ability to supply the oxygen, nutrition, and energy needed.

Conclusion: There are no $12-$ foot$-$giants, and some of the reasons are nothing more, or less, than the geometry of similar figures.

For further reading: *On Being the Right Size*, by J. B. S. Haldane, also available at http://irl.cs.ucla.edu/papers/right-size.html.

Lesson Summary

In his lesson we focused on one main point: The areas of similar polygons have a ratio that is the **square** of the scale factor. We also used ideas about similar figures to analyze scale drawings and scale models, which are actually similar representations of actual objects.

Points to Consider

You have now learned quite a bit about the lengths of sides and areas of polygons. Next we'll build on knowledge about polygons to come to a conclusion about the "perimeter" of the "ultimate polygon," which is the circle.

Suppose we constructed regular polygons that are all inscribed in the same circle.

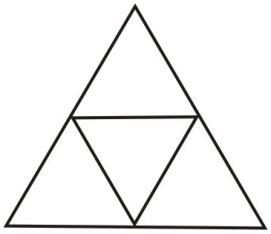
- Think about polygons that have more and more sides.
- How would the perimeter of the polygons change as the number of sides increases?

The answers to these questions will lead us to an understanding of the formula for the circumference (perimeter) of a circle.

Review Questions

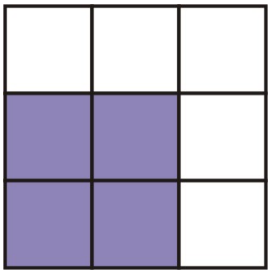
The figure below is made from small congruent equilateral triangles.

4 congruent small triangles fit together to make a bigger, similar triangle.

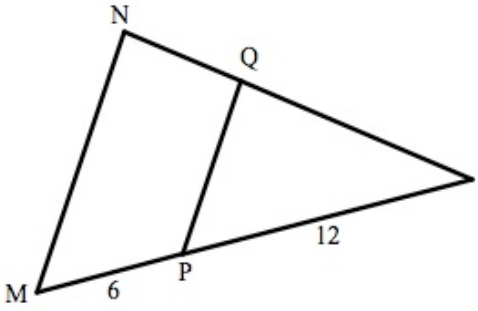


1. What is the scale factor of the large and small triangles?
2. If the area of the large triangle is 20 square units, what is the area of a small triangle?

The smallest squares in the diagram below are congruent.



3. What is the scale factor of the shaded square and the largest square?
4. If the area of the shaded square is 50 square units, what is the area of he largest square?
5. Frank drew two equilateral triangles. Each side of one triangle is 2.5 times as long as a side of the other triangle. The perimeter of the smaller triangle is 40 cm. What is the perimeter of the larger triangle? In the diagram below, $.\overline{MN} : \overline{PQ}$.



6. What is the scale factor of the small triangle and the large triangle?
7. If the perimeter of the large triangle is 42, what is the perimeter of the small triangle?
8. If the area of the small triangle is A, write an expression for the area of the large triangle.

9. If the area of the small triangle is K, write an expression for the area of the trapezoid.
10. The area of one square on a game board is exactly twice the area of another square. Each side of the larger square is 50 mm long. How long is each side of the smaller square?
11. The distance from Charleston to Morgantown is 160 miles. The distance from Fairmont to Elkins is 75 miles. Charleston and Morgantown are 5 inches apart on a map. How far apart are Fairmont and Elkins on the same map?

Marlee is making models of historic locomotives (train engines). She uses the same scale for all of her models.

- The $S1$ locomotive was 140 feet long. The model is 8.75 inches long.
- The 520 Class locomotive was 87 feet long.

12. What is the scale of Marlee's models?
13. How long is the model of the 520 Class locomotive?

Review Answers

1. 2
2. 5
3. $\frac{4}{9}$ or $4:9$
4. 112.5
5. 100 cm
6. $\frac{2}{3}$
7. 28
8. $\frac{9}{4}A$ or $\frac{9A}{4}$
9. $\frac{5}{4}K$ or $\frac{5K}{4}$
10. 35.4 mm
11. 2.3 inches
12. 1 inch = 16 feet or equivalent
13. 5.4 inches

10.4 Circumference and Arc Length

Learning Objectives

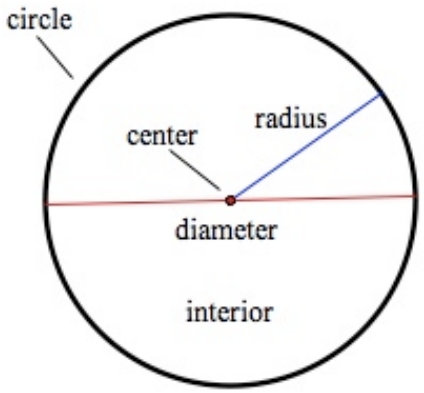
- Understand the basic idea of a limit.
- Calculate the circumference of a circle.
- Calculate the length of an arc of a circle.

Introduction

In this lesson, we extend our knowledge of perimeter to the perimeter—or circumference—of a circle. We'll use the idea of a limit to derive a well-known formula for the circumference. We'll also use common sense to calculate the length of part of a circle, known as an arc.

The Parts of a Circle

A **circle** is the set of all points in a plane that are a given distance from another point called the center. Flat round things, like a bicycle tire, a plate, or a coin, remind us of a circle.



The diagram reviews the names for the "parts" of a circle.

- The center
- The circle: the points that are a given distance from the center (which does not include the center or interior)
- The interior: all the points (including the center) that are inside the circle
- circumference: the distance around a circle (exactly the same as perimeter)
- radius: any segment from the center to a point on the circle (sometimes "radius" is used to mean the length of the segment and it is usually written as r)
- diameter: any segment from a point on the circle, through the center, to another point on the circle (sometimes "diameter" is used to mean the length of the segment and it is usually written as d)

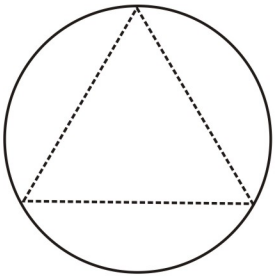
If you like formulas, you can already write one for a circle:

$d = 2r$ or $(r = \frac{d}{2})$

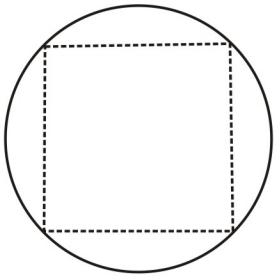
Circumference Formula

The formula for the circumference of a circle is a classic. It has been known, in rough form, for thousands of years. Let's look at one way to derive this formula.

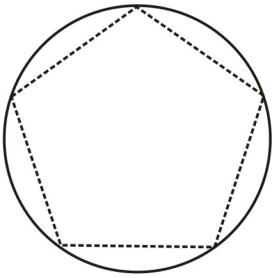
Start with a circle with a diameter of 1 unit. Inscribe a regular polygon in the circle. We'll inscribe regular polygons with more and more sides and see what happens. For each inscribed regular polygon, the perimeter will be given (how to figure that is in a review question).



3 sides
perimeter =2.598



4 sides
perimeter =2.828



5 sides
perimeter =2.939

What do you notice?

1. The more sides there are, the closer the polygon is to the circle itself.
2. The perimeter of the inscribed polygon increases as the number of sides increases.
3. The more sides there are, the closer the perimeter of the polygon is to the circumference of the circle.

Now imagine that we continued inscribing polygons with more and more sides. It would become nearly impossible to tell the polygon from the circle. The table below shows the results if we did this.

Regular Polygons Inscribed in a Circle with Diameter 1

Table 10.4:

Number of sides of polygon	Perimeter of polygon
3	2.598
4	2.828
5	2.939
6	3.000
8	3.062
10	3.090
20	3.129
50	3.140

Table 10.4: (continued)

Number of sides of polygon	Perimeter of polygon
100	3.141
500	3.141

As the number of sides of the inscribed regular polygon increases, the perimeter seems to approach a "limit." This limit, which is the **circumference** of the circle, is approximately 3.141. This is the famous and well-known number π. π is an endlessly non-repeating decimal number. We often use $\pi \approx 3.14$ as a value for π in calculations, but this is only an approximation.

Conclusion: The circumference of a circle with diameter 1 is π .

For Further Reading

Mathematicians have calculated the value of π to thousands, and even millions, of decimal places. You might enjoy finding some of these megadecimal numbers. Of course, all are approximately equal to 3.14.

- The article at the following URL shows more than a million digits of the decimal for π. http://wiki.answers.com/Q/What_is_the_exact_value_for_Pi_at_this_moment

Tech Note - Geometry Software

You can use geometry software to continue making more regular polygons inscribed in a circle with diameter 1 and finding their perimeters.

Can we extend this idea to other circles? First, recall that *all circles are similar to each other.* (This is also true for all equilateral triangles, all squares, all regular pentagons, etc.)

Suppose a circle has a diameter of d units.

- The scale factor of this circle and the one in the diagram and table above, with diameter 1, is $d : 1,$, $\frac{d}{1}$, or just d.
- You know how a scale factor affects linear measures, which include perimeter and circumference. If the scale factor is d, then the perimeter is d times as much.

This means that if the circumference of a circle with diameter 1 is π , then the circumference of a circle with diameter d is πd .

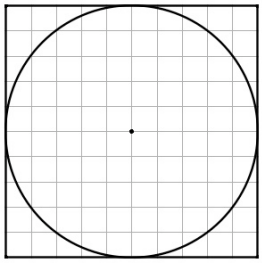
Circumference Formula

Let d be the diameter of a circle, and C the circumference.

$$C = \pi d$$

Example 1

A circle is inscribed in a square. Each side of the square is 10 cm *long. What is the circumference of the circle?*



Use $C = \pi d$. The length of a side of the square is also the diameter of the circle. $C = \pi d = 10\pi \approx 31.4$ cm

Note that sometimes an approximation is given using $\pi \approx 3.14$. In this example the circumference is 31.4 cm using that approximation. An exact is given in terms of π (leaving the symbol for π in the answer rather than multiplying it out. In this example the exact circumference is 10π cm.

Arc Length

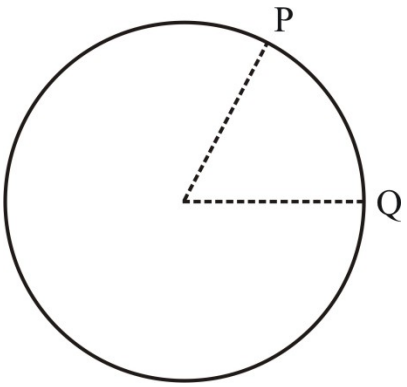
Arcs are measured in two different ways.

- **Degree measure:** The degree measure of an arc is the fractional part of a 360° complete circle that the arc is.
- **Linear measure:** This is the length, in units such as centimeters and feet, if you traveled from one end of the arc to the other end.

Example 2

Find the length of $\widehat{PQ}$.

$m\widehat{PQ} = 60°$. The radius of the circle is 9 inches.



Remember, 60° is the measure of the central angle associated with $m\widehat{PQ}$.

$m\widehat{PQ}$ is $\frac{60}{360}$ of a circle. The circumference of the circle is

$\pi d = 2\pi r = 2\pi(9) = 18\pi$ inches . The *arc length* of PQ : is $\frac{60}{360} \times 18\pi = \frac{1}{6} \times 18\pi = 3\pi \approx$ 9.42 inches.

In this lesson we study the second type of arc measure—the measure of an arc's length. Arc length is directly related to the degree measure of an arc.

Suppose a circle has:

- circumference C
- diameter d
- radius r

Also, suppose an arc of the circle has degree measure m.

Note that $\frac{m}{360}$ is the fractional part of the circle that the arc represents.

Arc length

$$\text{Arc Length} = \frac{m}{360} \times c = \frac{m}{360} \times \pi d = \frac{m}{360} \times 2\pi r$$

Lesson Summary

This lesson can be summarized with a list of the formulas developed.

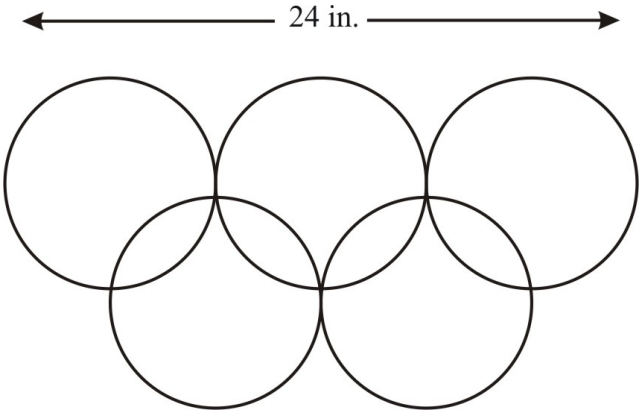
- Radius and diameter: $d = 2r$
- Circumference of a circle: $C = \pi d$
- Arc length $= \frac{m}{360} \times c = \frac{m}{360} \times \pi d = \frac{m}{360} \times 2\pi r$

Points to Consider

After perimeter and circumference, the next logical measure to study is area. In this lesson, we learned about the perimeter of a circle (circumference) and the arc length of a sector. In the next lesson we'll learn about the areas of circles and sectors.

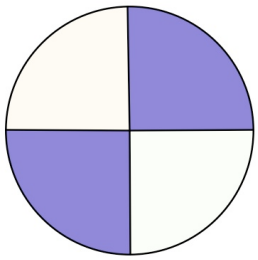
Review Questions

1. Prove: The circumference of a circle with radius r is $2\pi r$.
2. The Olympics symbol is five congruent circles arranged as shown below. Assume the top three circles are tangent to each other.



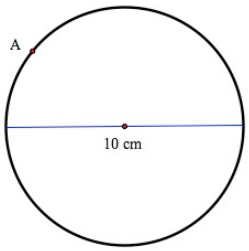
Brad is tracing the entire symbol for a poster. How far will his pen point travel?

3. A truck has tires that measure 14 inches from the center of the wheel to the outer edge of the tire.
   (a) How far forward does the truck travel every time a tire turns exactly once?
   (b) How many times will the tire turn when the truck travels 1 mile? (1 mile = 5280 feet).
4. The following wire sculpture was made from two perpendicular 50 cm segments that intersect each other at the center of a circle.



(a) If the radius of the circle is 25 cm, how much wire was used to outline the shaded sections?

5. The circumference of a circle is 300 feet. What is the radius of the circle?
6. A gear with a radius of 3 inches inches turns at a rate of 2000 RPM (revolutions per minute). How far does a point on the edge of the pulley travel in one second?
7. A center pivot irrigation system has a boom that is 400 m long. The boom is anchored at the center pivot. It revolves around the center pivot point once every three days. How far does the tip of the boom travel in one day?
8. The radius of Earth at the Equator is about $4,000$ miles. Belem (in Brazil) and the Galapagos Islands (in the Pacific Ocean) are on (or very near) the Equator. The approximate longitudes are Belem, $50°W$, and Galapagos Islands, $90°W$.
   (a) What is the *degree measure* of the **major** arc on the Equator from Belem to the Galapagos Islands?
   (b) What is the distance from Belem to the Galapagos Islands on the Equator the "long way around?"
9. A regular polygon inscribed in a circle with diameter 1 has n sides. Write a formula that expresses the perimeter, p, of the polygon in terms of n. (Hint: Use trigonometry.)
10. The pulley shown below revolves at a rate of 800 RPM.



   (a) How far does point A travel in one hour?

Review Answers

1. $C = \pi d, d = 2r, C = \pi(2r) = 2\pi r$
2. $40\pi \approx 125.6$ inches
3. (a) $28\pi \approx 87.92$ inches
   (b) Approximately 721 times
4. $100 + 25\pi \approx 178.5$ cm
5. Approximately 47.8 feet
6. Approximately 628 inches
7. Approximately 837 m
8. (a) $320°$
   (b) Approximately $22,329$ miles
9. $p = n \sin\left(\frac{180}{n}\right)$ or equivalent

10. $480,000\pi \approx 1,507,200$ cm

10.5 Circles and Sectors

Learning Objectives

- Calculate the area of a circle.
- Calculate the area of a sector.
- Expand understanding of the limit concept.

Introduction

In this lesson we complete our area toolbox with formulas for the areas of circles and sectors. We'll start with areas of regular polygons, and work our way to the limit, which is the area of a circle. This may sound familiar; it's exactly the same approach we used to develop the formula for the *circumference* of a circle.

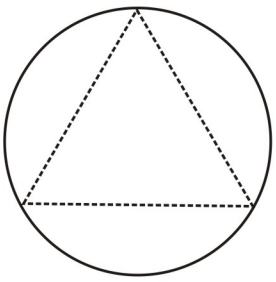
Area of a Circle

The big idea:

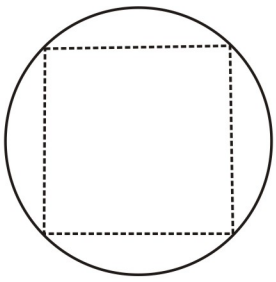
- Find the areas of regular polygons with radius 1.
- Let the polygons have more and more sides.
- See if a limit shows up in the data.
- Use similarity to generalize the results.

The details:

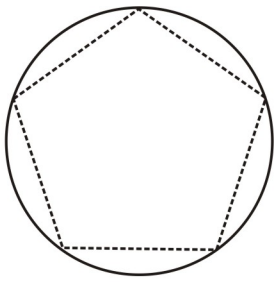
Begin with polygons having $3, 4$, and 5 sides, inscribed in a circle with a *radius* of 1.



3 sides
area = 1.2990



4 sides
area = 2.0000



5 sides
area = 2.3776

Now imagine that we continued inscribing polygons with more and more sides. It would become nearly impossible to tell the polygon from the circle. The table below shows the results if we did this.

Regular Polygons Inscribed in a Circle with Radius 1

Table 10.5:

Number of sides of polygon	Area of polygon
3	1.2990
4	2.0000
5	2.3776
6	2.5981
8	2.8284
10	2.9389
20	3.0902
50	3.1333
100	3.1395
500	3.1415
1000	3.1416
2000	3.1416

As the number of sides of the inscribed regular polygon increases, the area seems to approach a "limit." This limit is approximately 3.1416, which is π.

Conclusion: The area of a circle with radius 1 is π.

Now we extend this idea to other circles. You know that *all circles are similar to each other.*

Suppose a circle has a radius of r units.

- The scale factor of this circle and the one in the diagram and table above, with radius 1, is $r : 1$, $\frac{r}{1}$, or just r.
- You know how a scale factor affects area measures. If the scale factor is r, then the area is r^2 times as much.

This means that if the area of a circle with radius 1 is π , then the area of a circle with radius r is πr^2.

Area of a Circle Formula

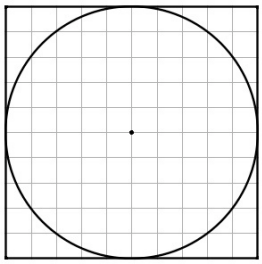
Let r be the radius of a circle, and A the area.

$$A = \pi r^2$$

You probably noticed that the reasoning about **area** here is very similar to the reasoning in an earlier lesson when we explored the *perimeter* of polygons and the *circumference* of circles.

Example 1

A circle is inscribed in a square. Each side of the square is 10 cm *long. What is the area of the circle?*



Use $A = \pi r^2$. The length of a side of the square is also the diameter of the circle. The radius is 5 cm.

$$A = \pi r^2 = \pi(5^2) = 25\pi \approx 78.5$$

The area is $25\pi \approx 78.5$ cm^2.

Area of a Sector

The area of a sector is simply an appropriate fractional part of the area of the circle. Suppose a sector of a circle with radius r and circumference C has an arc with a degree measure of m° and an arc length of s units.

- The sector is $\frac{m}{360}$ of the circle.
- The sector is also $\frac{s}{c} = \frac{s}{2\pi r}$ of the circle.

To find the area of the sector, just find one of these fractional parts of the area of the circle. We know that the area of the circle is πr^2. Let A be the area of the sector.

$$A = \frac{m}{360} \times \pi r^2$$

Also, $A = \frac{s}{c} \times \pi r^2 = \frac{s}{2\pi r} \times \pi r^2 = \frac{1}{2} sr$.

Area of a Sector

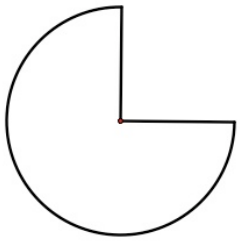
A circle has radius r. A sector of the circle has an arc with degree measure m° and arc length s units.

The area of the sector is A square units.

$$A = \frac{m}{360} \times \pi r^2 = \frac{1}{2} sr$$

Example 2

Mark drew a sheet metal pattern made up of a circle with a sector cut out. The pattern is made from an arc of a circle and two perpendicular $6 -$ inch *radii.*



How much sheet metal does Mark need for the pattern?

The measure of the arc of the piece is 270°, which is $\frac{270}{360} = \frac{3}{4}$ of the circle.

The area of the sector (pattern) is $= \frac{3}{4}\pi r^2 = \frac{3}{4}\pi \times 6^2 = 27\pi \approx 84.8$ sq in.

Lesson Summary

We used the idea of a limit again in this lesson. That enabled us to find the area of a circle by studying polygons with more and more sides. Our approach was very similar to the one used earlier for the circumference of a circle. Once the area formula was developed, the area of a sector was a simple matter of taking the proper fractional part of the whole circle.

Summary of Formulas:

Area Formula

Let r be the radius of a circle, and A the area.

$$A = \pi r^2$$

Area of a Sector

A circle has radius r. A sector of the circle has an arc with degree measure m° and arc length s units.

The area of the sector is A square units.

$$A = \frac{m}{360} \times \pi r^2 = \frac{1}{2} sr$$

Points to Consider

When we talk about a limit, for example finding the limit of the areas of regular polygons, how many sides do we mean when we talk about "more and more?" As the polygons have more and more sides, what happens to the length of each side? Is a circle a polygon with an infinite number of sides? And is each "side" of a circle infinitely small? Now that's small!

In the next lesson you'll see where the formula comes from that gives us the areas of regular polygons. This is the formula that was used to produce the table of areas in this lesson.

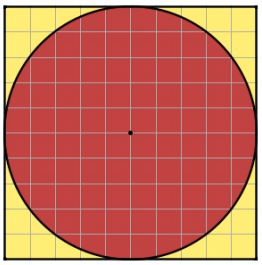
Review Questions

Complete the table of radii and areas of circles. Express your answers in terms of π.

Table 10.6:

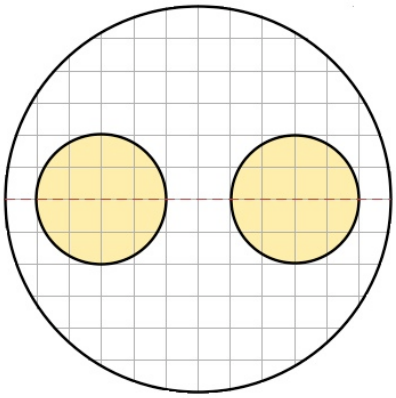
	Radius (units)	Area (square units)
1a.	10	?
1b.	?	2.25π
1c.	?	9
1d.	5π	?

1.
2. Prove: The area of a circle with diameter d is $\frac{\pi d^2}{4}$.
3. A circle is inscribed in a square.



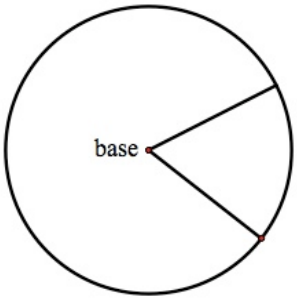
The yellow shaded area is what percent of the square?

4. The circumference of a circle is 300 feet. What is the area of the circle?
5. A center pivot irrigation system has a boom that is 400 m long. The boom is anchored at the center pivot. It revolves around the center pivot point once every three days, irrigating the ground as it turns. How many hectares of land are irrigated each day? (1 hectare $= 10,000$ m^2)
6. Vicki is cutting out a gasket in her machine shop. She made a large circle of gasket material, then cut out and removed the two small circles. The centers of the small circles are on a diameter of the large circle. Each square of the grid is 1 square inch.



How much gasket material will she use for the gasket?

7. A security system scans all points up to 100 m from is base. It scans back and forth through an angle of 65° .



How much space does the system cover?

8. A simplified version of the international radiation symbol is shown below.

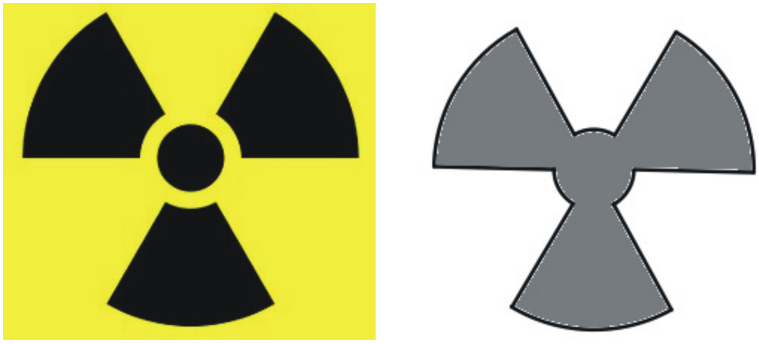


Figure 10.1

The symbol is made from two circles and three equally spaced diameters of the large circle. The diameter of the large circle is 12 inches, and the diameter of the small circle is 4 inches. What is the total area of the symbol?

9. Chad has 400 feet of fencing. He will use it all. Which would enclose the most space, a square fence or a circular fence? Explain your answer.

Review Answers

1. 1a. 100π 1b. 1.5 1c. $\frac{3\sqrt{\pi}}{\pi}$ 1d. $(25\pi)^2$
2. $A = \pi r^2, r = \frac{d}{2}$
   $A = \pi \left(\frac{d}{2}\right)^2 = \pi \left(\frac{d^2}{4}\right) = \frac{\pi d^2}{4}$
3. Approximately 21.5%
4. Approximately 7166 square feet
5. Approximately 16.7
6. Approximately 87.9 square inches
7. Approximately 5669 m^2
8. $20\pi \approx 62.8$ square inches
9. The circular fence has a greater area. Square:
   $P = 4S = 400, s = 100$
   $A = s^2 = 100^2 = 10,000 \text{ ft}^2$
   Circle:
   $C = \pi\ d = 2\pi r = 400$
   $r = \frac{400}{2\pi}$
   $A = \pi r^2 = \pi \left(\frac{400}{2\pi}\right)^2 = \frac{200^2}{\pi} \approx 12,739 \text{ ft}^2$

10.6 Regular Polygons

Learning Objectives

- Recognize and use the terms involved in developing formulas for regular polygons.
- Calculate the area and perimeter of a regular polygon.
- Relate area and perimeter formulas for regular polygons to the limit process in prior lessons.

Introduction

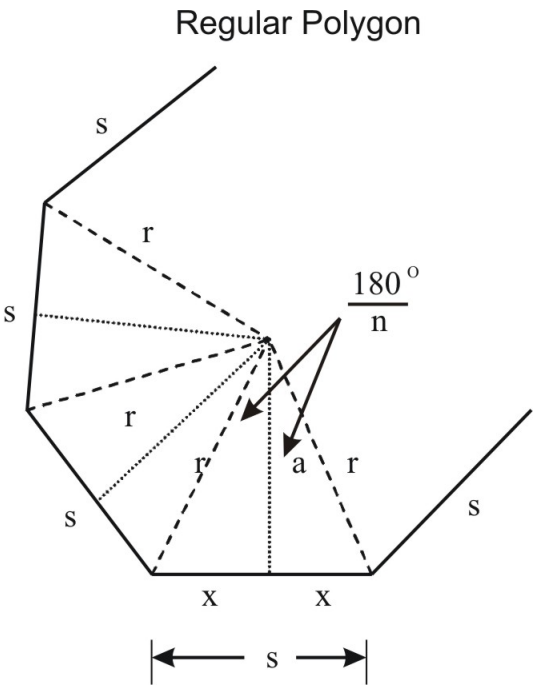
You've probably been asking yourself, "Where did the areas and perimeters of regular polygons in earlier lessons come from?" Or maybe not! You might be confident that the information presented then was accurate. In either case, in this lesson we'll fill in the missing link. We'll derive formulas for the perimeter and area of any regular polygon.

You already know how to find areas and perimeters of some figures—triangles, rectangles, etc. Not surprisingly, the new formulas in this lesson will build on those basic figures—in particular, the triangle. Note too that we will find an outstanding application of trigonometric functions in this lesson.

Parts and Terms for Regular Polygons

Let's start with some background on regular polygons.

Here is a general regular polygon with n sides; some of its sides are shown.



In the diagram, here is what each variable represents.

- s is the length of each side of the polygon.
- r is the length of a "radius" of the polygon, which is a segment from a vertex of the polygon to the center.
- x is the length of one-half of a side of the polygon ($2x = s$).
- a is the length of a segment called the **apothem**—a segment from the center to a side of the polygon, perpendicular to the side. (Notice that a is the altitude of each of the triangles formed by two radii and a side.)

The angle between two consecutive radii measures $\frac{360^\circ}{n}$ because n congruent central angles are formed by the radii from the center to each of the n vertices of the polygon. An apothem divides each of these central angles into two congruent halves; each of these half angles measures $\frac{1}{2} \times \frac{360^\circ}{n} = \frac{360^\circ}{2n} = \frac{180^\circ}{n}$.

Using Trigonometry with the Regular Polygon

Recall that in a right triangle:

$$\text{sine of an angle} = \frac{\text{opposite side}}{\text{hypotenuse}} \quad \text{cosine of an angle} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

In the diagram above, for the half angles mentioned,

- x is the length of the opposite side
- a is the length of the adjacent side
- r is the length of the hypotenuse

Now we can put these facts together:

- $\sin \frac{180^\circ}{n} = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{x}{r}$
- $x = r \sin \frac{180^\circ}{n}$
- $\cos \frac{180^\circ}{n} = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{a}{r}$
- $a = r \cos \frac{180^\circ}{n}$

Perimeter of a Regular Polygon

We continue with the regular polygon diagrammed above. Let P be the perimeter. In simplest terms,

$$P = ns$$

Here is an alternate version of the perimeter formula.

$$P = ns = n(2x) = 2nx$$
$$P = 2nr \sin \frac{180^\circ}{n}$$

Perimeter of a regular polygon with n sides and a radius r units long:

$$P = 2nr \sin \frac{180^\circ}{n}$$

One more version of the perimeter formula applies when the polygon is inscribed in a "unit circle," which is a circle with a radius of 1.

$$P = 2nr \sin \frac{180^\circ}{n} = 2n(1) \sin \frac{180^\circ}{n} = 2n \sin \frac{180^\circ}{n}$$

Perimeter of a regular polygon with n sides inscribed in a unit circle:

$$P = 2n \sin \frac{180^\circ}{n}$$

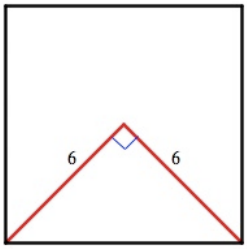
Example 1

A square has a radius of 6 inches. *What is the perimeter of the square?*

Use $P = 2nr \sin \frac{180^\circ}{n}$, with $n = 4$ and $r = 6$.

$$P = 2nr \sin \frac{180^\circ}{n} = 2(4)(6) \sin 45^\circ = 48 \left(\frac{\sqrt{2}}{2} \right) \approx 33.9 \text{ in.}$$

Notice that a side and two radii make a right triangle.



The legs are 6 inches long, and the hypotenuse, which is a side of the square, is $6\sqrt{2}$ inches long.

Use $P = ns$.

$$P = ns = 4(6\sqrt{2}) = 24\sqrt{2} \approx 33.9 \text{ inches.}$$

The purpose of this example is not to calculate the perimeter, but to verify that the formulas developed above "work."

Area of a Regular Polygon

The next logical step is to complete our study of regular polygons by developing area formulas.

Take another look at the regular polygon figure above. Here's how we can find its area, A.

Two radii and a side make a triangle with base s and altitude a.

- There are n of these triangles.
- The area of each triangle is $\frac{1}{2}bh = \frac{1}{2}sa$.

The entire area is $A = n\left(\frac{1}{2}sa\right) = \frac{1}{2}(ns)a = \frac{1}{2}Pa$.

Area of a regular polygon with apothem a:

$$A = \frac{1}{2}Pa$$

We can use trigonometric functions to produce a different version of the area formula.

$A = \frac{1}{2}Pa = \frac{1}{2}(ns)a = \frac{1}{2}n(2x)a = nxa$ (remember that $s = 2x$)

$A = n\left(r\sin\frac{180^\circ}{n}\right)\left(r\cos\frac{180^\circ}{n}\right)$ (remember that $x = r\sin\frac{180^\circ}{n}$ and $a = r\cos\frac{180^\circ}{n}$)

$A = nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$

Area of a regular polygon with n sides and radius r:

$$A = nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$$

One more version of the area formula applies when the polygon is inscribed in a unit circle.

$$A = nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} = n(1^2)\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} = n\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$$

(remember that $r = 1$)

Area of a regular polygon with n sides inscribed in a unit circle:

$$A = n\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$$

Example 2

A square is inscribed in a unit circle. What is the area of the square?

Use $A = n\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n}$ with $n = 4$.

$A = n\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} = 4\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} = 4\sin 45\cos 45 = 4(0.5) = 2$

The square is a rhombus with diagonals 2 units long. Use the area formula for a rhombus.

$$A = \frac{1}{2}d_1d_2 = \frac{1}{2}(2)(2) = \frac{1}{2} \times 4 = 2$$

Comments: As in example 1, the purpose of this example is to show that the new area formulas *do* work. We can confirm that the area formula gives a correct answer because we have another way to confirm that the area is correct.

Lesson Summary

The lesson can be summarized with a review of the formulas we derived.

Table 10.7:

	Perimeter	**Area**
Any regular polygon	$P = ns$	$A = \frac{1}{2}Pa$
Any regular polygon	$P = 2nr \sin \frac{180^\circ}{n}$	$A = nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}$
Regular polygon inscribed in a unit circle	$P = 2n \sin \frac{180^\circ}{n}$	$A = n \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}$

Points to Consider

We used the concept of a limit in an earlier lesson. In the Lesson Exercises, you will have an opportunity to use the formulas from this lesson to "confirm" the circumference and area formulas for a circle, which is the "ultimate" regular polygon (with many, many sides that are very short).

Review Questions

Each side of a regular hexagon is 5 inches long.

1. What is the radius of the hexagon?
2. What is the perimeter of the hexagon?
3. What is the area of the hexagon?

A regular $50-$ gon and a regular $100-$ gon are inscribed in a circle with a radius of 10 centimeters.

4. Which polygon has the greater perimeter?

5. How much greater is the perimeter?
6. Which polygon has the greater area?
7. How much greater is the area?
8. A regular $n-$gon is inscribed in a unit circle. The area of the $n-$gon, rounded to the nearest hundredth, is 3.14. What is the smallest possible value of n?

Review Answers

1. 5 inches
2. 30 inches
3. 65.0 square inches
4. The $100-$gon
5. 0.031 cm
6. The $100-$gon
7. 0.62 cm^2
8. 56

10.7 Geometric Probability

Learning Objectives

- Identify favorable outcomes and total outcomes.
- Express geometric situations in probability terms.
- Interpret probabilities in terms of lengths and areas.

Introduction

You've probably studied probability before now (pun intended). We'll start this lesson by reviewing the basic concepts of probability.

Once we've reviewed the basic ideas of probability, we'll extend them to situations that are represented in geometric settings. We focus on probabilities that can be calculated based on lengths and areas. The formulas you learned in earlier lessons will be very useful in figuring these geometric probabilities.

Basic Probability

Probability is a way to assign specific numbers to how likely, or unlikely, an event is. We need to know two things:

- the total number of possible outcomes for an event. Let's call this t.
- the number of "favorable" outcomes for the event. Let's call this f.

The probability of the event, call it P, is the ratio of the number of favorable outcomes to the total number of outcomes.

Definition of Probability

$$P = \frac{f}{t}$$

Example 1

Nabeel's company has 12 *holidays each year. Holidays are always on weekdays (not weekends). This year there are* 260 *weekdays. What is the probability that any weekday is a holiday?*

There are 260 weekdays in all.

$$t = 260$$

12 of the weekdays are holidays

$$f = 12$$
$$P = \frac{f}{t} = \frac{12}{260} \approx 0.05$$

Comments: Probabilities are often expressed as fractions, decimals, and percents. Nabeel can say that there is a 5% chance of any weekday being a holiday. Note that this is (unfortunately?) a relatively low probability.

Example 2

Charmane has four coins in a jar: two nickels, a dime, and a quarter. She mixes them well. Charmane takes out two of the coins without looking. What is the probability that the coins she takes have a total value of more than \$0.25?

t in this problem is the total number of two-coin combinations. We can just list them all. To make it easy to keep track, use these codes: $N1$ (one of the nickels), $N2$ (the other nickel), D (the dime), and Q (the quarter).

Two-coin combinations:

$$N1, N2 \qquad N1, D \qquad N1, Q \qquad N2, D \qquad N2, Q \qquad D, Q$$

There are six two-coin combinations.

$$t = 6$$

Of the six two-coin combinations, three have a total value of more than \$0.25. They are:

$$N1, Q(\$0.30) \qquad N2, Q(\$0.30) \qquad D, Q(\$0.35)$$

$$f = 3$$

The probability that the two coins will have a total value of more than \$0.25 is $P = \frac{f}{t} = \frac{3}{6} = \frac{1}{2} = 0.5 = 50\%$.

The probability is usually written as $\frac{1}{2}, 0.5$, or 50%. Sometimes this is expressed as "a $50-50$ chance" because the probability of success and of failure are both 50%.

Geometric Probability

The values of t and f that determine a probability can be lengths and areas.

Example 3

Sean needs to drill a hole in a wall that is 14 feet *wide and* 8 feet *high. There is a* $2-$ foot$-by-3-$ foot *rectangular mirror on the other side of the wall so that Sean can't see the mirror. If Sean drills at a random location on the wall, what is the probability that he will hit the mirror?*

The area of the wall is $14 \times 8 = 112$ square feet. This is t.

The area of the mirror is $2 \times 3 = 6$ square feet. This is f.

The probability is $P = \frac{6}{112} \approx 0.05$.

Example 4

Ella repairs an electric power line that runs from Acton to Dayton through Barton and Canton. The distances in miles between these towns are as follows.

- Barton to Canton = 8 miles.
- Acton to Canton = 12 miles.
- Canton to Dayton = 2 miles.

If a break in the power line happens, what is the probability that the break is between Barton and Dayton?

Approximately 71%.

$$t = \text{the distance from Acton to Dayton} = 4 + 8 + 2 = 14 \text{ miles.}$$
$$f = \text{the distance from Barton to Dayton} = 8 + 2 = 10 \text{ miles.}$$
$$P = \frac{f}{t} = \frac{10}{14} = \frac{5}{7} \approx 0.71 = 71\%$$

Lesson Summary

Probability is a way to measure how likely or unlikely an event is. In this section we saw how to use lengths and areas as models for probability questions. The basic probability ideas are the same as in non-geometry applications, with probability defined as:

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Points to Consider

Some events are more likely, and some are less likely. No event has a negative probability! Can you think of an event with an extremely low, or an extremely high, probability? What are the ultimate extremes—the greatest and the least values possible for a probability? In ordinary language these are called "impossible" (least possible probability) and "certain" or a "sure thing" (greatest possible probability).

The study of probability originated in the seventeenth century as mathematicians analyzed games of chance.

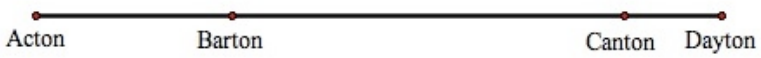
For Further Reading

- French mathematicians Pierre de Fermat and Blaise Pascal are credited as the "inventors" of mathematical probability. The reference below is an easy introduction to their ideas. http://mathforum.org/isaac/problems/prob1.html

Review Questions

1. Rita is retired. For her, every day is a holiday. What is the probability that tomorrow is a holiday for Rita?
2. Chaz is "on call" any time, any day. He never has a holiday. What is the probability that tomorrow is a holiday for Chaz?
3. The only things on Ray's refrigerator door are 4 green magnets and 6 yellow magnets. Ray takes one magnet off without looking.
   (a) What is the probability that the magnet is green?
   (b) What is the probability that the magnet is yellow?
   (c) What is the probability that the magnet is purple?

   Ray takes off two magnets without looking.
   (a) What is the probability that both magnets are green?
   (b) What is the probability that Ray takes off one green and one yellow magnet?
4. Reed uses the diagram below as a model of a highway.

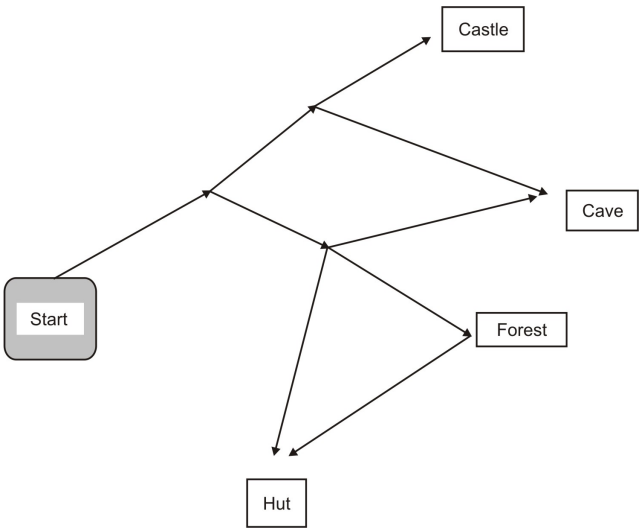


   (a) What is the probability that the accident is **not** between Canton and Dayton?
   (b) What is the probability that the accident is closer to Canton than it is to Barton?

Reed got a call about an accident at an unknown location between Acton and Dayton.

5. A tire has an outer diameter of 26 inches. Nina noticed a weak spot on the tire. She marked the weak spot with chalk. The chalk mark is 4 inches along the outer edge of the tire. What is the probability that part of the weak spot is in contact with the ground at any time?
6. Mike set up a rectangular landing zone that measures 200 feet by 500 feet. He marked a circular helicopter pad that measured 50 feet across at its widest in the landing zone. As a test, Mike dropped a package that landed in the landing zone. What is the probability that the package landed outside the helicopter pad?
7. Fareed made a target for a game. The target is a 4−foot-by−4-foot square. To win a player must hit a smaller square in the center of the target. If the probability that players who hit the target win is 20%, what is the length of a side of the smaller square?

8. Amazonia set off on a quest. She followed the paths shown by the arrows in the map.



Every time a path splits, Amazonia takes a new path at random. What is the probability that she ends up in the cave?

Review Answers

1. $1, 100\%$, or equivalent
2. 0
3. (a) $\frac{2}{5}, 0.4, 40\%$, or equivalent
   (b) $\frac{3}{5}, 0.6, 60\%$, or equivalent
   (c) 0
   (d) $\frac{2}{15} \approx 0.13$ or equivalent
   (e) $\frac{4}{15} \approx 0.27$ or equivalent
4. (a) $\frac{12}{14} = \frac{6}{7} \approx 0.86 = 86\%$
   (b) $\frac{6}{14} = \frac{3}{7} \approx 0.43 = 43\%$
5. Approximately $0.05 = 5\%$
6. Approximately 98%
7. Approximately 1.78 feet
8. $\frac{5}{12} \approx 0.42$ or equivalent

Image Sources

(1) http://upload.wikimedia.org/wikipedia/commons/0/0b/Radiation_warning_symbol.svg. Public Domain.

Chapter 11

Surface Area and Volume

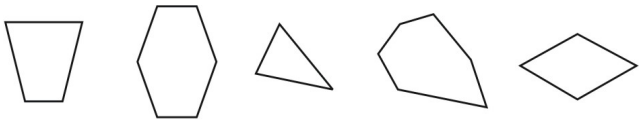
11.1 The Polyhedron

Learning Objectives

- Identify polyhedra.
- Understand the properties of polyhedra.
- Use Euler's formula solve problems.
- Identify regular (Platonic) polyhedra.

Introduction

In earlier chapters you learned that a **polygon** is a two-dimensional (planar) figure that is made of three or more points joined together by line segments. Examples of polygons include triangles, quadrilaterals, pentagons, or octagons. In general, an $n-$ gon is a polygon with n sides. So a triangle is a $3-$ gon, or 3−sided polygon, a pentagon is a $5-gon$, or 5−sided polygon.



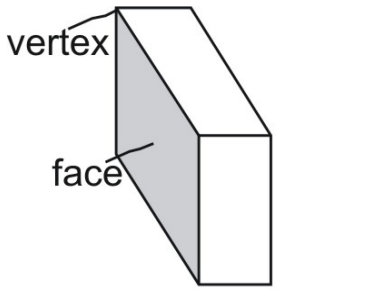
polygons

You can use polygons to construct a 3-dimensional figure called a **polyhedron** (plural: **polyhedra**). A polyhedron is a 3-dimensional figure that is made up of polygon faces. A cube is an example of a polyhedron and its faces are squares (quadrilaterals).

Polyhedron or Not

A polyhedron has the following properties:

- It is a 3−dimensional figure.
- It is made of polygons and only polygons. Each polygon is called a **face** of the polyhedron.
- Polygon faces join together along segments called **edges.**
- Each edge joins exactly two faces.
- Edges meet in points called **vertices.**
- There are no gaps between edges or vertices.



a polyhedron
(prism)

Example 1

Is the figure a polyhedron?

Yes. A figure is a polyhedron if it has all of the properties of a polyhedron. This figure:

- Is 3−dimensional.
- Is constructed entirely of flat polygons (triangles and rectangles).
- Has faces that meet in edges and edges that meet in vertices.
- Has no gaps between edges.

- Has no non-polygon faces (e.g., curves).
- Has no concave faces.

Since the figure has all of the properties of a polyhedron, it is a polyhedron.

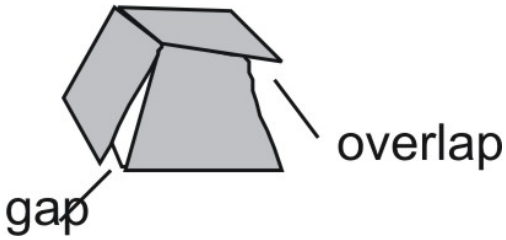
Example 2

Is the figure a polyhedron?

No. This figure has faces, edges, and vertices, but all of its surfaces are *not* flat polygons. Look at the end surface marked A. It is flat, but it has a curved edge so it is not a polygon. Surface B is not flat (planar).

Example 3

Is the figure a polyhedron?



No. The figure is made up of polygons and it has faces, edges, and vertices. But the faces do not fit together—the figure has gaps. The figure also has an overlap that creates a concave surface. For these reasons, the figure is not a polyhedron.

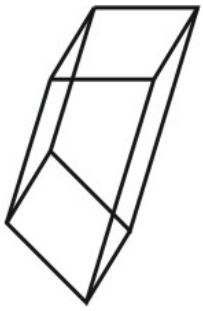
Face, Vertex, Edge, Base

As indicated above, a polyhedron joins faces together along edges, and edges together at vertices. The following statements are true of any polyhedron:

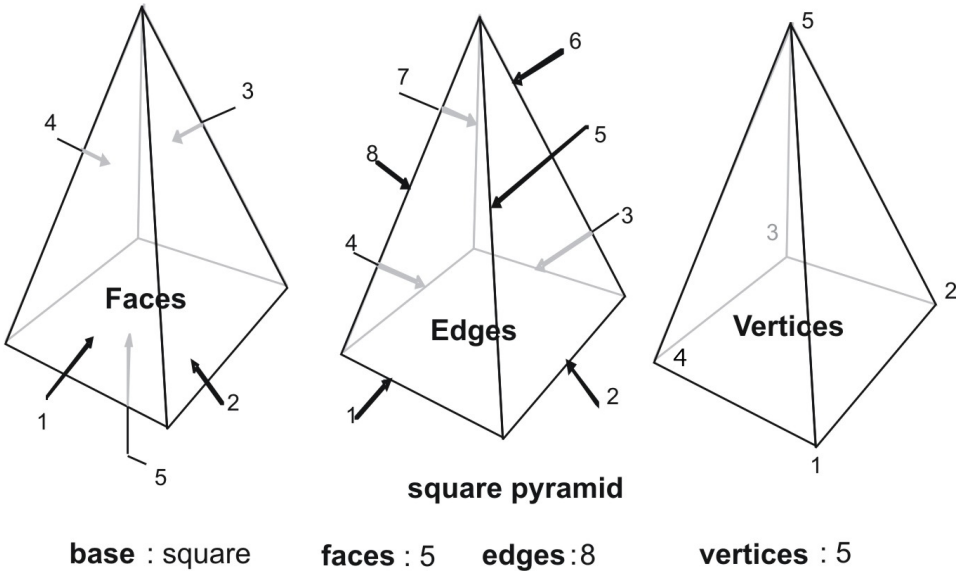
- Each edge joins exactly two faces.

- Each edge joins exactly two vertices.

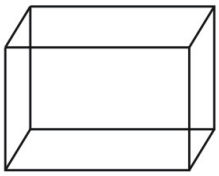
To see why this is true, take a look at this prism. Each of its edges joins two faces along a single line segment. Each of its edges includes exactly two vertices.



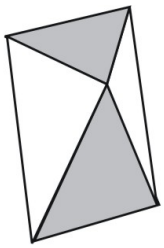
Let's count the number of faces, edges, and vertices in a few typical polyhedra. The square pyramid gets its name from its base, which is a square. It has 5 faces, 8 edges, and 5 vertices.



Other figures have a different number of faces, edges, and vertices.



rectangular prism
base: rectangle
faces: 6
edges: 12
vertices: 8



octahedron
base: triangle
faces: 8
edges: 12
vertices: 6

pentagonal prism
base: pentagon
faces: 7
edges: 15
vertices: 10

If we make a table that summarizes the data from each of the figures we get:

Table 11.1:

Figure	Vertices	Faces	Edges
Square pyramid	5	5	8
Rectangular prism	8	6	12
Octahedron	6	8	12
Pentagonal prism	10	7	15

Do you see a pattern? Calculate the sum of the number of vertices and edges. Then compare that sum to the number of edges:

Table 11.2:

Figure	V	F	E	$V + F$
square pyramid	5	5	8	10
rectangular prism	8	6	12	14
octahedron	6	8	12	14
pentagonal prism	10	7	15	17

Do you see the pattern? The formula that summarizes this relationship is named after mathematician Leonhard Euler. Euler's formula says, for any polyhedron:

Euler's Formula for Polyhedra

$$\text{vertices} + \text{faces} = \text{edges} + 2$$

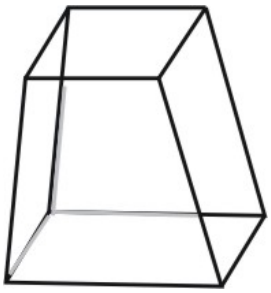
or

$$v + f = e + 2$$

You can use Euler's formula to find the number of edges, faces, or vertices in a polyhedron.

Example 4

Count the number of faces, edges, and vertices in the figure. Does it conform to Euler's formula?



There are 6 faces, 12 edges, and 8 vertices. Using the formula:

$$v + f = e + 2$$
$$8 + 6 = 12 + 2$$

So the figure conforms to Euler's formula.

Example 5

In a $6-$*faced polyhedron, there are* 10 edges. *How many vertices does the polyhedron have?*

Use Euler's formula.

$v + f = e + 2$	Euler's formula
$v + 6 = 10 + 2$	Substitute values for f and e
$v = 6$	Solve

There are 6 vertices in the figure.

Example 6

A 3-dimensional figure has 10 vertices, 5 faces, *and* 12 edges. *It is a polyhedron? How do you know?*

Use Euler's formula.

$v + f = e + 2$	Euler's formula
$10 + 5 \neq 12 + 2$	Substitute values for $v, f,$ and e
$15 \neq 14$	Evaluate

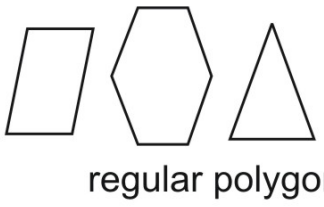
The equation does not hold so Euler's formula does not apply to this figure. Since all polyhedra conform to Euler's formula, this figure must not be a polyhedron.

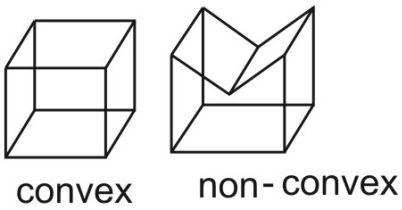
Regular Polyhedra

Polyhedra can be named and classified in a number of ways—by side, by angle, by base, by number of faces, and so on. Perhaps the most important classification is whether or not a polyhedron is **regular** or not. You will recall that a **regular polygon** is a polygon whose sides and angles are all congruent.

A polyhedron is regular if it has the following characteristics:

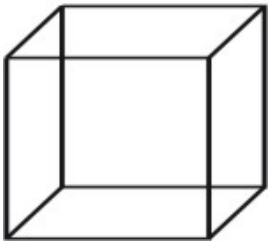
- All faces are the same.
- All faces are congruent regular polygons.
- The same number of faces meet at every vertex.
- The figure has no gaps or holes.
- The figure is convex—it has no indentations.





Example 7

Is a cube a regular polyhedron?



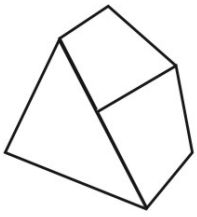
All faces of a cube are regular polygons—squares. The cube is convex because it has no indented surfaces. The cube is simple because it has no gaps. Therefore, a cube is a regular polyhedron.

A polyhedron is **semi-regular** if all of its faces are regular polygons and the same number of faces meet at every vertex.

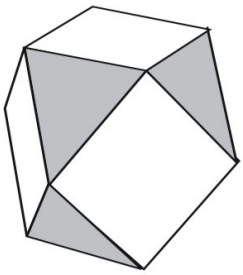
- Semi-regular polyhedra often have two different kinds of faces, both of which are regular polygons.
- **Prisms** with a regular polygon base are one kind of semi-regular polyhedron.
- Not all semi-regular polyhedra are prisms. An example of a non-prism is shown below.

semi-regular:
prism equilateral
triangles and
squares

Completely irregular polyhedra also exist. They are made of different kinds of regular and irregular polygons.



Not-regular



semi-regular: squares and equilateral triangles

So now a question arises. Given that a polyhedron is regular if all of its faces are congruent regular polygons, it is convex and contains no gaps or holes. How many *regular* polyhedra actually exist?

In fact, you may be surprised to learn that only five regular polyhedra can be made. They are known as the **Platonic** (or noble) solids.

tetrahedron cube octahedron dodecahedron icosahedron

Note that no matter how you try, you can't construct any other regular polyhedra besides the ones above.

Example 8

How many faces, edges, and vertices does a tetrahedron (see above) have?

Faces : 4, edges : 6, vertices : 4

Example 9

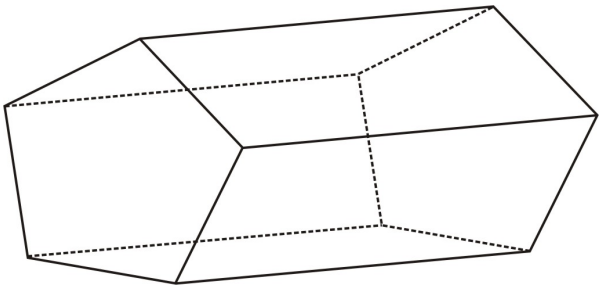
Which regular polygon does an icosahedron feature?

An equilateral triangle

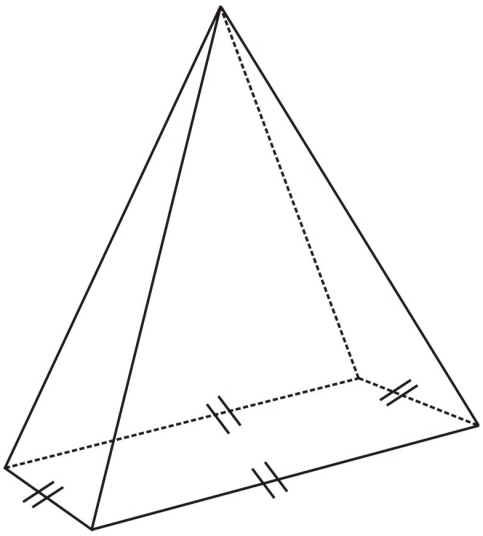
Review Questions

Identify each of the following three-dimensional figures:

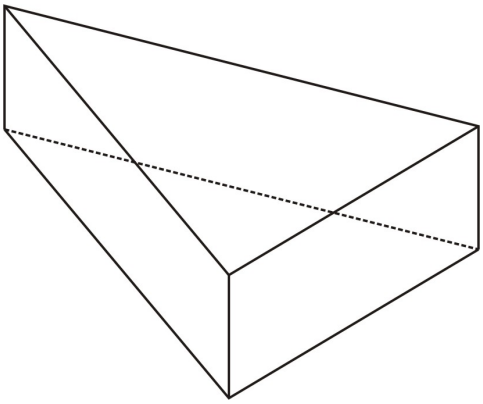
1.



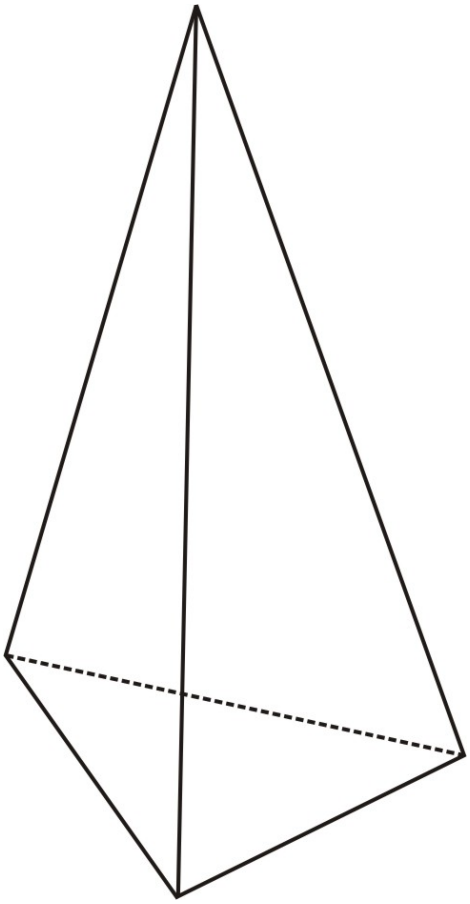
2.



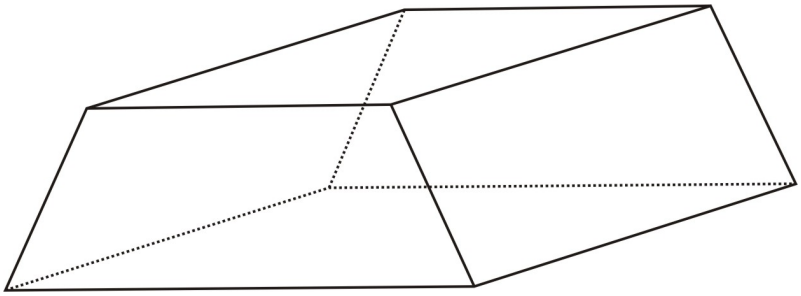
3.



4.



5.



6. Below is a list of the properties of a polyhedron. Two of the properties are not correct. Find the incorrect ones and correct them.

   - It is a 3 dimensional figure.
   - Some of its faces are polygons.
   - Polygon faces join together along segments called edges.
   - Each edge joins three faces.
   - There are no gaps between edges and vertices.

   Complete the table and verify Euler's formula for each of the figures in the problem.

Table 11.3:

Figure	# vertices	# edges	# faces
Pentagonal prism			
Rectangular pyramid			
Triangular prism			
Trapezoidal prism			

7.

Review Answers

Identify each of the following three dimensional figures:

1. ***pentagonal prism***
2. ***rectangular pyramid***
3. ***triangular prism***
4. ***triangular pyramid***
5. ***trapezoidal prism***
6. Below is a list of the properties of a polyhedron. Two of the properties are not correct. Find the incorrect ones and correct them.
   - It is a 3 dimensional figure.
   - Some of its faces are polygons. ***All of its faces are polygons.***
   - Polygon faces join together along segments called edges.
   - Each edge joins three faces. ***Each edge joins two faces.***
   - There are no gaps between edges and vertices.

   Complete the table and verify Euler's formula for each of the figures in the problem.

Table 11.4:

Figure	# vertices	# edges	# faces
Pentagonal prism	10	15	7
Rectangular pyramid	5	8	5
Triangular prism	6	9	5
Trapezoidal prism	8	12	6

7. ***In all cases*** vertices + faces = edges + 2

11.2 Representing Solids

Learning Objectives

- Identify isometric, orthographic, cross-sectional views of solids.
- Draw isometric, orthographic, cross-sectional views of solids.
- Identify, draw, and construct nets for solids.

Introduction

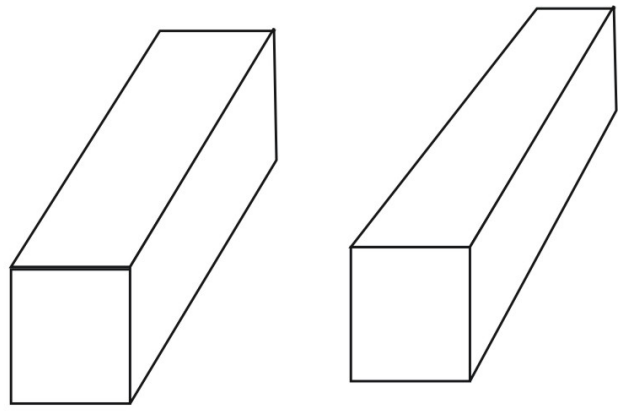
The best way to represent a three-dimensional figure is to use a solid model. Unfortunately, models are sometimes not available. There are four primary ways to represent solids in two dimensions on paper. These are:

- An isometric (or perspective) view.
- An orthographic or blow-up view.
- A cross-sectional view.
- A net.

Isometric View

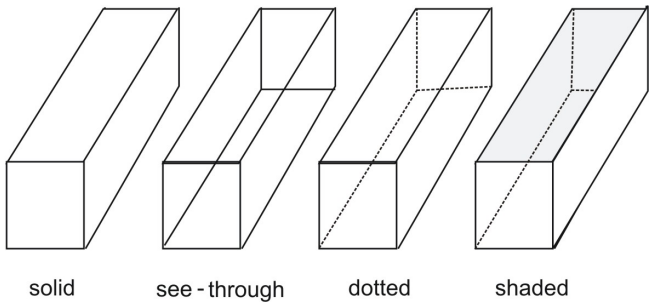
The typical three-dimensional view of a solid is the **isometric** view. Strictly speaking, an isometric view of a solid does not include **perspective.** Perspective is the illusion used by artists to make things in the distance look smaller than things nearby by using a vanishing point where parallel lines converge.

The figures below show the difference between an isometric and perspective view of a solid.



As you can see, the perspective view looks more "real" to the eye, but in geometry, isometric representations are useful for measuring and comparing distances.

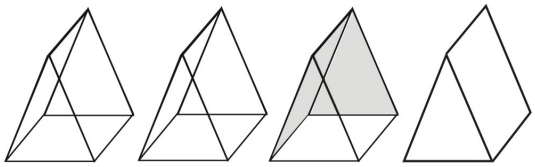
The isometric view is often shown in a transparent "see-through" form.



Color and shading can also be added to help the eye visualize the solid.

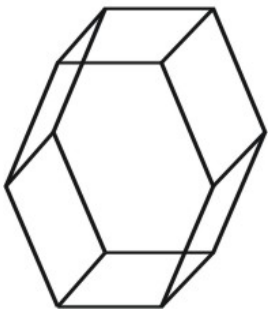
Example 1

Show isometric views of a prism with an equilateral triangle for its base.



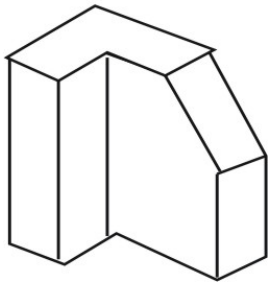
Example 2

Show a see-through isometric view of a prism with a hexagon for a base.

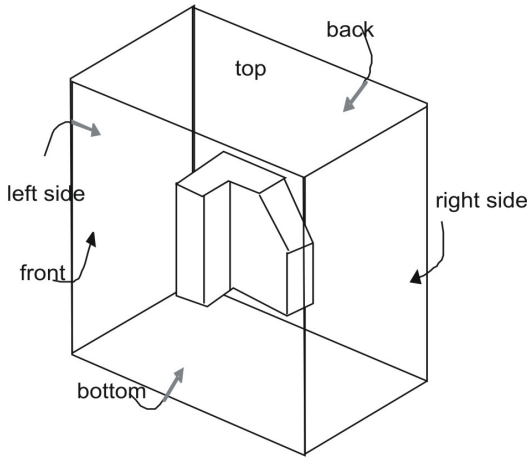


Orthographic View

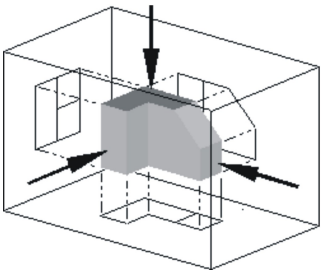
An **orthographic** projection is a blow-up view of a solid that shows a flat representation of each of the figure's sides. A good way to see how an orthographic projection works is to construct one. The (non-convex) polyhedron shown has a different projection on every side.



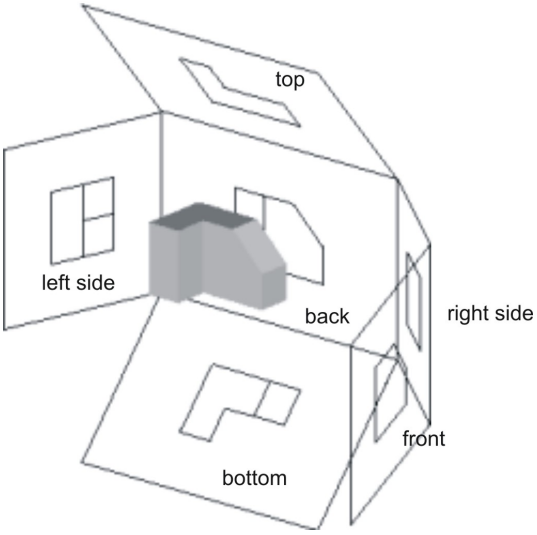
To show the figure in an orthographic view, place it in an imaginary box.



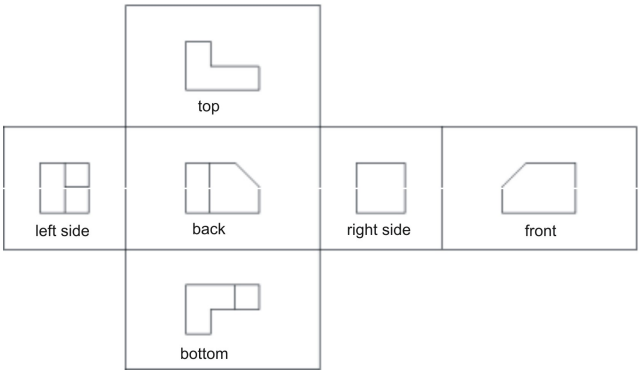
Now project out to each of the walls of the box. Three of the views are shown below.



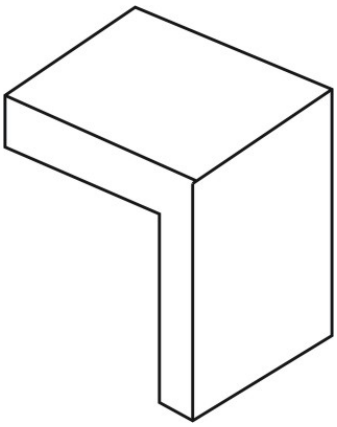
A more complete orthographic blow-up shows the image of the side on each of the six walls of the box.



The same image looks like this in fold out view.

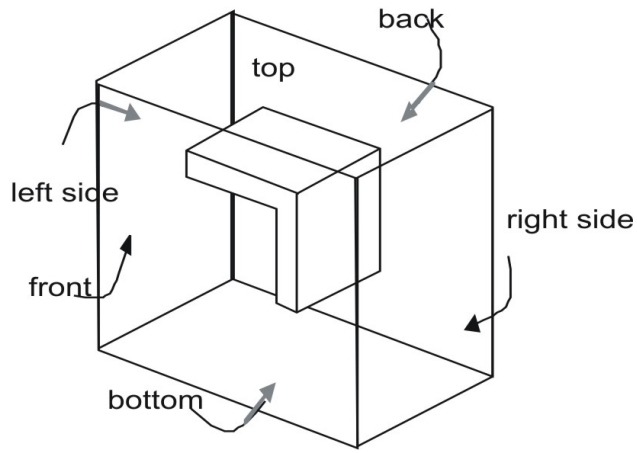


Example 3

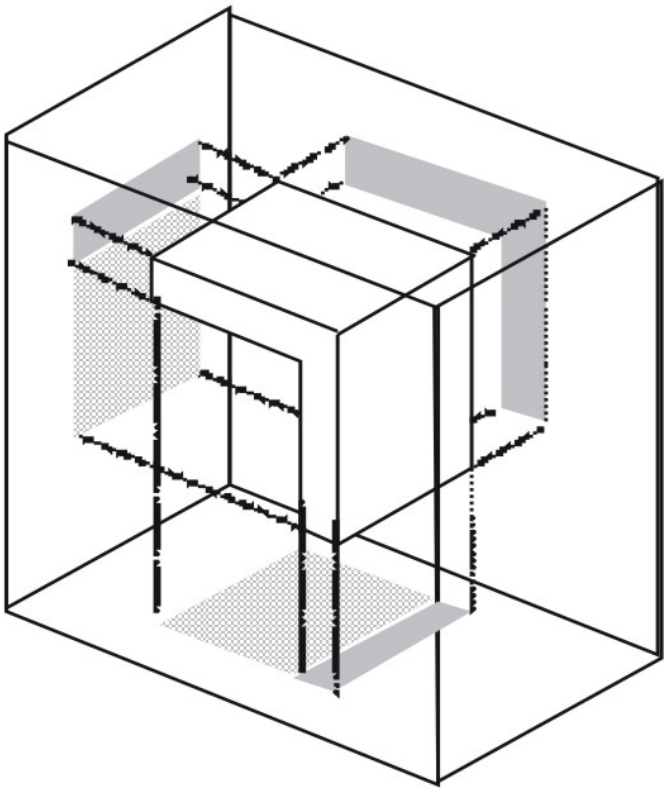


Show an orthographic view of the figure.

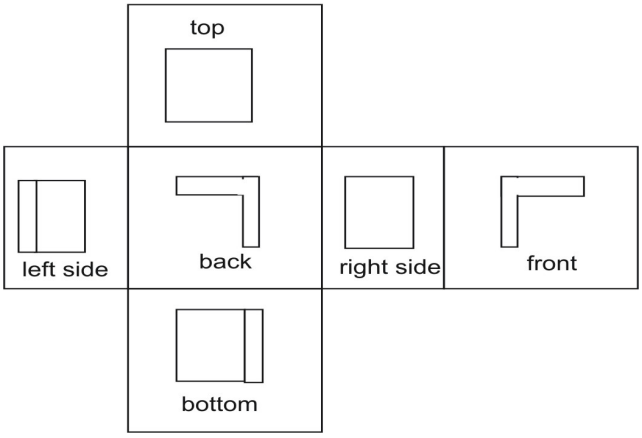
First, place the figure in a box.



Now project each of the sides of the figure out to the walls of the box. Three projections are shown.

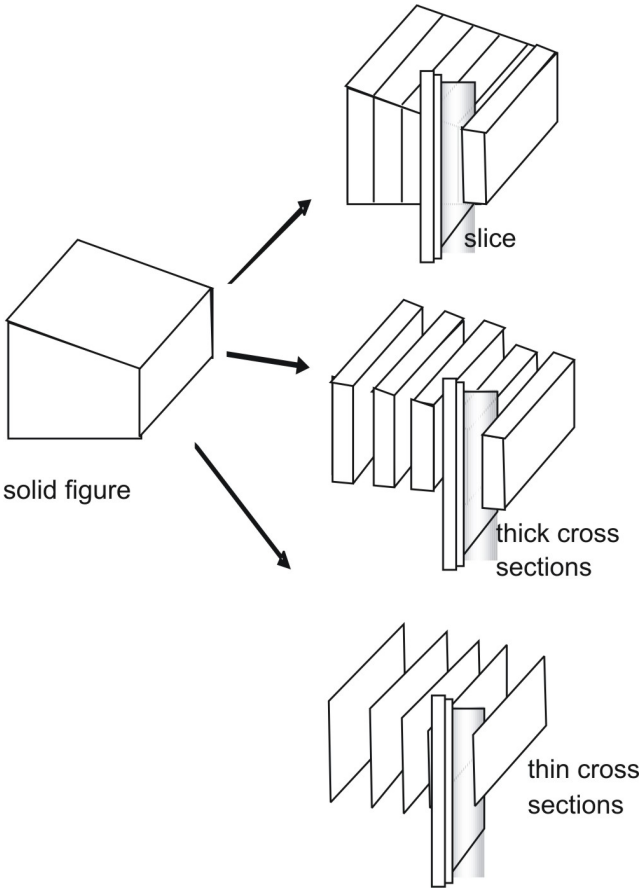


You can use this image to make a fold-out representation of the same figure.



Cross Section View

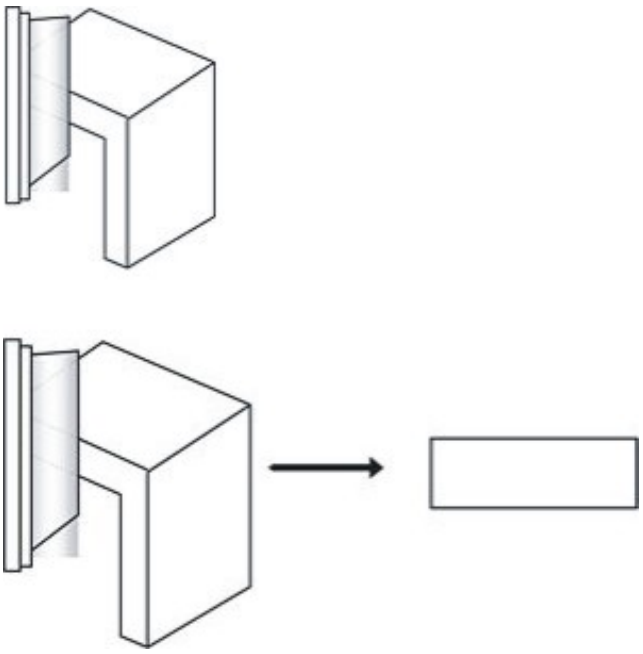
Imagine slicing a three-dimensional figure into a series of thin slices. Each slice shows a **cross-section** view.



The cross section you get depends on the angle at which you slice the figure.

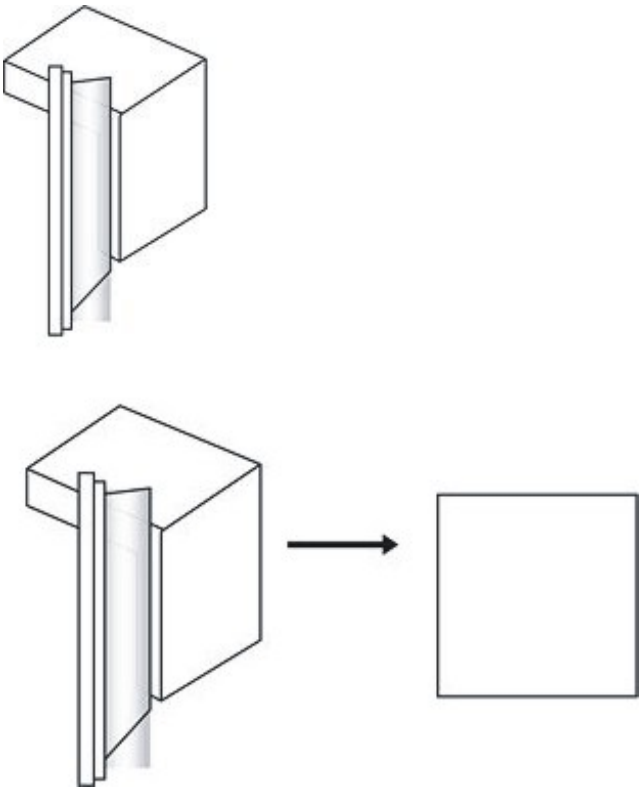
Example 4

What kind of cross section will result from cutting the figure at the angle shown?



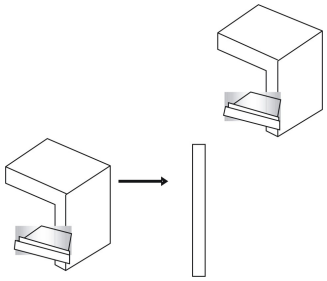
Example 5

What kind of cross section will result from cutting the figure at the angle shown?



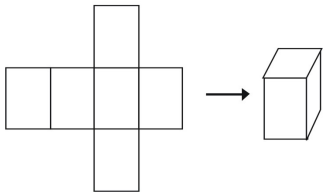
Example 6

What kind of cross section will result from cutting the figure at the angle shown?

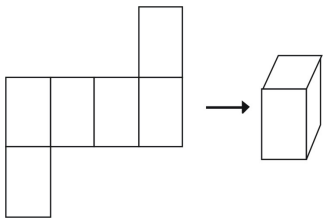


Nets

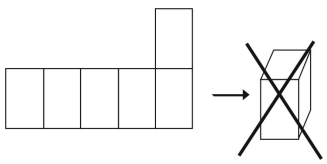
One final way to represent a solid is to use a net. If you cut out a net you can fold it into a model of a figure. Nets can also be used to analyze a single solid. Here is an example of a net for a cube.



There is more than one way to make a net for a single figure.

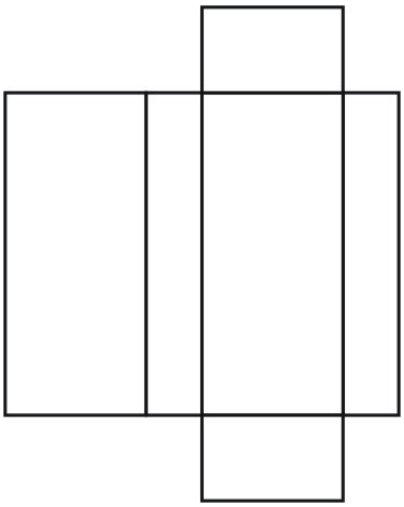


However, not all arrangements will create a cube.

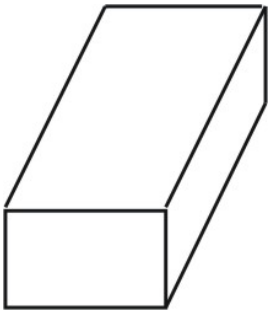


Example 7

What kind of figure does the net create? Draw the figure.

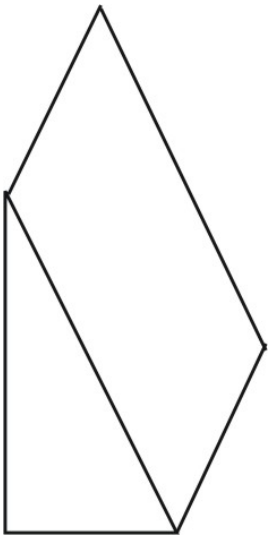


The net creates a box-shaped rectangular prism as shown below.

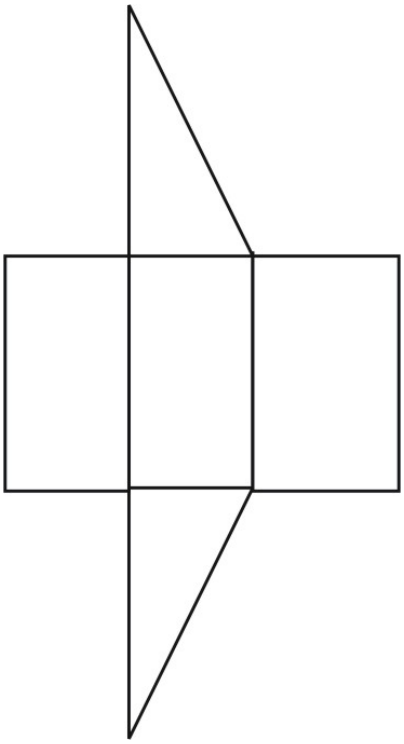


Example 8

What kind of net can you draw to represent the figure shown? Draw the net.



A net for the prism is shown. Other nets are possible.

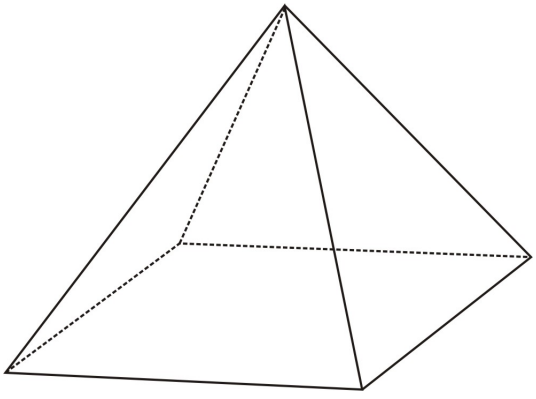


Multimedia Link The applet here animates how four solids are made from nets. There are two unique nets for the cube and two for the dodecahedron. Unfolding Polyhedra.

Review Questions

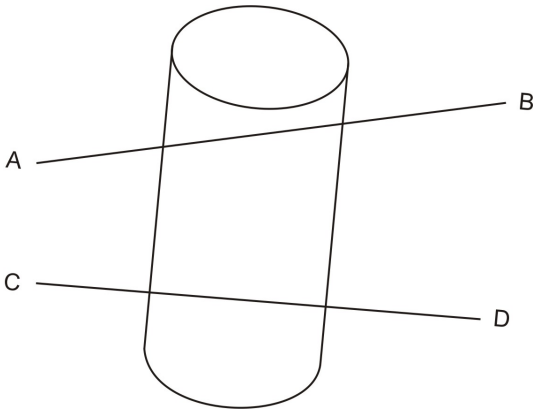
1. Name four different ways to represent solids in two dimensions on paper.
2. Show an isometric view of a prism with a square base.

Given the following pyramid:

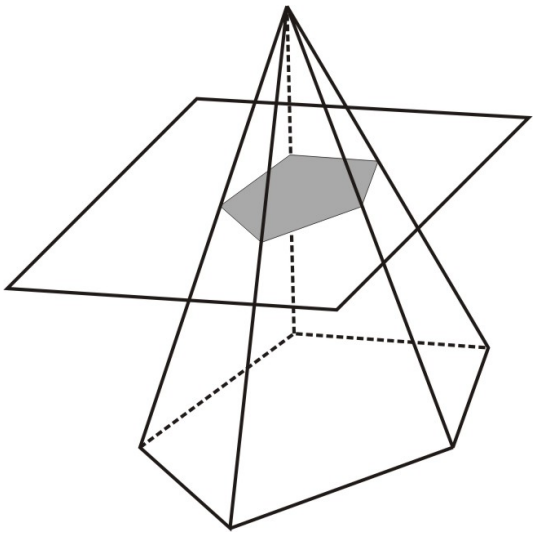


3. If the pyramid is cut with a plane parallel to the base, what is the cross section?
4. If the pyramid is cut with a plane passing through the top vertex and perpendicular to the base, what is the cross section?
5. If the pyramid is cut with a plane perpendicular to the base but not through the top vertex, what is the cross section?

Sketch the shape of the plane surface at the cut of this solid figure.

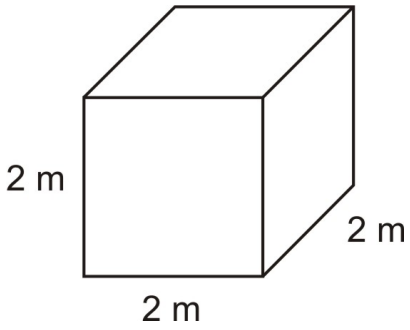


6. Cut AB
7. Cut CD
8. For this figure, what is the cross section?

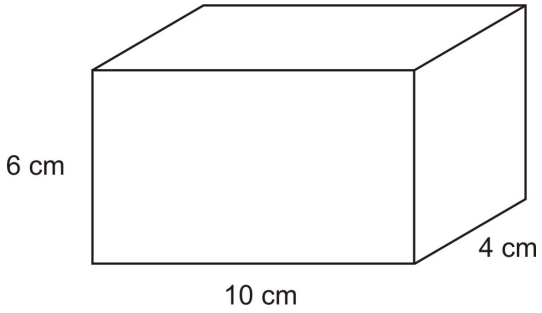


Draw a net for each of the following:

9.



10.

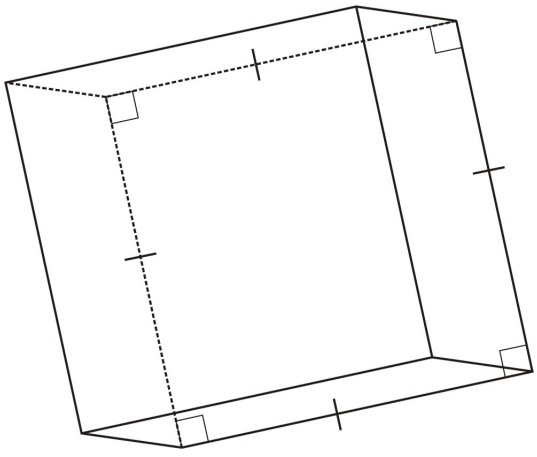


Review Answers

1. Name four different ways to represent solids in two dimensions on paper.

Isometric, orthographic, cross sectional, net

2. Show an isometric view of a prism with a square base.

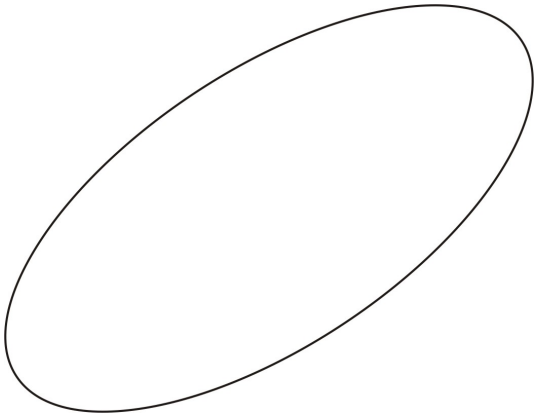


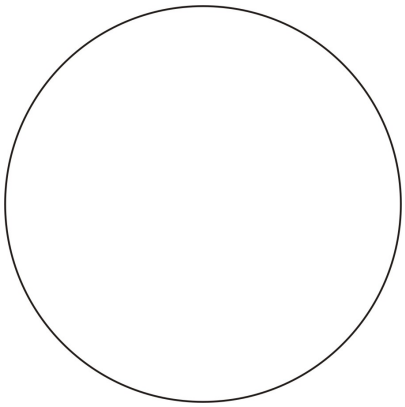
Given the following pyramid:

3. If the pyramid is cut with a plane parallel to the base, what is the cross section?
   square

4. If the pyramid is cut with a plane passing through the top vertex and perpendicular to the base, what is the cross section? ***triangle***
5. If the pyramid is cut with a plane perpendicular to the base but not through the top vertex, what is the cross section? ***trapezoid***

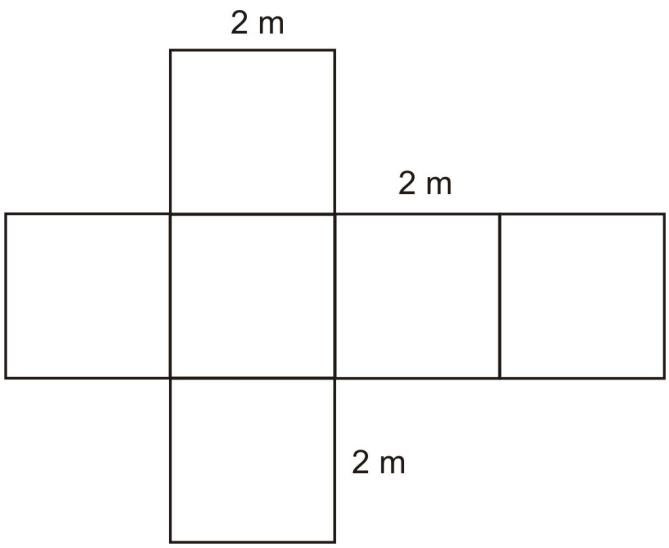
Sketch the shape of the plane surface at the cut of this solid figure.

6. 

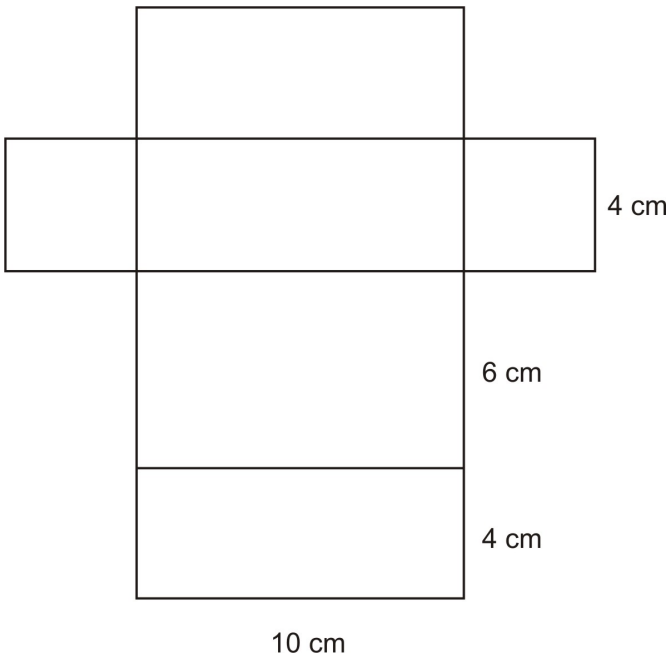
7. 

8. ***pentagon***

9.



10.



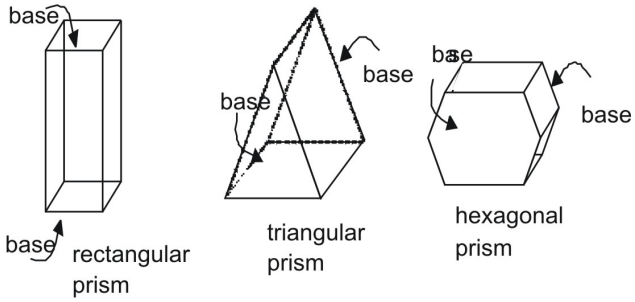
11.3 Prisms

Learning Objectives

- Use nets to represent prisms.
- Find the surface area of a prism.
- Find the volume of a prism.

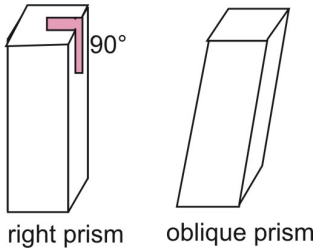
Introduction

A prism is a three-dimensional figure with a pair of parallel and congruent ends, or bases. The sides of a prism are parallelograms. Prisms are identified by their bases.



Surface Area of a Prism Using Nets

The prisms above are **right prisms**. In a right prism, the lateral sides are perpendicular to the bases of prism. Compare a right prism to an **oblique prism**, in which sides and bases are not perpendicular.



Two postulates that apply to area are the Area Congruence Postulate and the Area Addition Postulate.

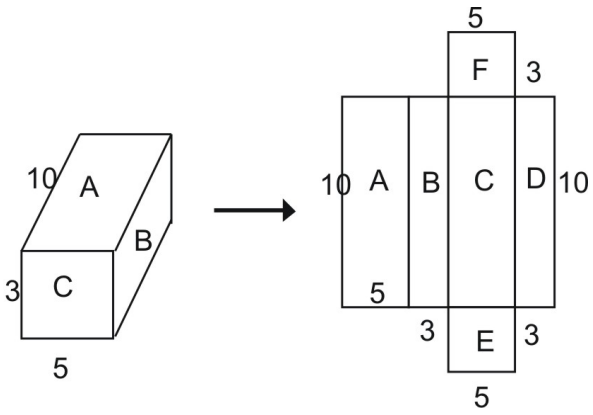
Area Congruence Postulate:

If two polygons (or plane figures) are congruent, then their areas are congruent.

Area Addition Postulate:

The surface area of a three-dimensional figure is the sum of the areas of all of its non-overlapping parts.

You can use a net and the Area Addition Postulate to find the surface area of a right prism.



From the net, you can see that that the surface area of the entire prism equals the sum of the figures that make up the net:

$$\text{Total surface area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E + \text{area } F$$

Using the formula for the area of a rectangle, you can see that the area of rectangle A is:

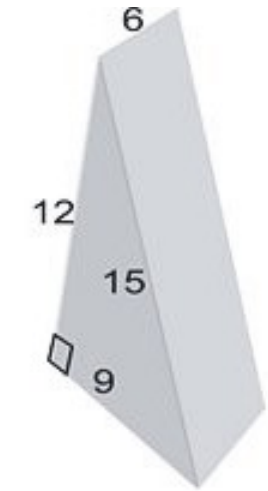
$$A = l \cdot w$$
$$A = 10 \cdot 5 = 50 \text{ square units}$$

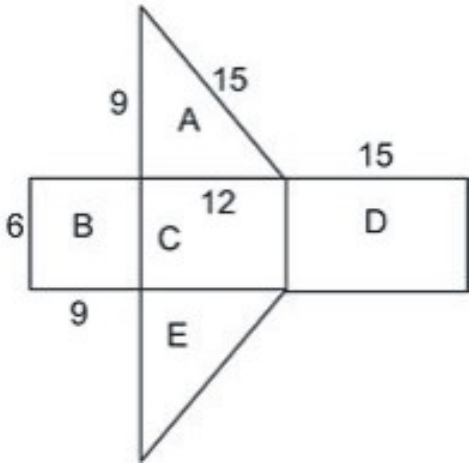
Similarly, the areas of the other rectangles are inserted back into the equation above.

$$\text{Total surface area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E + \text{area } F$$
$$\text{Total surface area} = (10 \cdot 5) + (10 \cdot 3) + (10 \cdot 5) + (10 \cdot 3) + (5 \cdot 3) + (5 \cdot 3)$$
$$\text{Total surface area} = 50 + 30 + 50 + 30 + 15 + 15$$
$$\text{Total surface area} = 190 \text{ square units}$$

Example 9

Use a net to find the surface area of the prism.





The area of the net is equal to the surface area of the figure. To find the area of the triangle, we use the formula:

$A = 1/2$ hb where h is the height of the triangle and b is its base.

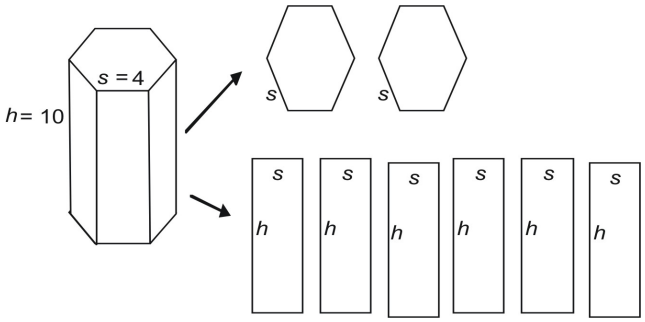
Note that triangles A and E are congruent so we can multiply the area of triangle A by 2.

$$\begin{aligned}
\text{area} &= \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E \\
&= 2(\text{area } A) + \text{area } B + \text{area } C + \text{area } D \\
&= 2\left[\frac{1}{2}(9 \cdot 12)\right] + (6 \cdot 9) + (6 \cdot 12) + (6 \cdot 15) \\
&= 108 + 54 + 72 + 90 \\
&= 324
\end{aligned}$$

Thus, the surface area is 324 square units .

Surface Area of a Prism Using Perimeter

This hexagonal prism has two regular hexagons for bases and six sides. Since all sides of the hexagon are congruent, all of the rectangles that make up the lateral sides of the three-dimensional figure are also congruent. You can break down the figure like this.



The surface area of the rectangular sides of the figure is called the **lateral area** of the figure. To find the lateral area, you could add up all of the areas of the rectangles.

$$\begin{aligned} \text{lateral area} &= 6 \text{ (area of one rectangle)} \\ &= 6 \ (s \cdot h) \\ &= 6sh \end{aligned}$$

Notice that $6s$ is the perimeter of the base. So another way to find the lateral area of the figure is to multiply the perimeter of the base by h, the height of the figure.

$$\begin{aligned} \text{lateral area} &= 6sh \\ &= (6s) \cdot h \\ &= (\text{perimeter})h \\ &= Ph \end{aligned}$$

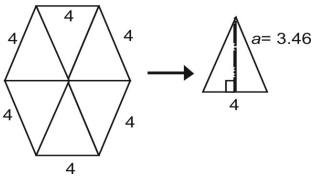
Substituting P, the perimeter, for $6s$, we get the formula for any lateral area of a right prism:

$$\text{lateral area of a prism} = Ph$$

Now we can use the formula to calculate the total surface area of the prism. Using P for the perimeter and B for the area of a base:

$$\begin{aligned}\text{Total surface area} &= \text{lateral area} + \text{area of 2 bases} \\ &= (\text{perimeter of base} \cdot \text{height}) + 2\ (\text{area of base}) \\ &= Ph + 2\ B\end{aligned}$$

To find the surface area of the figure above, first find the area of the bases. The regular hexagon is made of six congruent small triangles. The altitude of each triangle is the **apothem** of the polygon. Note: be careful here—we are talking about the altitude of the triangles, not the height of the prism. We find the length of the altitude of the triangle using the Pythagorean Theorem, $a = \sqrt{4^2 - 2^2} \approx 3.46$



So the area of each small triangle is:

$$\begin{aligned}A\ (\text{triangle}) &= \frac{1}{2}ab \\ &= \frac{1}{2}(3.46)(4) \\ &= 6.92\end{aligned}$$

The area of the entire hexagon is therefore:

$$\begin{aligned}A\ (\text{base}) &= 6\ (\text{area of triangle}) \\ &= 6 \cdot 6.92 \\ &= 41.52\end{aligned}$$

You can also use the formula for the area of a regular polygon to find the area of each base:

$$\begin{aligned}A\ (\text{polygon}) &= \frac{1}{2}aP \\ &= \frac{1}{2}(3.46)(24) \\ &= 41.52\end{aligned}$$

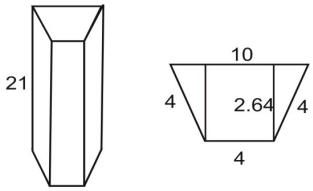
Now just substitute values to find the surface area of the entire figure above.

$$\begin{aligned}(\text{total surface area}) &= Ph + 2B \\ &= [(6 \cdot 4) \cdot 10] + 2(41.52) \\ &= (24 \cdot 10) + 83.04 \\ &= 240 + 83.04 \\ &= 323.04 \text{ square units}\end{aligned}$$

You can use the formula $A = Ph + 2B$ to find the surface area of any right prism.

Example 10

Use the formula to find the total surface area of the trapezoidal prism.



The dimensions of the trapezoidal base are shown. Set up the formula. We'll call the height of the entire prism H to avoid confusion with h, the height of each trapezoidal base.

$$\text{Total surface area} = PH + 2B$$

Now find the area of each trapezoidal base. You can do this by using the formula for the area of a trapezoid. (Note that the height of the trapezoid, 2.46 is small h.)

$$\begin{aligned}A &= \frac{1}{2}h(b_1 + b_2) \\ &= \frac{1}{2}(2.64)[10 + 4] \\ &= 18.48 \text{ square units}\end{aligned}$$

Now find the perimeter of the base.

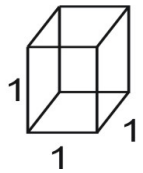
$$\begin{aligned}P &= 10 + 4 + 4 + 4 \\ &= 22\end{aligned}$$

Now find the total surface area of the solid.

$$\begin{aligned}(\text{total surface area}) &= Ph + 2B \\ &= (22)(21) + 2(18.48) \\ &= 462 + 36.96 \\ &= 498.96 \text{ square units}\end{aligned}$$

Volume of a Right Rectangular Prism

Volume is a measure of how much space a three-dimensional figure occupies. In everyday language, the volume tells you how much a three-dimensional figure can hold. The basic unit of volume is the cubic unit—cubic centimeter, cubic inch, cubic meter, cubic foot, and so on. Each basic cubic unit has a measure of 1 for its length, width, and height.



1 cubic unit

Two postulates that apply to volume are the Volume Congruence Postulate and the Volume Addition Postulate.

Volume Congruence Postulate

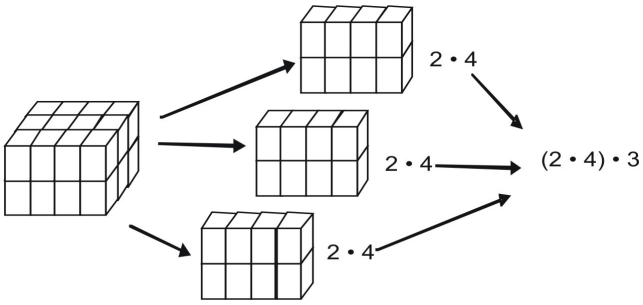
If two polyhedrons (or solids) are congruent, then their volumes are congruent.

Volume Addition Postulate

The volume of a solid is the sum of the volumes of all of its non-overlapping parts.

A right rectangular prism is a prism with rectangular bases and the angle between each base and its rectangular lateral sides is also a right angle. You can recognize a right rectangular prism by its "box" shape.

You can use the Volume Addition Postulate to find the volume of a right rectangular prism by counting boxes. The box below measures 2 units in height, 4 units in width, and 3 units in depth. Each layer has 2×4 cubes or 8 cubes.



Together, you get three groups of $2 \cdot 4$ so the total volume is:

$$\begin{aligned} V &= 2 \cdot 4 \cdot 3 \\ &= 24 \end{aligned}$$

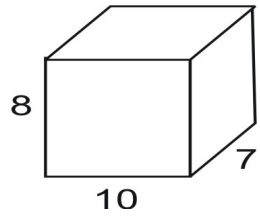
The volume is 24 cubic units.

This same pattern holds for any right rectangular prism. Volume is giving by the formula:

$$\text{Volume} = l \cdot w \cdot h$$

Example 11

Find the volume of this box.



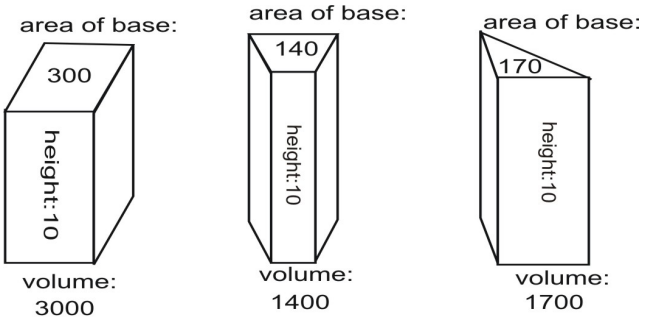
Use the formula for volume of a right rectangular prism.

$$\begin{aligned} V &= l \cdot w \cdot h \\ V &= 8 \cdot 10 \cdot 7 \\ V &= 560 \end{aligned}$$

So the volume of this rectangular prism is 560 cubic units.

Volume of a Right Prism

Looking at the volume of right prisms with the same height and different bases you can see a pattern. The computed area of each base is given below. The height of all three solids is the same, 10.



Putting the data for each solid into a table, we get:

Table 11.5:

Solid	Height	Area of base	Volume
Box	10	300	3000
Trapezoid	10	140	1400
Triangle	10	170	1700

The relationship in each case is clear. This relationship can be proved to establish the following formula for any right prism:

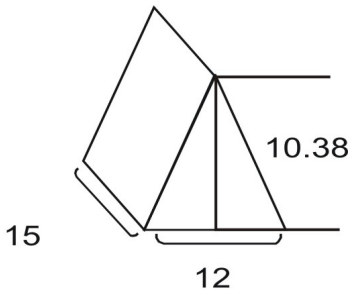
Volume of a Right Prism

The volume of a right prism is $V = Bh$.

where B is the area of the base of the three-dimensional figure, and h is the prism's height (also called altitude).

Example 12

Find the volume of the prism with a triangular equilateral base and the dimensions shown in centimeters.



To find the volume, first find the area of the base. It is given by:

$$A = \frac{1}{2}bh$$

The height (altitude) of the triangle is 10.38 cm. Each side of the triangle measures 12 cm. So the triangle has the following area.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(10.38)(12) \\ &= 62.28 \end{aligned}$$

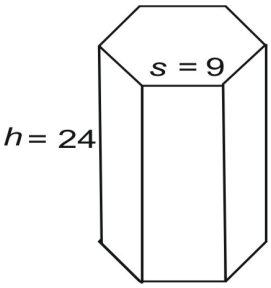
Now use the formula for the volume of the prism, $V = Bh$, where B is the area of the base (i.e., the area of the triangle) and h is the height of the prism. Recall that the "height" of the prism is the distance between the bases, so in this case the height of the prism is 15 cm. You can imagine that the prism is lying on its side.

$$\begin{aligned} V &= Bh \\ &= (62.28)(15) \\ &= 934.2 \end{aligned}$$

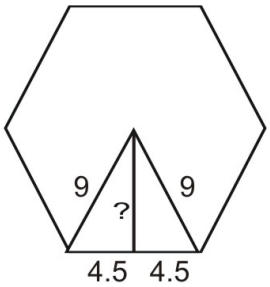
Thus, the volume of the prism is 934.2 cm^3.

Example 13

Find the volume of the prism with a regular hexagon for a base and $9 -$ inch *sides.*



You don't know the apothem of the figure's base. However, you do know that a regular hexagon is divided into six congruent equilateral triangles.



You can use the Pythagorean Theorem to find the apothem. The right triangle measures 9 by 4.5 by a, the apothem.

$$\begin{aligned} 9^2 &= 4.5^2 + n^2 \\ 81 &= 20.25 + n^2 \\ 60.75 &= n^2 \\ 7.785 &= n \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}(\text{apothem})\ (\text{perimeter}) \\ &= \frac{1}{2}(7.785)(6 \cdot 9) \\ &= 210.195 \text{ sq in} \end{aligned}$$

Thus, the volume of the prism is given by:

$$\begin{aligned} V &= Bh \\ &= 210.195 \cdot 24 \\ &= 5044.7 \text{ cu in} \end{aligned}$$

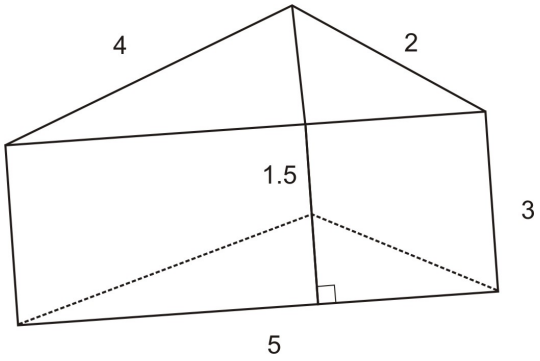
Review Questions

For each of the following find the surface area using

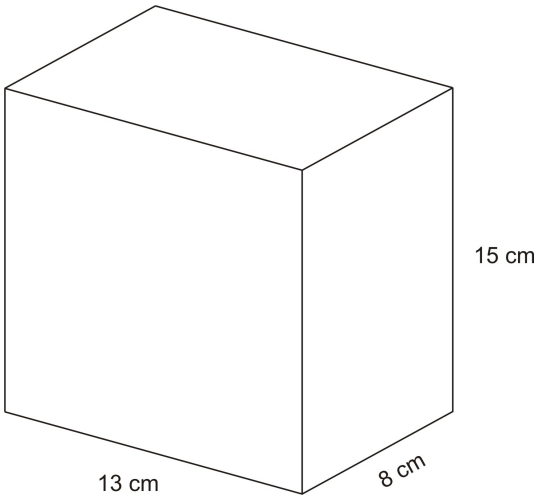
a. the method of nets and

b. the perimeter.

1.



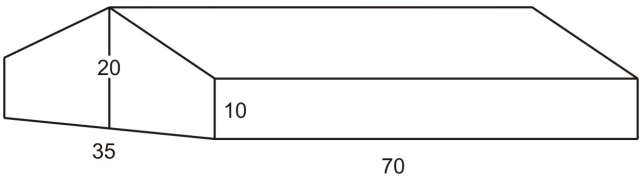
2.



3. The base of a prism is a right triangle whose legs are 3 and 4 and show height is 20. What is the total area of the prism?
4. A right hexagonal prism is 24 inches tall and has bases that are regular hexagons measuring 8 inches on a side. What is the total surface area?
5. What is the volume of the prism in problem #4?

For problems 6 and 7:

A barn is shaped like a pentagonal prism with dimensions shown in feet:



6. How many square feet (excluding the roof) are there on the surface of the barn to be painted?
7. If a gallon of paint covers 250 square feet , how many gallons of paint are needed to paint the barn?
8. A cardboard box is a perfect cube with an edge measuring 17 inches. How many cubic feet can it hold?
9. A swimming pool is 16 feet wide, 32 feet long and is uniformly 4 feet deep. How many cubic feet of water can it hold?
10. A cereal box has length 25 cm, width 9 cm and height 30 cm. How much cereal can it hold?

Review Answers

1. 40.5 in^2
2. 838 cm^2
3. 252 square units
4. 1484.6 square units
5. 7981.3 cubic units
6. 2450 square feet
7. 10 gallons of paint
8. 2.85 cubic feet (be careful here. The units in the problem are given in inches but the question asks for feet.)
9. 2048 cubic feet
10. 6750 cm^3

11.4 Cylinders

Learning Objectives

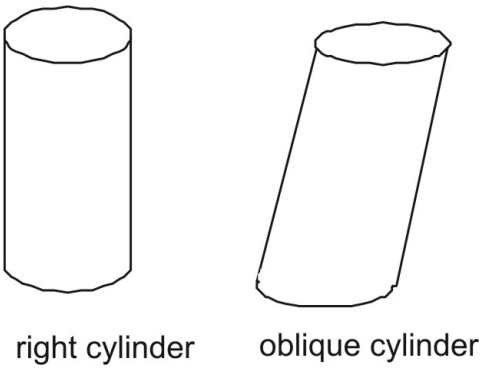
- Find the surface area of cylinders.
- Find the volume of cylinders.

- Find the volume of composite three-dimensional figures.

Introduction

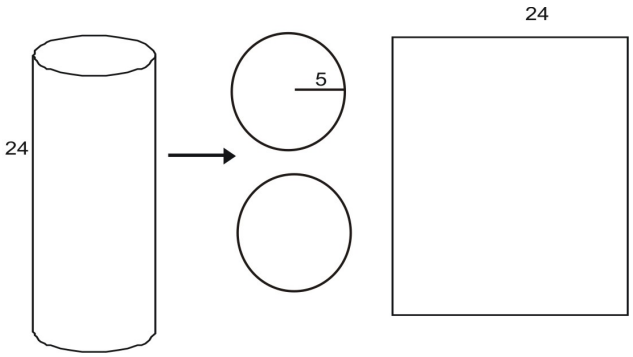
A **cylinder** is a three-dimensional figure with a pair of parallel and congruent circular ends, or **bases**. A cylinder has a single curved side that forms a rectangle when laid out flat.

As with prisms, cylinders can be *right* or *oblique*. The side of a right cylinder is perpendicular to its circular bases. The side of an oblique cylinder is not perpendicular to its bases.



Surface Area of a Cylinder Using Nets

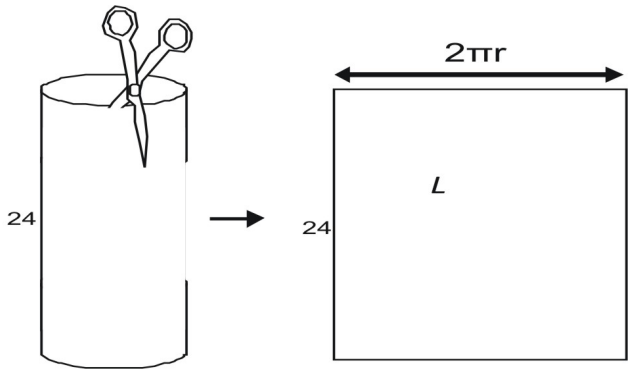
You can deconstruct a cylinder into a net.



The area of each base is given by the area of a circle:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(5)^2 \\ &= 25\pi \\ &\approx (25)(3.14) = 78.5 \end{aligned}$$

The area of the rectangular lateral area L is given by the product of a width and height. The height is given as 24. You can see that the width of the area is equal to the circumference of the circular base.



To find the width, imagine taking a can-like cylinder apart with a scissors. When you cut the lateral area, you see that it is equal to the circumference of the can's top. The circumference of a circle is given by $C = 2\pi r$,

the lateral area, L, is

$$\begin{aligned} L &= 2\pi rh \\ &= 2\pi(5)(24) \\ &= 240\pi \\ &\approx (240)(3.14) = 753.6 \end{aligned}$$

Now we can find the area of the entire cylinder using $A =$ (area of two bases)+(area of lateral side).

$$\begin{aligned} A &= 2(75.36) + 753.6 \\ &= 904.32 \end{aligned}$$

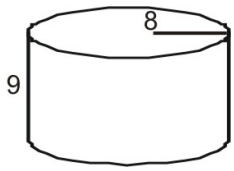
You can see that the formula we used to find the total surface area can be used for any right cylinder.

Area of a Right Cylinder

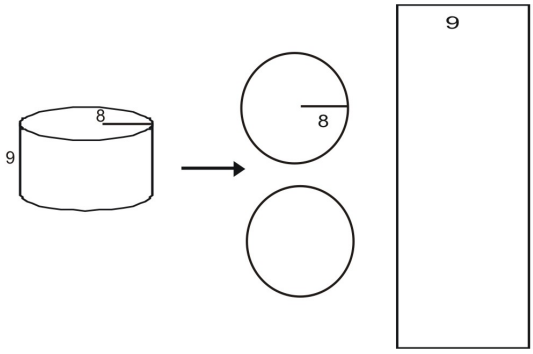
The surface area of a right cylinder, with radius r and height h is given by $A = 2B + L$, where B is the area of each base of the cylinder and L is the lateral area of the cylinder.

Example 1

Use a net to find the surface area of the cylinder.



First draw and label a net for the figure.



Calculate the area of each base.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(8)^2 \\ &= 64\pi \\ &\approx (64)(3.14) = 200.96 \end{aligned}$$

Calculate L.

$$\begin{aligned} L &= 2\pi rh \\ &= 2\pi(8)(9) \\ &= 144\pi \\ &\approx (240)(3.14) = 452.16 \end{aligned}$$

Find the area of the entire cylinder.

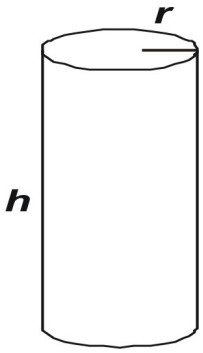
$$\begin{aligned} A &= 2(200.96) + 452.16 \\ &= 854.08 \end{aligned}$$

Thus, the total surface area is approximately 854.08 square units

Surface Area of a Cylinder Using a Formula

You have seen how to use nets to find the total surface area of a cylinder. The postulate can be broken down to create a general formula for all right cylinders.

$$A = 2B + L$$



Notice that the base, B, of any cylinder is:

$$B = \pi r^2$$

The lateral area, L, for any cylinder is:

$$\begin{aligned} L &= \text{width of lateral area} \cdot \text{height of cylinder} \\ &= \text{circumference of base} \cdot \text{height of cylinder} \\ &= 2\pi r \cdot h \end{aligned}$$

Putting the two equations together we get:

$$\begin{aligned} A &= 2B + L \\ &= 2(\pi r^2) + 2\pi r \cdot h \end{aligned}$$

Factoring out a $2\pi r$ from the equation gives:

$$A = 2\pi r(r + h)$$

The Surface Area of a Right Cylinder

A right cylinder with radius r and height h can be expressed as:

$$A = 2\pi r^2 + 2\pi rh$$

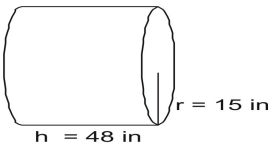
or:

$$A = 2\pi r(r + h)$$

You can use the formulas to find the area of any right cylinder.

Example 2

Use the formula to find the surface area of the cylinder.

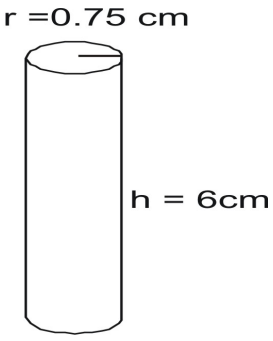


Write the formula and substitute in the values and solve.

$$\begin{aligned} A &= 2(\pi r^2) + 2\pi rh \\ &= 2(3.14)(15)(15) + 2(3.14)(15)(48) \\ &= 1413 + 4521.6 \\ &= 5934.6 \text{ square inches} \end{aligned}$$

Example 3

Find the surface area of the cylinder.



Write the formula and substitute in the values and solve.

$$\begin{aligned} A &= 2\pi r(r+h) \\ &= 2(3.14)(0.75)[0.75+6] \\ &= 31.7925 \text{ square inches} \end{aligned}$$

Example 4

Find the height of a cylinder that has radius 4 cm and *surface area of* 226.08 sq cm.

Write the formula with the given information and solve for h.

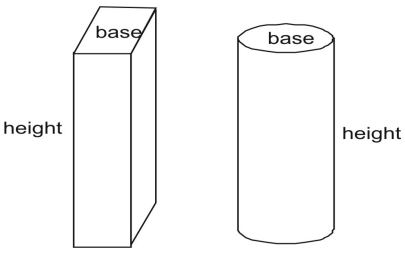
$$\begin{aligned} A &= 2\pi r(r+h) \\ 226.08 &= 2(3.14)(4)[4+h] \\ 226.08 &= 25.12[4+h] \\ 226.08 &= 100.48 + 25.12h \\ 5 &= h \end{aligned}$$

Volume of a Right Cylinder

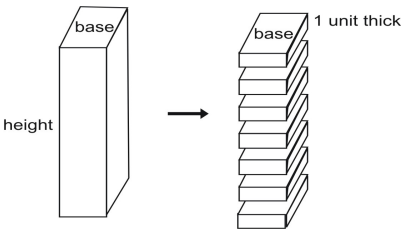
You have seen how to find the volume of any right prism.

$$V = Bh$$

where B is the area of the prism's base and h is the height of the prism.



As you might guess, right prisms and right cylinders are very similar with respect to volume. In a sense, a cylinder is just a "prism with round bases." One way to develop a formula for the volume of a cylinder is to compare it to a prism. Suppose you divided the prism above into slices that were 1 unit thick.



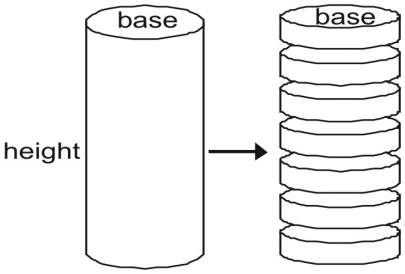
The volume of each individual slice would be given by the product of the area of the base and the height. Since the height for each slice is 1, the volume of a single slice would be:

$$\begin{aligned} V \text{ (single slice)} &= \text{area of base} \cdot \text{height} \\ &= B \cdot 1 \\ &= B \end{aligned}$$

Now it follows that the volume of the entire prism is equal to the area of the base multiplied by the number of slices. If there are h slices, then:

$$\begin{aligned} V \text{ (whole prism)} &= B \cdot \text{number of slices} \\ &= Bh \end{aligned}$$

Of course, you already know this formula from prisms. But now you can use the same idea to obtain a formula for the volume of a cylinder.



Since the height of each unit slice of the cylinder is 1, each slice has a volume of $B \cdot (1)$, or B. Since the base has an area of πr^2, each slice has a volume of πr^2 and:

$$\begin{aligned} V \text{ (whole cylinder)} &= B \cdot \text{number of slices} \\ &= Bh \\ &= \pi r^2 h \end{aligned}$$

This leads to a postulate for the volume of any right cylinder.

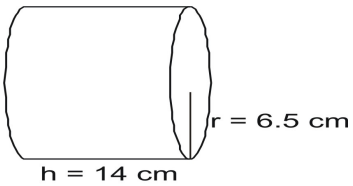
Volume of a Right Cylinder

The volume of a right cylinder with radius r and height h can be expressed as:

$$\text{Volume} = \pi r^2 h$$

Example 5

Use the postulate to find the volume of the cylinder.



Write the formula from the postulate. Then substitute in the values and solve.

$$\begin{aligned} V &= \pi r^2 h \\ &= (3.14)(6.5)(6.5)(14) \\ &= 1857.31 \text{ cubic inches} \end{aligned}$$

Example 6

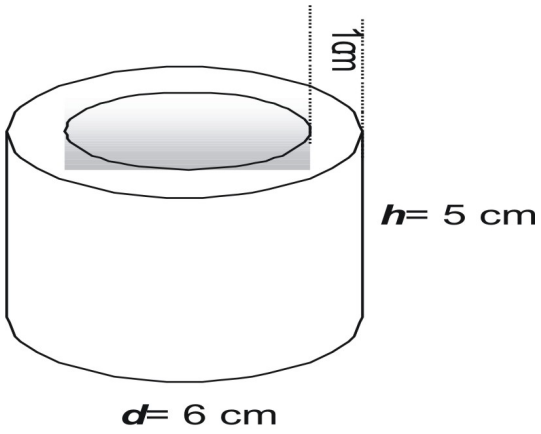
What is the radius of a cylinder with height 10 cm *and a volume of* 250π*?*

Write the formula. Solve for r.

$$\begin{aligned} V &= \pi r^2 h \\ 250\pi &= \pi r^2 (10) \\ 250\pi / 10\pi &= r^2 \\ 25 &= r^2 \\ 5 &= r \end{aligned}$$

Composite Solids

Suppose this pipe is made of metal. How can you find the volume of metal that the pipe is made of?



The basic process takes three steps.

Step 1: Find the volume of the entire cylinder as if it had no hole.

Step 2: Find the volume of the hole.

Step 3: Subtract the volume of the hole from the volume of the entire cylinder.

Here are the steps carried out. First, use the formula to find the volume of the entire cylinder. Note that since d, the diameter of the pipe, is 6 cm, the radius is half of the diameter, or 3 cm.

$$\begin{aligned} V &= \pi r^2 h \\ &= (3.14)(3)(3)(5) \\ &= 141.3 \text{ cubic inches} \end{aligned}$$

Now find the volume of the inner empty "hole" in the pipe. Since the pipe is 1 inch thick, the diameter of the hole is 2 inches less than the diameter of the outer part of the pipe.

$$\begin{aligned} d \text{ (inner pipe)} &= d \text{ (outer pipe)} - 2 \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The radius of the hole is half of 4 or 2.

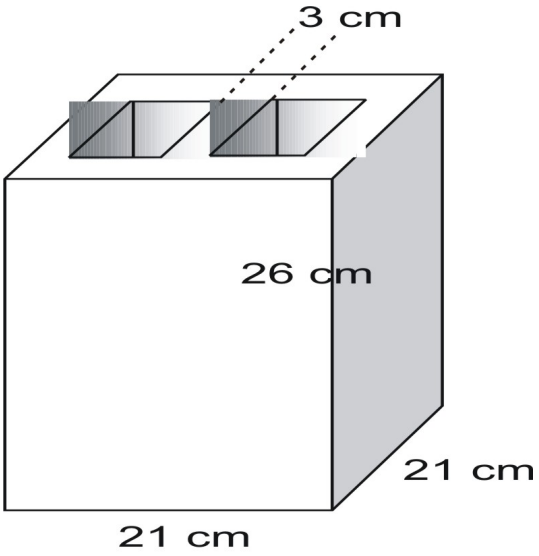
$$\begin{aligned} V &= \pi r^2 h \\ &= (3.14)(2)(2)(5) \\ &= 62.8 \text{ cubic inches} \end{aligned}$$

Now subtract the hole from the entire cylinder.

$$\begin{aligned} V(\text{pipe}) &= V\ (\text{cylinder}) - V\ (\text{hole}) \\ &= 141.3 - 62.8 \\ &= 78.5 \text{ cubic inches} \end{aligned}$$

Example 7

Find the solid volume of this cinder block. Its edges are 3 cm *thick all around. The two square holes are identical in size.*



Find the volume of the entire solid block figure. Subtract the volume of the two holes.

To find the volume of the three-dimensional figure:

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 21 \cdot 21 \cdot 26 \\ &= 11,466 \text{ cubic cm} \end{aligned}$$

Now find the length of the sides of the two holes. The width of the entire block is 21 cm. This is equal to:

$$\begin{aligned}\text{width of block} &= 3 \text{ edges} + 2 \text{ holes} \\ 21 &= 3(3 \text{ cm}) + 2n \\ 21 &= 9 + 2n \\ 12 &= 2n \\ 6 &= n\end{aligned}$$

So the sides of the square holes are 6 cm by 6 cm.

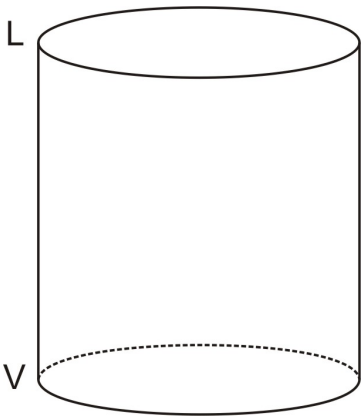
Now the volume of each square hole is:

$$\begin{aligned}V &= l \cdot w \cdot h \\ &= 6 \cdot 6 \cdot 26 \\ &= 936 \text{ cubic cm}\end{aligned}$$

Finally, subtract the volume of the two holes from the volume of the entire brick.

$$\begin{aligned}V(\text{block}) &= V(\text{solid}) - V(\text{holes}) \\ &= 11,466 - 2(936) \\ &= 9,594 \text{ cubic cm}\end{aligned}$$

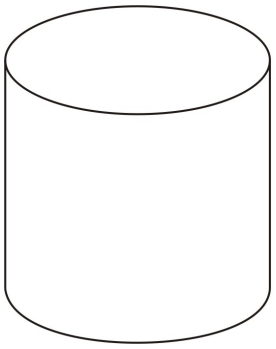
Review Questions



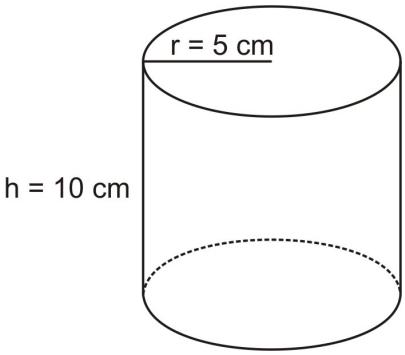
Complete the following sentences. They refer to the figure above.

1. The figure above is a _____

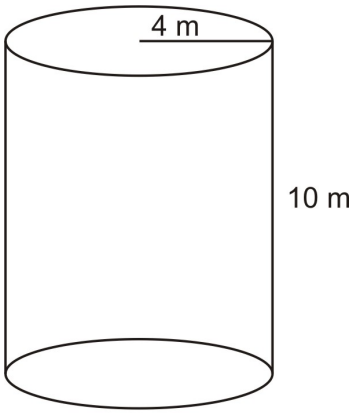
2. The shape of the lateral face of the figure is _____-_____
3. The shape of a base is a(n) _____
4. Segment LV is the _____
5. Draw the net for this cylinder and use the net to find the surface area of the cylinder.



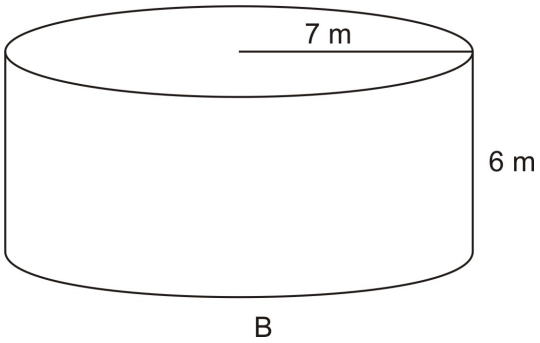
6. Use the formula to find the volume of this cylinder.



7. Matthew's favorite mug is a cylinder that has a base area of 9 square inches and a height of 5 inches. How much coffee can he put in his mug?
8. Given the following two cylinders which of the following statements is true:



A

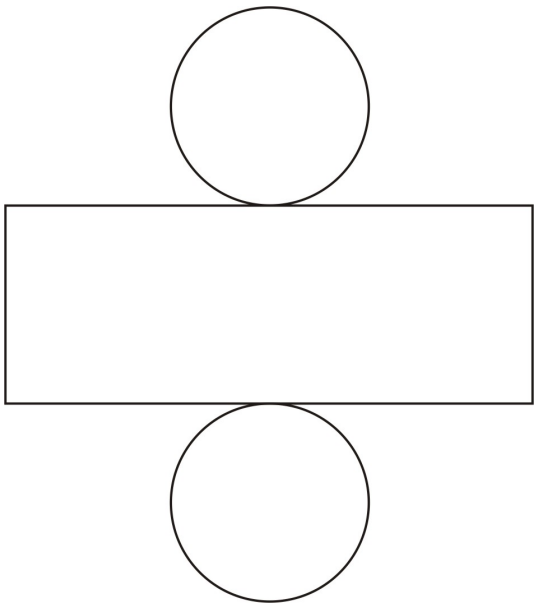


(a) Volume of $A <$ Volume of B
(b) Volume of $A >$ Volume of B
(c) Volume of $A =$ Volume of B

9. Suppose you work for a company that makes cylindrical water tanks. A customer wants a tank that measures 9 meters in height and 2 meters in diameter. How much metal should you order to make this tank?
10. If the radius of a cylinder is doubled what effect does the doubling have on the volume of this cylinder? Explain your answer.

Review Answers

1. Cylinder
2. Rectangle
3. Circle
4. Height
5. Surface area $= 266\pi$ in^2



6. 250π cm^2
7. Volume $= 45$ in^3
8. Volume of $A <$ volume of B
9. 18π m^2
10. The volume will be quadrupled

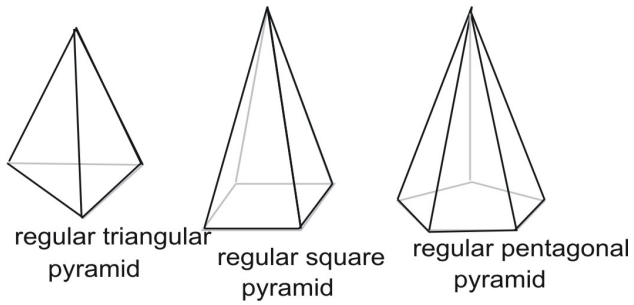
11.5 Pyramids

Learning Objectives

- Identify pyramids.
- Find the surface area of a pyramid using a net or a formula.
- Find the volume of a pyramid.

Introduction

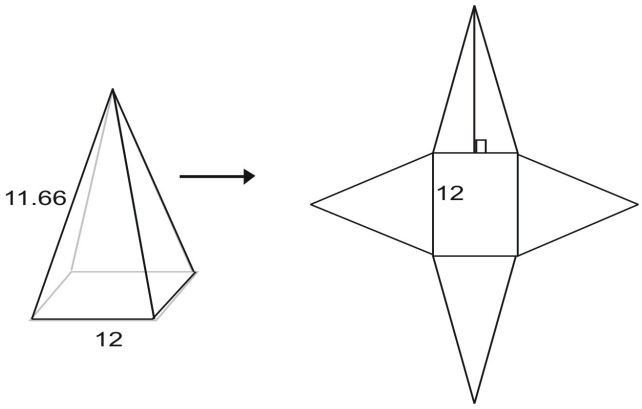
A pyramid is a three-dimensional figure with a single base and a three or more non-parallel sides that meet at a single point above the base. The sides of a pyramid are triangles.



A **regular pyramid** is a pyramid that has a regular polygon for its base and whose sides are all congruent triangles.

Surface Area of a Pyramid Using Nets

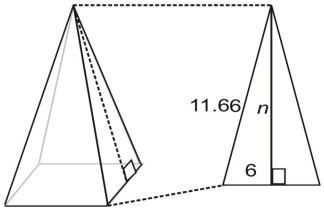
You can deconstruct a pyramid into a net.



To find the surface area of the figure using the net, first find the area of the base:

$$\begin{aligned} A &= s^2 \\ &= (12)(12) \\ &= 144 \text{ square units} \end{aligned}$$

Now find the area of each isosceles triangle. Use the Pythagorean Theorem to find the height of the triangles. This height of each triangle is called the **slant height** of the pyramid. The slant height of the pyramid is the altitude of one of the triangles. Notice that the slant height is larger than the altitude of the triangle.



We'll call the slant height n for this problem. Using the Pythagorean Theorem:

$$\begin{aligned} (11.66)^2 &= 6^2 + n^2 \\ 136 &= 36 + n^2 \\ 100 &= n^2 \\ 10 &= n \end{aligned}$$

Now find the area of each triangle:

$$\begin{aligned} A &= \frac{1}{2}hb \\ &= \frac{1}{2}(10)(12) \\ &= 60 \text{ square units} \end{aligned}$$

As there are 4 triangles:

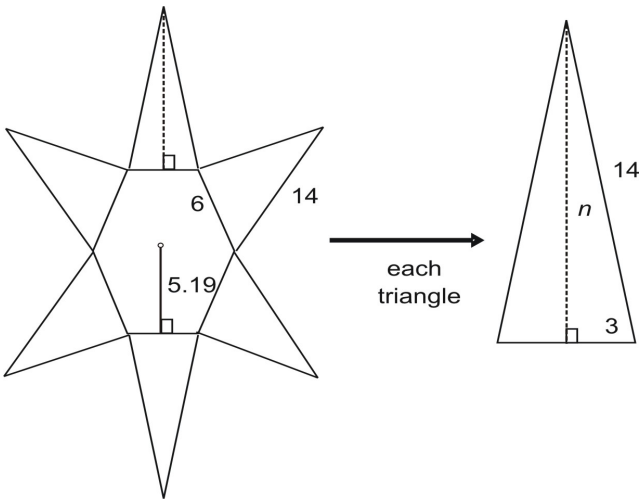
$$\begin{aligned} A(\text{triangles}) &= 4(60) \\ &= 240 \text{ square units} \end{aligned}$$

Finally, add the total area of the triangles to the area of the base.

$$\begin{aligned} A(\text{total}) &= A(\text{triangles}) + A(\text{base}) \\ &= 240 + 144 \\ &= 384 \text{ square units} \end{aligned}$$

Example 1

Use the net to find the total area of the regular hexagonal pyramid with an apothem of 5.19. *The dimensions are given in centimeters.*



The area of the hexagonal base is given by the formula for the area of a regular polygon. Since each side of the hexagon measures 6 cm, the perimeter is $6 \cdot 6$ or 36 cm. The apothem, or perpendicular distance to the center of the hexagon is 5.19 cm.

$$\begin{aligned} A &= \frac{1}{2}(\text{apothem})(\text{perimeter}) \\ &= \frac{1}{2}(5.19)(36) \\ &= 93.42 \text{ square cm} \end{aligned}$$

Using the Pythagorean Theorem to find the slant height of each lateral triangle.

$$\begin{aligned} (14)^2 &= 3^2 + n^2 \\ 196 &= 9 + n^2 \\ 187 &= n^2 \\ 13.67 &= n \end{aligned}$$

Now find the area of each triangle:

$$\begin{aligned} A &= \frac{1}{2}hb \\ &= \frac{1}{2}(13.67)(6) \\ &= 41 \text{ square cm} \end{aligned}$$

Together, the area of all six triangles that make up the lateral sides of the pyramid are

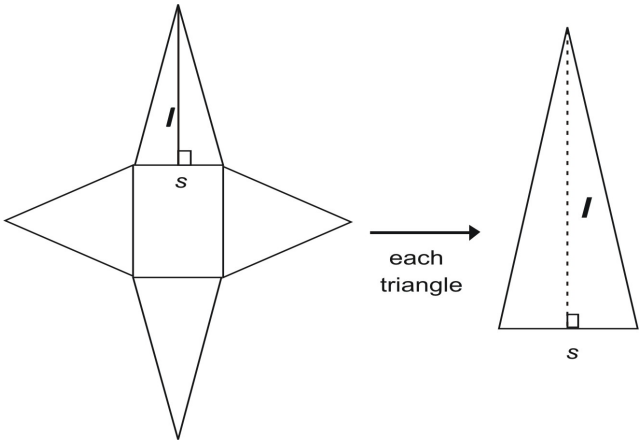
$$\begin{aligned} A &= 6(\text{area of each triangle}) \\ &= 6 \cdot 41 \\ &= 246 \text{ square cm} \end{aligned}$$

Add the area of the lateral sides to the area of the hexagonal base.

$$\begin{aligned} A(\text{total}) &= A(\text{triangles}) + A(\text{base}) \\ &= 246 + 93.42 \\ &= 339.42 \text{ square cm} \end{aligned}$$

Surface Area of a Regular Pyramid

To get a general formula for the area of a regular pyramid, look at the net for this square pyramid.



The slant height of each lateral triangle is labeled l (the lowercase letter L), and the side of the regular polygon is labeled s. For each lateral triangle, the area is:

$$A = \frac{1}{2}ls$$

There are n triangles in a regular polygon—e.g., $n = 3$ for a triangular pyramid, $n = 4$ for a square pyramid, $n = 5$ for a pentagonal pyramid. So the total area, L, of the lateral triangles is:

$$\begin{aligned} L &= n \cdot (\text{area of each lateral triangle}) \\ &= n\left(\frac{1}{2}ls\right) \end{aligned}$$

If we rearrange the above equation we get:

$$L = \left(\frac{1}{2}ln \cdot s\right)$$

Notice that $n \cdot s$ is just the perimeter, P, of the regular polygon. So we can substitute P into the equation to get the following postulate.

$$L = \left(\frac{1}{2}lP\right)$$

To get the total area of the pyramid, add the area of the base, B, to the equation above.

$$A = \frac{1}{2}lP + B$$

Area of a Regular Pyramid

The surface area of a regular pyramid is

$$A = \frac{1}{2}lP + B$$

where l is the slant height of the pyramid and P is the perimeter of the regular polygon that forms the pyramid's base, and B is the area of the base.

Example 2

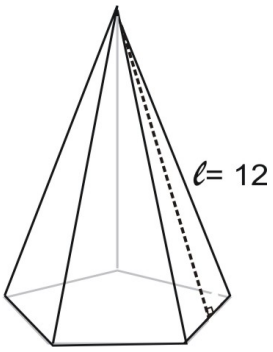
A tent without a bottom has the shape of a hexagonal pyramid with a slant height l of 30 feet. *The sides of the hexagonal perimeter of the figure each measure* 8 feet. *Find the surface area of the tent that exists above ground.*

For this problem, B is zero because the tent has no bottom. So simply calculate the lateral area of the figure.

$$\begin{aligned} A &= \frac{1}{2}lP + B \\ &= \frac{1}{2}lP + 0 \\ &= \frac{1}{2}lP \\ &= \frac{1}{2}(30)(6 \cdot 8) \\ &= 720 \text{ square feet} \end{aligned}$$

Example 3

A pentagonal pyramid has a slant height l of 12 cm. *The sides of the pentagonal perimeter of the figure each measure* 9 cm. *The apothem of the figure is* 6.19 cm. *Find the surface area of the figure.*



First find the lateral area of the figure.

$$\begin{aligned} L &= \frac{1}{2}lP \\ &= \frac{1}{2}(12)(5 \cdot 9) \\ &= 270 \text{ square cm} \end{aligned}$$

Now use the formula for the area of a regular polygon to find the area of the base.

$$\begin{aligned} A &= \frac{1}{2}(\text{apothem})(\text{perimeter}) \\ &= \frac{1}{2}(6.19)(5 \cdot 9) \\ &= 139.3605 \text{ square cm} \end{aligned}$$

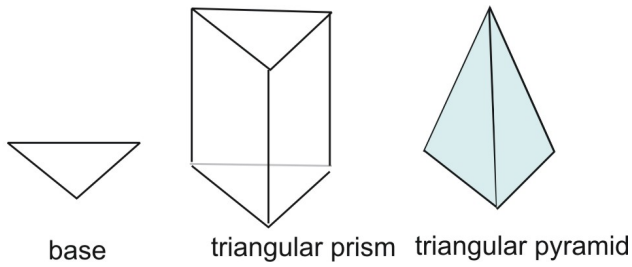
Finally, add these together to find the total surface area.

$$139.3605 + 270 \approx 409.36 \text{ square centimeters}$$

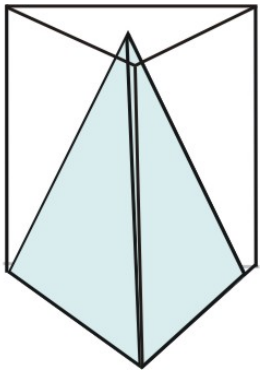
Estimate the Volume of a Pyramid and Prism

Which has a greater volume, a prism or a pyramid, if the two have the same base and height? To find out, compare prisms and pyramids that have congruent bases and the same height.

Here is a base for a triangular prism and a triangular pyramid. Both figures have the same height. Compare the two figures. Which one appears to have a greater volume?



The prism may appear to be greater in volume. But how can you *prove* that the volume of the prism is greater than the volume of the pyramid? Put one figure inside of the other. The figure that is smaller will fit inside of the other figure.

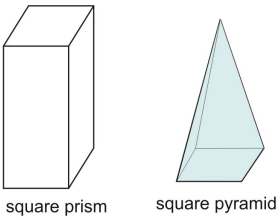


This is shown in the diagram on the above. Both figures have congruent bases and the same height. The pyramid clearly fits inside of the prism. So the volume of the pyramid must be smaller.

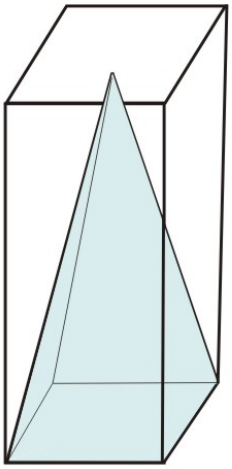
Example 4

Show that the volume of a square prism is greater than the volume of a square pyramid.

Draw or make a square prism and a square pyramid that have congruent bases and the same height.



Now place the one figure inside of the other. The pyramid fits inside of the prism. So when two figures have the same height and the same base, the prism's volume is greater.

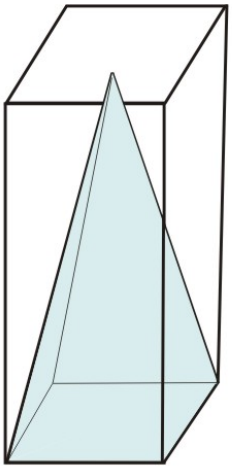


In general, when you compare two figures that have congruent bases and are equal in height, the prism will have a greater volume than the pyramid.

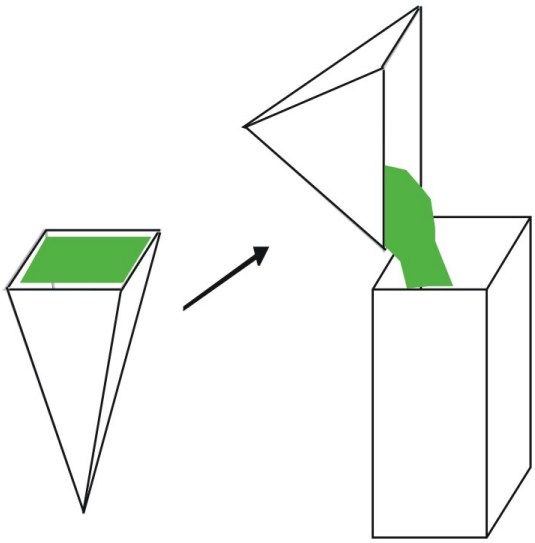
The reason should be obvious. At the "bottom," both figures start out the same—with a square base. But the pyramid quickly slants inward, "cutting away" large amounts of material while the prism does not slant.

Find the Volume of a Pyramid and Prism

Given the figure above, in which a square pyramid is placed inside of a square prism, we now ask: how many of these pyramids would fit inside of the prism?



To find out, obtain a square prism and square pyramid that are both hollow, both have no bottom, and both have the same height and congruent bases.



Now turn the figures upside down. Fill the pyramid with liquid. How many full pyramids of liquid will fill the prism up to the top?

In fact, it takes exactly three full pyramids to fill the prism. Since the volume of the prism is:

$$V = Bh$$

where B stands for the area of the base and h is the height of the prism, we can write:

$$3 \cdot (\text{volume of square pyramid}) = (\text{volume of square prism})$$

or:

$$(\text{volume of square pyramid}) = \frac{1}{3}(\text{volume of square prism})$$

And, since the volume of a square prism is Bh, we can write:

$$V = \frac{1}{3}Bh$$

This can be written as the Volume Postulate for pyramids.

Volume of a Pyramid

Given a right pyramid with a base that has area B and height h:

$$V = \frac{1}{3}Bh$$

Example 5

Find the volume of a pyramid with a right triangle base with sides that measure 5 cm, 8 cm, *and* 9.43 cm. *The height of the pyramid is* 15 cm.

First find the area of the base. The longest of the three sides that measure 5 cm, 8 cm, and 9.43 cm must be the hypotenuse, so the two shorter sides are the legs of the right triangle.

$$\begin{aligned} A &= \frac{1}{2}hb \\ &= \frac{1}{2}(5)(8) \\ &= 20 \text{ square cm} \end{aligned}$$

Now use the postulate for the volume of a pyramid.

$$\begin{aligned} V(\text{pyramid}) &= \frac{1}{3}Bh \\ &= \frac{1}{3}(20)(15) \\ &= 100 \text{ cubic cm} \end{aligned}$$

Example 6

Find the altitude of a pyramid with a regular pentagonal base. The figure has an apothem of 10.38 cm, 12 cm *sides, and a volume of* 2802.6 cu cm.

First find the area of the base.

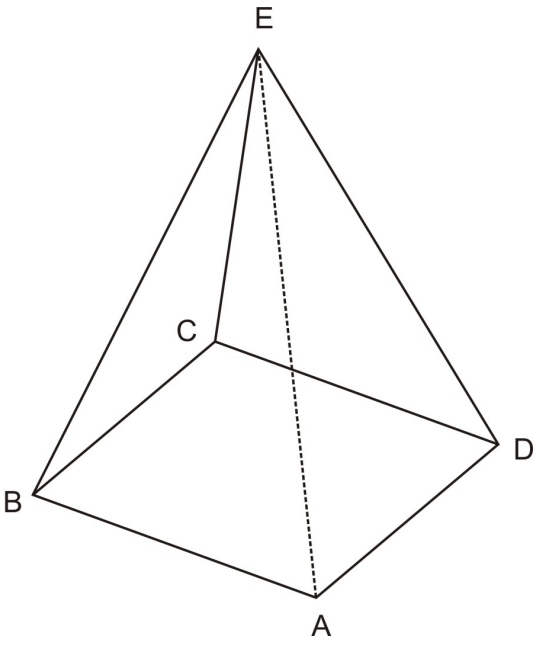
$$\begin{aligned} A(\text{base}) &= \frac{1}{2}aP \\ &= \frac{1}{2}(10.38)(5 \cdot 12) \\ &= 311.4 \text{ square cm} \end{aligned}$$

Now use the value for the area of the base and the postulate to solve for h.

$$\begin{aligned} V(\text{pyramid}) &= \frac{1}{3}Bh \\ 2802.6 &= \frac{1}{3}(311.4)h \\ 27 &= h \end{aligned}$$

Review Questions

Consider the following figure in answering questions 1 – 4.

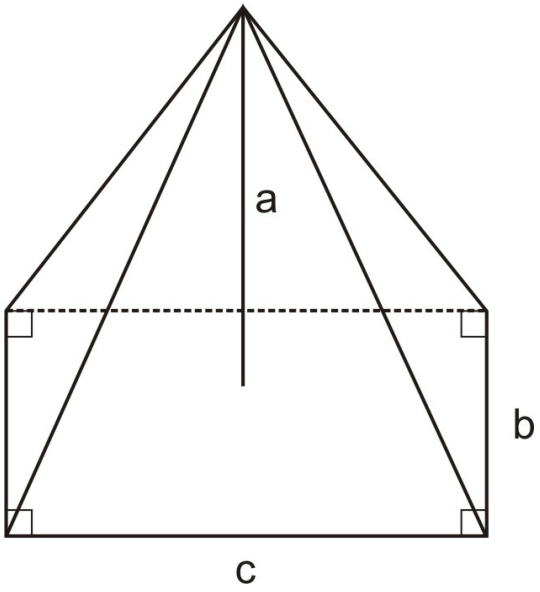


1. What type of pyramid is this?
2. Triangle ABE is what part of the figure?
3. Segment AE is $a(n)$ _____ of the figure.
4. Point E is the _____
5. How many faces are there on a pyramid whose base has 16 sides?

A right pyramid has a regular hexagon for a base. Each edge measures $2\sqrt{22}$. Find

6. The lateral surface area of the pyramid
7. The total surface area of the pyramid
8. The volume of the pyramid

9. The Transamerica Building in San Francisco is a pyramid. The length of each edge of the square base is 149 feet and the slant height of the pyramid is 800 feet. What is the lateral area of the pyramid? How tall is the building?
10. Given the following pyramid:



With c = 22 mm, b = 17 mm and volume = 1433.67 mm^3 what is the value of a?

Review Answers

1. Rectangular pyramid
2. Lateral face
3. Edge
4. Apex
5. 16
6. 135.6 square units
7. 200.55 square units
8. 171.84 cubic units
9. Lateral surface area = 238,400 square feet Height = 796.5 feet
10. $A = 11.5$ mm

11.6 Cones

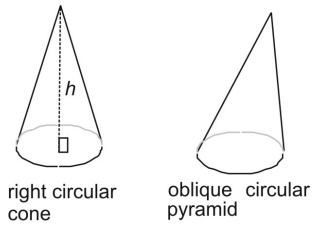
Learning Objectives

- Find the surface area of a cone using a net or a formula.

- Find the volume of a cone.

Introduction

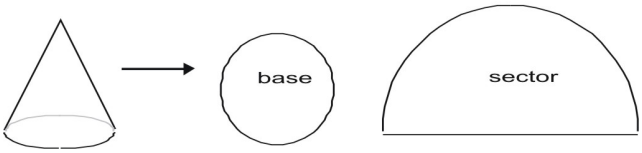
A cone is a three-dimensional figure with a single curved base that tapers to a single point called an *apex*. The base of a cone can be a circle or an oval of some type. In this chapter, we will limit the discussion to circular cones. The apex of a right cone lies above the center of the cone's circle. In an oblique cone, the apex is not in the center.



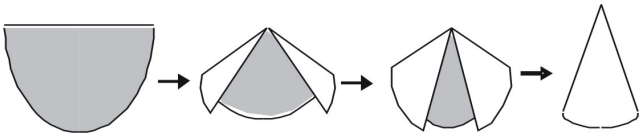
The height of a cone, h, is the perpendicular distance from the center of the cone's base to its apex.

Surface Area of a Cone Using Nets

Most three-dimensional figures are easy to deconstruct into a net. The cone is different in this regard. Can you predict what the net for a cone looks like? In fact, the net for a cone looks like a small circle and a sector, or part of a larger circle.



The diagrams below show how the half-circle sector folds to become a cone.



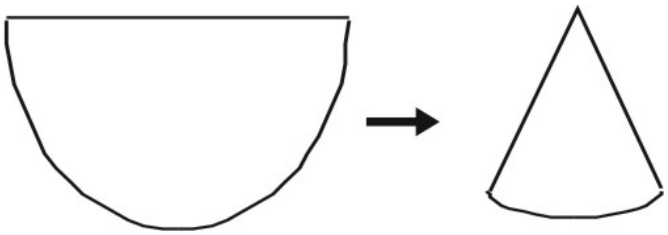
Note that the circle that the sector is cut from is much larger than the base of the cone.

Example 1

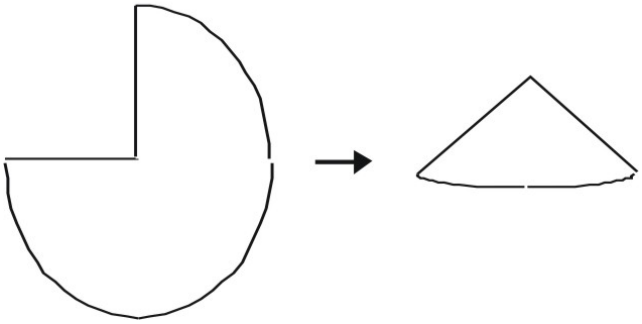
Which sector will give you a taller cone—a half circle or a sector that covers three-quarters of a circle? Assume that both sectors are cut from congruent circles.

Make a model of each sector.

The half circle makes a cone that has a height that is about equal to the radius of the semi-circle.



The three-quarters sector gives a cone that has a wider base (greater diameter) but its height as not as great as the half-circle cone.



Example 2

Predict which will be greater in height—a cone made from a half-circle sector or a cone made from a one-third-circle sector. Assume that both sectors are cut from congruent circles.

The relationship in the example above holds true—the greater (in degrees) the sector, the smaller in height of the cone. In other words, the fraction 1/3 is less than 1/2, so a one-third sector will create a cone with greater height than a half sector.

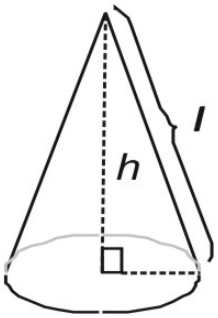
Example 3

Predict which will be greater in diameter—a cone made from a half-circle sector or a cone made from a one-third-circle sector. Assume that the sectors are cut from congruent circles

Here you have the opposite relationship—the larger (in degrees) the sector, the *greater* the diameter of the cone. In other words, 1/2 is greater than 1/3, so a one-half sector will create a cone with greater diameter than a one-third sector.

Surface Area of a Regular Cone

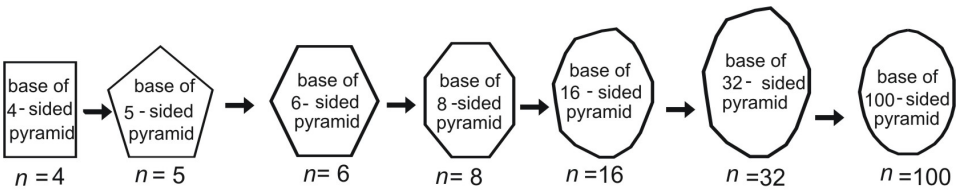
The surface area of a regular pyramid is given by:



$$A = \left(\frac{1}{2}lP\right) + B$$

where l is the slant height of the figure, P is the perimeter of the base, and B is the area of the base.

Imagine a series of pyramids in which n, the number of sides of each figure's base, increases.



As you can see, as n increases, the figure more and more resembles a circle. So in a sense, a circle approaches a polygon with an infinite number of sides that are infinitely small.

In the same way, a cone is like a pyramid that has an infinite number of sides that are infinitely small in length.

Given this idea, it should come as no surprise that the formula for finding the total surface area of a cone is similar to the pyramid formula. The only difference between the two is that the pyramid uses P, the perimeter of the base, while a cone uses C, the circumference of the base.

$$A(\text{pyramid}) = \frac{1}{2}lP + B$$
$$A(\text{cone}) = \frac{1}{2}lC + B$$

Surface Area of a Right Cone

The surface area of a right cone is given by:

$$A = \frac{1}{2}lC + B$$

Since the circumference of a circle is $2\pi r$:

$$\begin{aligned} A(\text{cone}) &= \frac{1}{2}lC + B \\ &= \frac{1}{2}l(2\pi r) + B \\ &= \pi rl + B \end{aligned}$$

You can also express B as πr^2 to get:

$$\begin{aligned} A(\text{cone}) &= \pi rl + B \\ &= \pi rl + \pi r^2 \\ &= \pi r(l + r) \end{aligned}$$

Any of these forms of the equation can be used to find the surface area of a right cone.

Example 4

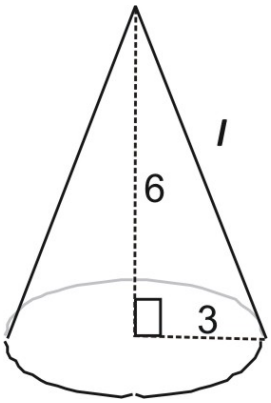
Find the total surface area of a right cone with a radius of 8 cm *and a slant height of* 10 cm.

Use the formula:

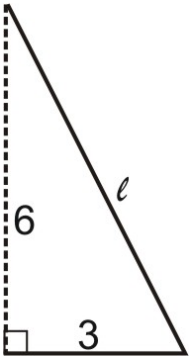
$$\begin{aligned} A(\text{cone}) &= \pi r(l + r) \\ &= (3.14)(8)[10 + 8] \\ &= 452.16 \text{ square cm} \end{aligned}$$

Example 5

Find the total surface area of a right cone with a radius of 3 feet *and an altitude (not slant height) of* 6 feet.



Use the Pythagorean Theorem to find the slant height:



$$\begin{aligned} l^2 &= r^2 + h^2 \\ &= (3)^2 + (6)^2 \\ &= 45 \\ l &= \sqrt{45} \\ &= 3\sqrt{5} \end{aligned}$$

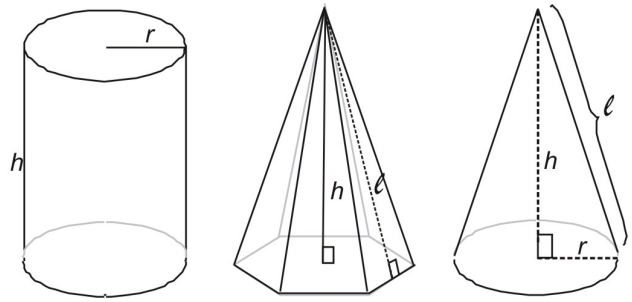
Now use the area formula.

$$\begin{aligned} A(\text{cone}) &= \pi r(l + r) \\ &= (3.14)(3)[3\sqrt{5} + 3] \\ &\approx 91.5 \text{ square cm} \end{aligned}$$

Volume of a Cone

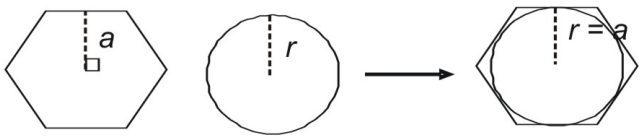
Which has a greater volume, a pyramid, cone, or cylinder if the figures have bases with the same "diameter" (i.e., distance across the base) and the same altitude? To find out, compare pyramids, cylinders, and cones that have bases with equal diameters and the same altitude.

Here are three figures that have the same dimensions—cylinder, a right regular hexagonal pyramid, and a right circular cone. Which figure appears to have a greater volume?



It seems obvious that the volume of the cylinder is greater than the other two figures. That's because the pyramid and cone taper off to a single point, while the cylinder's sides stay the same width.

Determining whether the pyramid or the cone has a greater volume is not so obvious. If you look at the bases of each figure you see that the apothem of the hexagon is congruent to the radius of the circle. You can see the relative size of the two bases by superimposing one onto the other.



From the diagram you can see that the hexagon is slightly larger in area than the circle. So it follows that the volume of the right hexagonal regular pyramid would be greater than the volume of a right circular cone. And indeed it is, but only because the area of the base of the hexagon is slightly greater than the area of the base of the circular cone.

The formula for finding the volume of each figure is virtually identical. Both formulas follow the same basic form:

$$V = \frac{1}{3}Bh$$

Since the base of a circular cone is, by definition, a circle, you can substitute the area of a

circle, πr^2 for the base of the figure. This is expressed as a volume postulate for cones.

Volume of a Right Circular Cone

Given a right circular cone with height h and a base that has radius r:

$$\begin{aligned} V &= \frac{1}{3}Bh \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

Example 6

Find the volume of a right cone with a radius of 9 cm *and a height of* 16 cm.

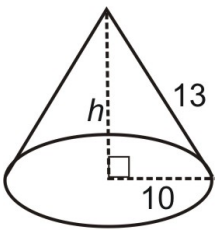
Use the formula:

$$\begin{aligned} V &= \frac{1}{3}Bh \\ &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}(3.14)(9)(9)(16) \\ &= 1356.48 \text{ cubic cm} \end{aligned}$$

Example 7

Find the volume of a right cone with a radius of 10 feet *and a slant height of* 13 feet.

Use the Pythagorean theorem to find the height:



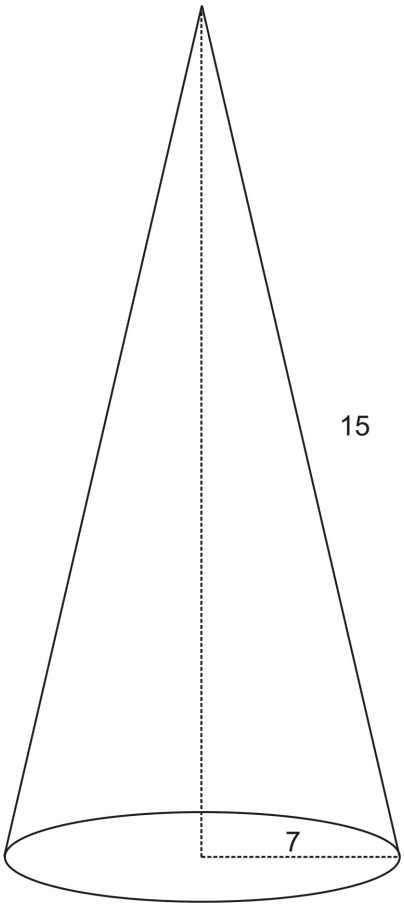
$$\begin{aligned} r^2 + h^2 &= l^2 \\ (10)^2 + h^2 &= (13)^2 \\ h^2 &= (13)^2 - (10)^2 \\ h^2 &= 69 \\ h &= 8.31 \end{aligned}$$

Now use the volume formula.

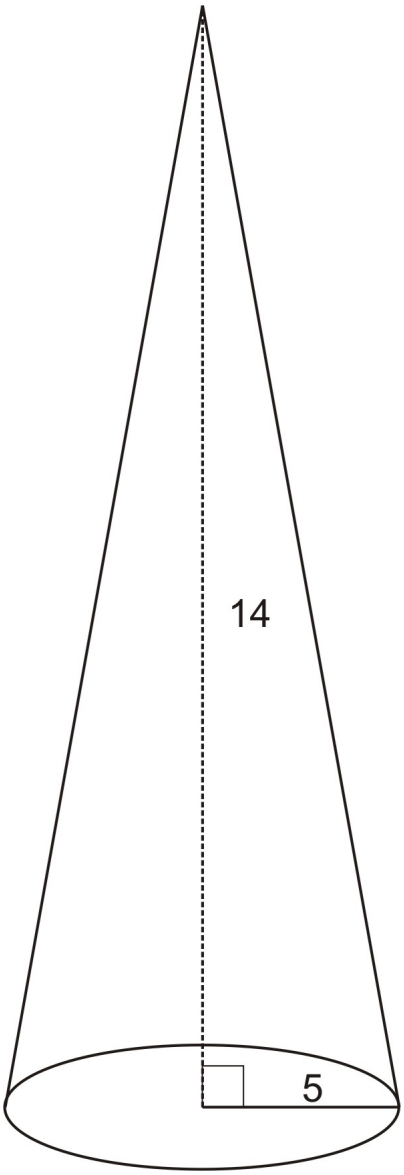
$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}(3.14)(10)(10)(8.31) \\ &= 869.78 \text{ cubic feet} \end{aligned}$$

Review Questions

1. Find the surface area of



2. Find the surface area of

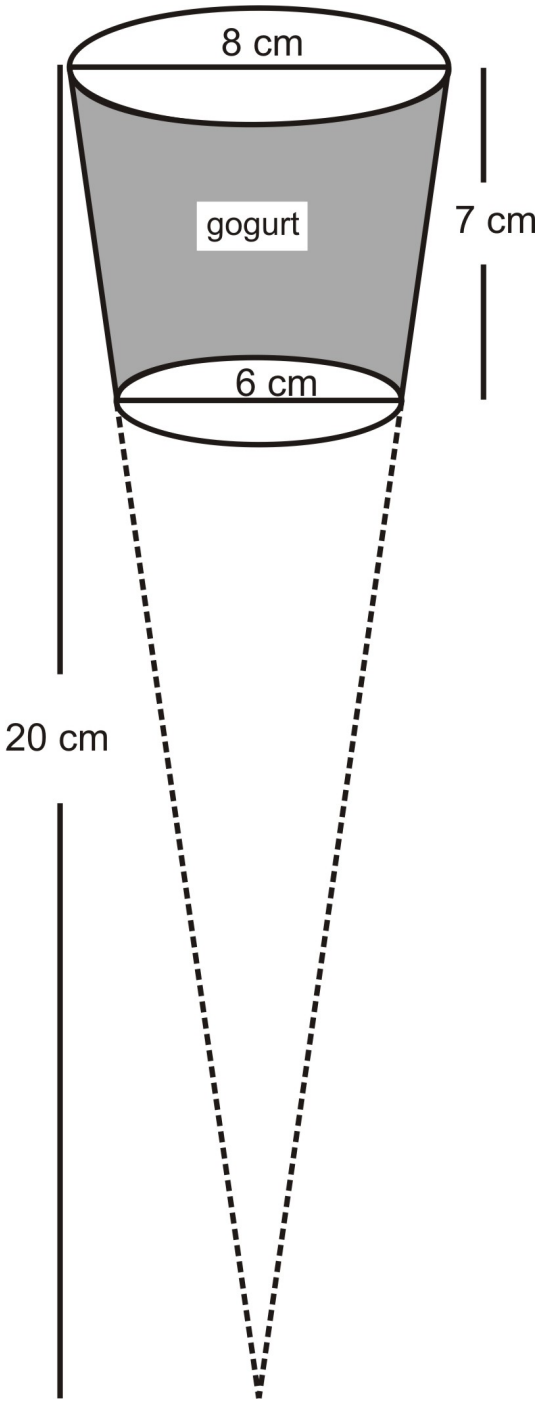


3. Find the surface area of a cone with a height of 4 m and a base area of 281.2 m^2

In problems 4 and 5 find the missing dimension. Round to the nearest tenth of a unit.

4. Cone: volume = 424 cubic meters Diameter = 18 meters
   Height = _____
5. Cone: surface area = 153.5 in^2 Radius = 4 inches
   Slant height = _____
6. A cone shaped paper cup is 8 cm high with a diameter of 5 cm. If a plant needs 240 ml of water, about how many paper cups of water are needed to water it? (1 mL = 1 cubic cm)

In problems 7 and 8 refer to this diagram. It is a diagram of a yogurt container. The yogurt container is a truncated cone.



7. What is the surface area of the container?
8. What is the volume of the container?

9. Find the height of a cone that has a radius of 2 cm and a volume of 23 cm^3
10. A cylinder has a volume of 2120.6 cm^3 and a base radius of 5 cm. What is the volume of a cone with the same height but a base radius of 3 cm?

Review Answers

1. 483.8 square units
2. 312.6 square units
3. Surface area = 75.4 m^2
4. Height = 5 meters
5. Slant height = 8 inches
6. 1.2 cups (approximately)
7. Surface area of the container = 152.62 cm^2
8. The volume of the container = 212.58 cm^3
9. Height = 5.49 cm
10. Volume of the cone = 254.45 cm^3

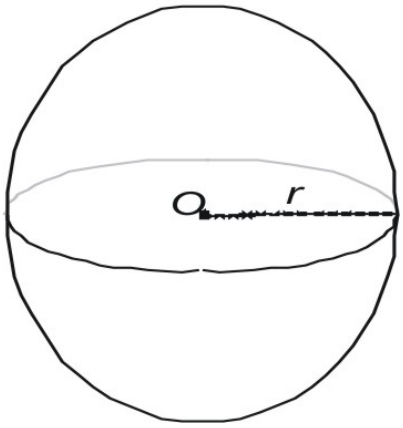
11.7 Spheres

Learning Objectives

- Find the surface area of a sphere.
- Find the volume of a sphere.

Introduction

A sphere is a three-dimensional figure that has the shape of a ball.



Spheres can be characterized in three ways.

- A sphere is the set of all points that lie a fixed distance r from a single center point O.
- A sphere is the surface that results when a circle is rotated about any of its diameters.

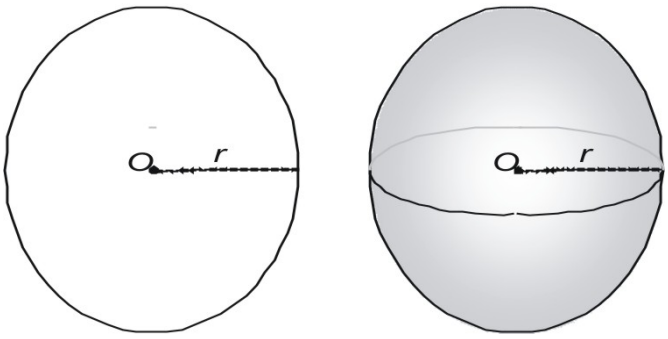
- A sphere results when you construct a polyhedron with an infinite number of faces that are infinitely small. To see why this is true, recall regular polyhedra.

As the number of faces on the figure increases, each face gets smaller in area and the figure comes more to resemble a sphere. When you imagine figure with an *infinite* number of faces, it would look like (and be!) a sphere.

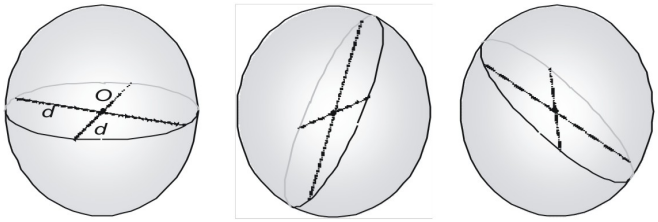
Parts of a Sphere

As described above, a sphere is the surface that is the set of all points a fixed distance from a center point O. Terminology for spheres is similar for that of circles.

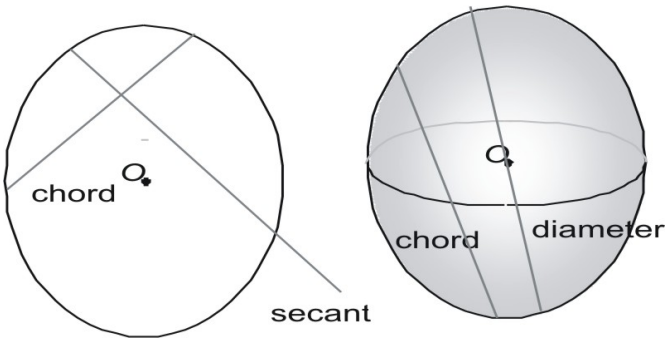
The distance from O to the surface of the sphere is r, the radius.



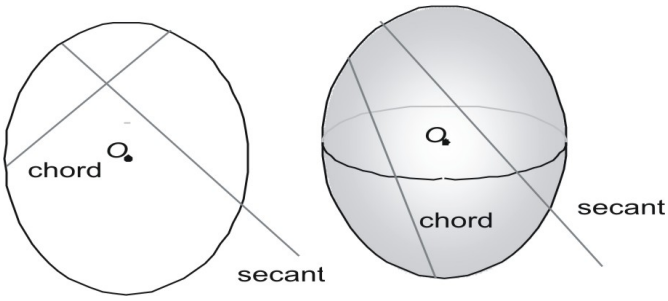
The **diameter**, d, of a sphere is the length of the segment connecting any two points on the sphere's surface and passing through O. Note that you can find a diameter (actually an infinite number of diameters) on any plane within the sphere. Two diameters are shown in each sphere below.



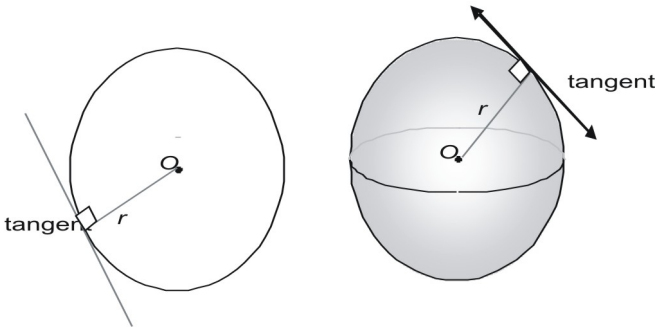
A **chord** for a sphere is similar to the chord of a circle except that it exists in three dimensions. Keep in mind that a diameter is a kind of chord—a special chord that intersects the center of the circle or sphere.



A **secant** is a line, ray, or line segment that intersects a circle or sphere in two places and extends outside of the circle or sphere.



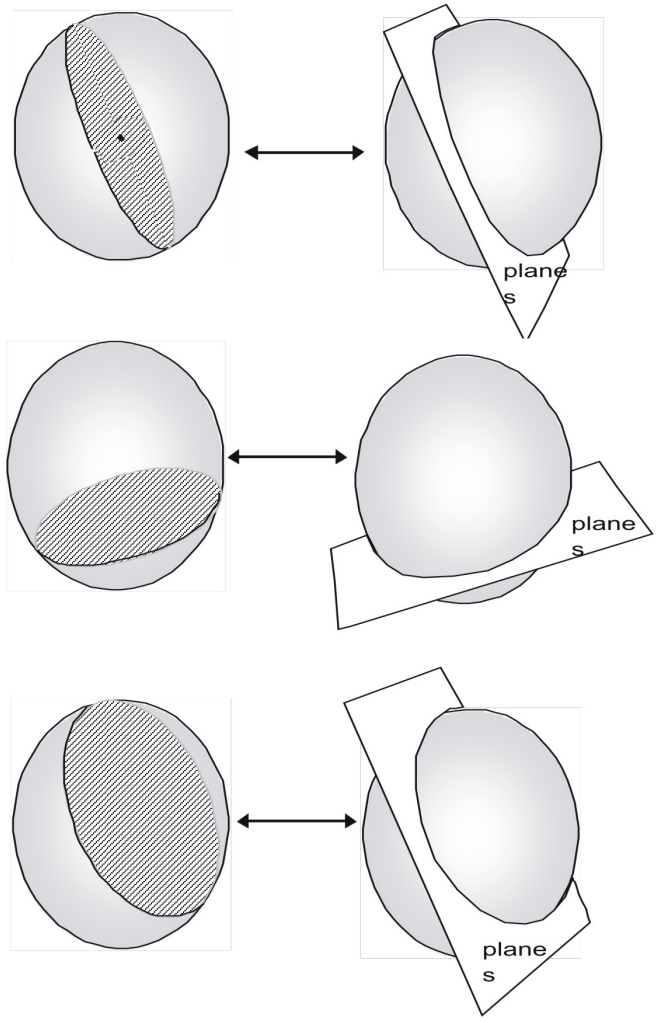
A **tangent** intersects the circle or sphere at only one point.



In a circle, a tangent is perpendicular to the radius that meets the point where the tangent intersects with the circle. The same thing is true for the sphere. All tangents are

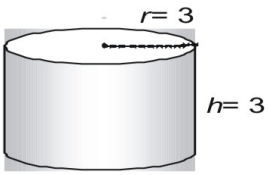
perpendicular to the radii that intersect with them.

Finally, a sphere can be sliced by an infinite number of different planes. Some planes include point O, the center of the sphere. Other points do not include the center.

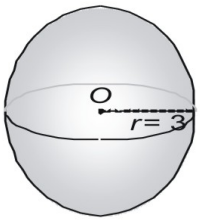


Surface Area of a Sphere

You can infer the formula for the surface area of a sphere by taking measurements of spheres and cylinders. Here we show a sphere with a radius of 3 and a right cylinder with both a radius and a height of 3 and express the area in terms of π.

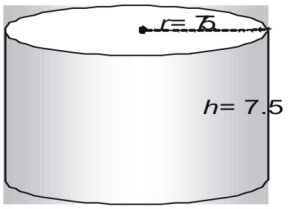


surface area = 36π

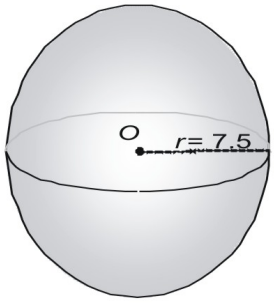


surface area = 36π

Now try a larger pair, expressing the surface area in decimal form.

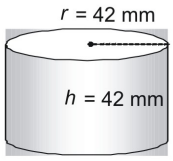


surface area = 706.5

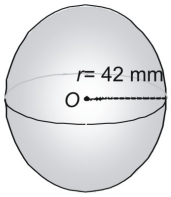


surface area = 706.5

Look at a third pair.



surface area = 22,155.84 sq mm



surface area = 22,155.84 sq mm

Is it a coincidence that a sphere and a cylinder whose radius and height are equal to the radius of the sphere have the exact same surface area? Not at all! In fact, the ancient Greeks used a method that showed that the following formula can be used to find the surface area of any sphere (or any cylinder in which $r = h$).

The Surface Area of a Sphere is given by:

$$A = 4\pi r^2$$

Example 1

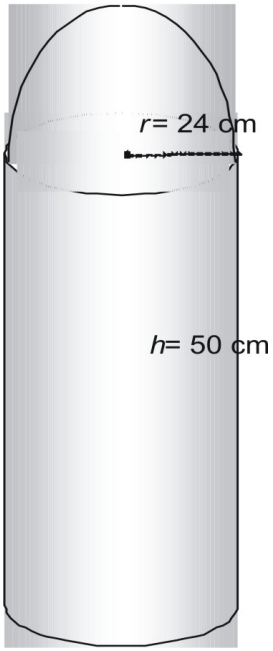
Find the surface area of a sphere with a radius of 14 feet.

Use the formula.

$$\begin{aligned} A &= 4\pi r^2 \\ &= 4\pi(14)^2 \\ &= 4\pi(196) \\ &= 784\pi \\ &= 2461.76 \text{ square feet} \end{aligned}$$

Example 2

Find the surface area of the following figure in terms of π.



The figure is made of one half sphere or hemisphere, and one cylinder without its top.

$$\begin{aligned} A(\text{half sphere}) &= \frac{1}{2}A(\text{sphere}) \\ &= \frac{1}{2} \cdot 4\pi r^2 \\ &= 2\pi(576) \\ &= 1152\pi \text{ square cm} \end{aligned}$$

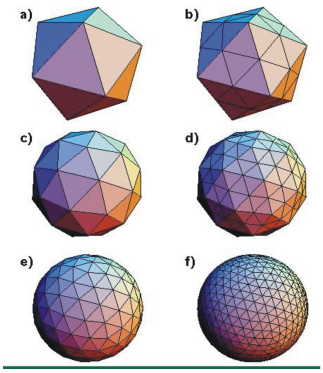
Now find the area of the cylinder without its top.

$$\begin{aligned} A(\text{topless cylinder}) &= A(\text{cylinder}) - A(\text{top}) \\ &= 2(\pi r^2) + 2\pi rh - \pi r^2 \\ &= \pi r^2 + 2\pi rh \\ &= \pi(576) + 2\pi(24)(50) \\ &= 2976\pi \text{ square cm} \end{aligned}$$

Thus, the total surface area is $1152\pi + 2976\pi = 4128\pi$ square cm

Volume of a Sphere

A sphere can be thought of as a regular polyhedron with an infinite number of congruent polygon faces. A series polyhedra with an increasing number of faces is shown.

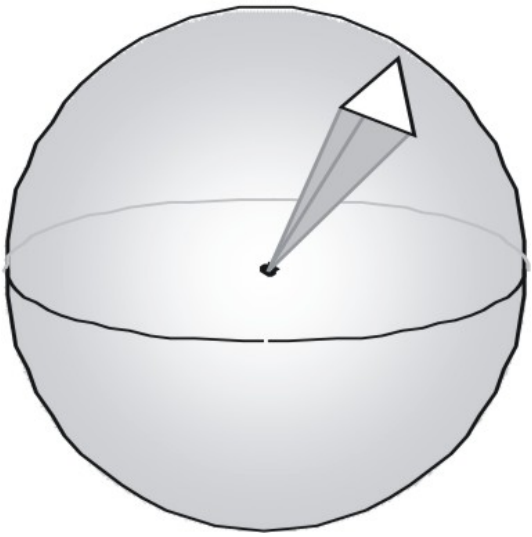


As n, the number of faces increases to an infinite number, the figure approaches becoming a sphere.

So a sphere can be thought of as a polyhedron with an infinite number faces. Each of those faces is the base of a pyramid whose vertex is located at O, the center of the sphere. This is shown below.

Each of the pyramids that make up the sphere would be congruent to the pyramid shown. The volume of this pyramid is given by:

$$V(\text{each pyramid}) = \frac{1}{3}Bh$$



To find the volume of the sphere, you simply need to add up the volumes of an infinite number of infinitely small pyramids.

$$\begin{aligned} V(\text{all pyramids}) &= V_1 + V_2 + V_3 + \ldots + V_n \\ &= \frac{1}{3}B_1h + \frac{1}{3}B_2h + \frac{1}{3}B_3h + \ldots + \frac{1}{3}B_nh \\ &= \frac{1}{3}h\left(B_1 + B_2 + B_3 + \ldots + B_n\right) \end{aligned}$$

Now, it should be obvious that the sum of all of the bases of the pyramids is simply the surface area of the sphere. Since you know that the surface area of the sphere is $4\pi r^2$, you can substitute this quantity into the equation above.

$$V(\text{all pyramids}) = \frac{1}{3}h\left(B_1 + B_2 + B_3 + \ldots + B_n\right)$$
$$= \frac{1}{3}h(4\pi r^2)$$

Finally, as n increases and the surface of the figure becomes more "rounded," h, the height of each pyramid becomes equal to r, the radius of the sphere. So we can substitute r for h. This gives:

$$V(\text{all pyramids}) = \frac{1}{3}r(4\pi r^2)$$
$$= \frac{4}{3}\pi r^3$$

We can write this as a formula.

Volume of a Sphere

Given a sphere that has radius r

$$V = \frac{4}{3}\pi r^3$$

Example 3

Find the volume of a sphere with radius 6.25 m.

Use the postulate above.

$$V(\text{sphere}) = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}(3.14)(6.25)(6.25)(6.25)$$
$$= 1022.14 \text{ cubic m}$$

Example 4

A sphere has a volume of $(85\pi)1/3$. *Find its diameter.*

Use the postulate above. Convert $(85)1/3$ to an improper fraction, $256/3$.

$$
\begin{aligned}
V(\text{sphere}) &= \frac{4}{3}\pi r^3 \\
\frac{256}{3}\pi &= \frac{4}{3}\pi r^3 \\
\left(\frac{3}{4}\right)\left(\frac{256}{3}\right)\pi &= \pi r^3 \\
\frac{192}{3} &= \pi r^3 \\
64 &= r^3 \\
4 &= r
\end{aligned}
$$

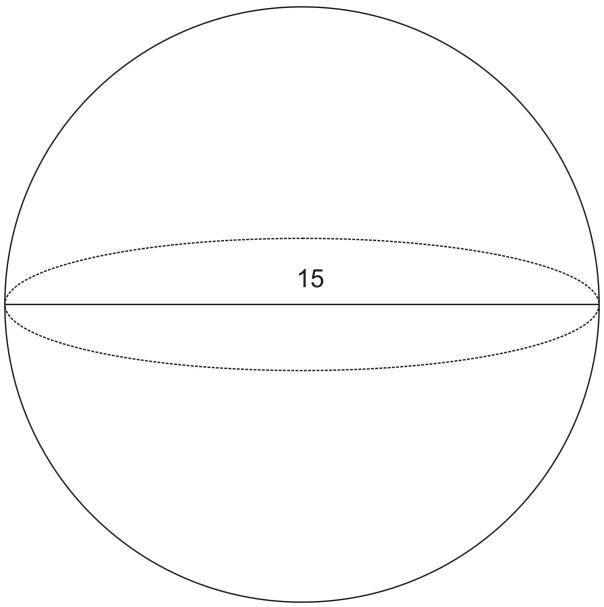
Since $r = 4$, the diameter is 8 units.

Review Questions

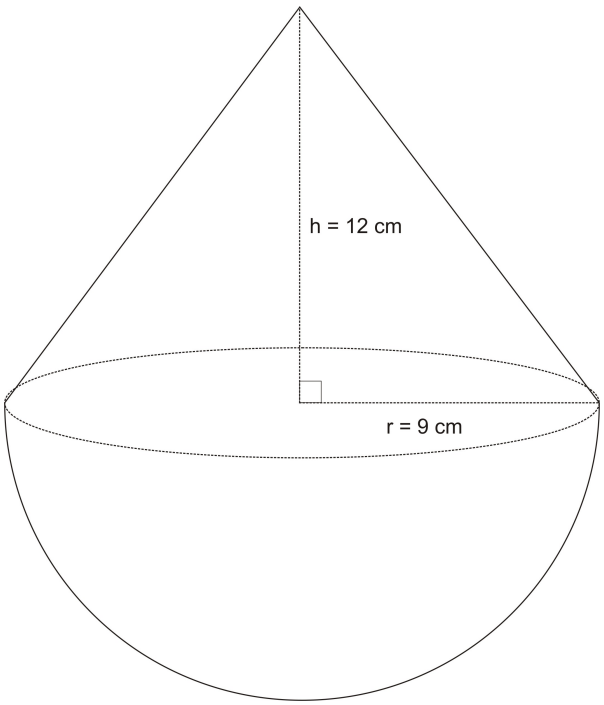
1. Find the radius of the sphere that has a volume of 335 cm^3.

Determine the surface area and volume of the following shapes:

2.



3.



4. The radius of a sphere is 4. Find its volume and total surface area.
5. A sphere has a radius of 5. A right cylinder, having the same radius has the same volume. Find the height and total surface area of the cylinder.

In problems 6 and 7 find the missing dimension.

6. Sphere : volume = 296 cm^3
   Diameter = _____
7. Sphere : surface area is 179 in^2.
   Radius = _____
8. Tennis balls with a diameter of 3.5 inches are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space not occupied by the tennis balls?
9. A sphere has surface area of 36π in^2. Find its volume.
10. A giant scoop, operated by a crane, is in the shape of a hemisphere of radius = 21 inches. The scoop is filled with melted hot steel. When the steel is poured into a cylindrical storage tank that has a radius of 28 inches, the melted steel will rise to a height of how many inches?

Review Answers

1. Radius = 4.39 cm

2. Surface area = 706.86 cm^2
   Volume = 1767.15 cm^3
3. Surface area = surface area of hemisphere + surface area of cone = 678.58 in^2
   Volume = 2544.69 in^3
4. Volume = 268.08 $units^3$
   Surface area = 201.06 $units^2$
5. Height = 20/3 units total surface area = 366.52 $units^2$
6. Diameter = 8.27 inches
7. Radius = 3.77 inches
8. Volume of cylinder = 32.16π in^3 volume of tennis balls = 21.44π in^3
   Volume of space not occupied by tennis balls = 33.67 in^3
9. Volume = 113.10 in^3
10. Height of molten steel in cylinder will be 7.88 inches

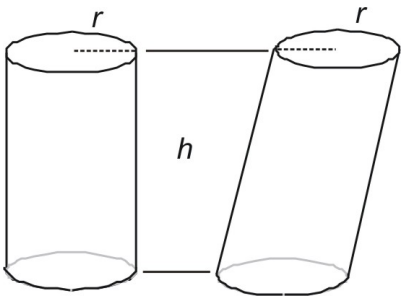
11.8 Similar Solids

Learning Objectives

- Find the volumes of solids with bases of equal areas.

Introduction

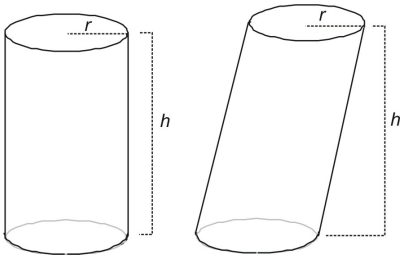
You've learned formulas for calculating the volume of different types of solids—prisms, pyramids, cylinders, and spheres. In most cases, the formulas provided had special conditions. For example, the formula for the volume of a cylinder was specific for a *right* cylinder.



Now the question arises: What happens when you consider the volume of two cylinders that have an equal base but one cylinder is non-right—that is, oblique. Does an oblique cylinder have the same volume as a right cylinder if the two share bases of the same area?

Parts of a Solid

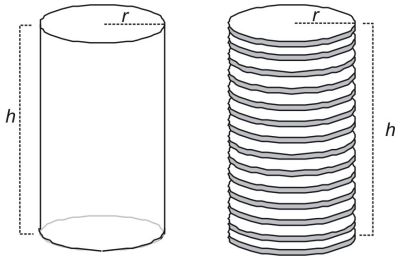
Given, two cylinders with the same height and radius. One cylinder is a right cylinder, the other is oblique. To see if the volume of the oblique cylinder is equivalent to the volume of the right cylinder, first observe the two solids.



Since they both have the same circular radius, they both have congruent bases with area:

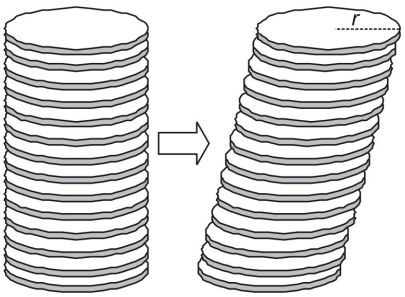
$$A = \pi r^2.$$

Now cut the right cylinder into a series of n cross-section disks each with height 1 and radius r.

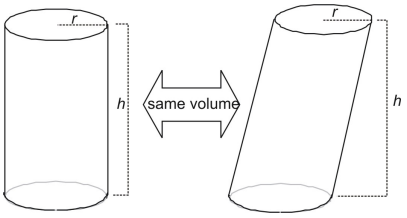


It should be clear from the diagram that the total volume of the n disks is equal to the volume of the original cylinder.

Now start with the same set of disks. Shift each disk over to the right. The volume of the shifted disks must be exactly the same as the unshifted disks, since both figures are made out of the same disks.



It follows that the volume of the oblique figure is equal to the volume of the original right cylinder.



In other words, if the radius and height of each figure are congruent:

$$V(\text{right cylinder}) = V(\text{oblique cylinder})$$
$$V(\text{any cylinder}) = \frac{1}{3}Bh$$
$$V(\text{any cylinder}) = \frac{1}{3}\pi r^2 h$$

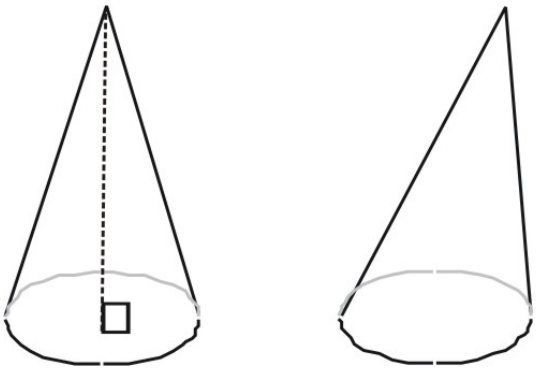
The principle shown above was developed in the seventeenth century by Italian mathematician Francisco Cavalieri. It is known as **Cavalieri's Principle.** (Liu Hui also discovered the same principle in third-century China, but was not given credit for it until recently.) The principle is valid for any solid studied in this chapter.

Volume of a Solid Postulate (Cavalieri's Principle):

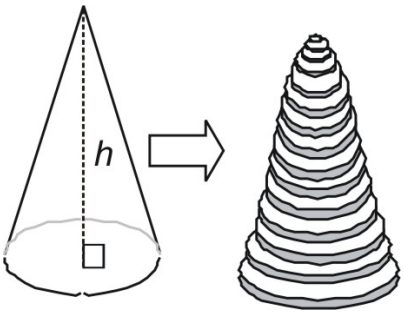
The volumes of two objects are equal if the areas of their **corresponding** cross-sections are in all cases equal. Two cross-sections **correspond** if they are intersections of the figure with planes equidistant from a chosen base plane.

Example 1

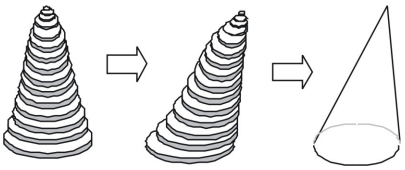
Prove (informally) that the two circular cones with the same radius and height are equal in volume.



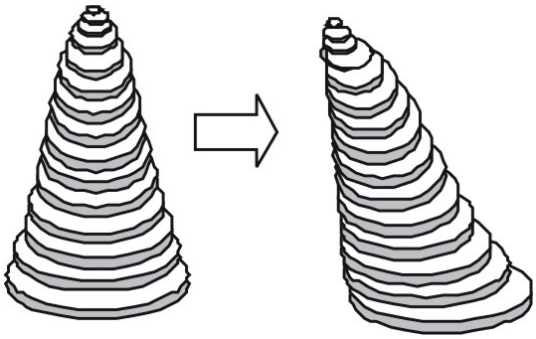
As before, we can break down the right circular cone into disks.



Now shift the disks over.



You can see that the shifted-over figure, since it uses the very same disks as the straight figure, must have the same volume. In fact, you can shift the disks any way you like. Since you are always using the same set of disks, the volume is the same. ♦



Keep in mind that Cavalieri's Principle will work for any two solids as long as their bases are equal in area (not necessarily congruent) and their cross sections change in the same way.

Example 2

A rectangular pyramid and a circular cone have the same height, and base area. Are their volumes congruent?

Yes. Even though the two figures are different, both can be computed by using the following formula:

$$V(\text{cone}) = \frac{1}{3}B_{circle}h \text{ and } V(\text{pyramid}) = \frac{1}{3}B_{rectangle}h$$

Since

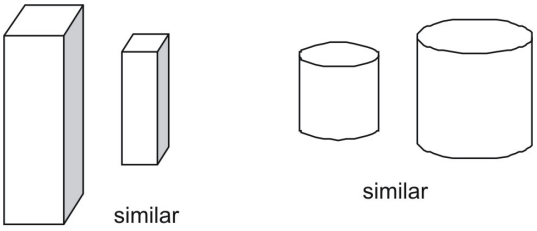
$$B_{circle} = B_{rectangle}$$

Then

$$V(\text{cone}) = V(\text{rectangle})$$

Similar or Not Similar

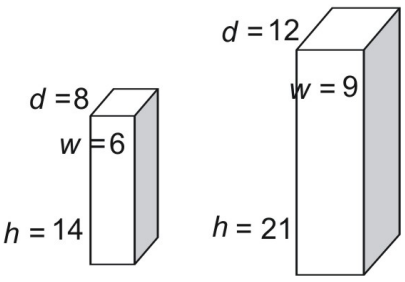
Two solids of the same type with **equal ratios** of corresponding linear measures (such as heights or radii) are called similar solids.



To be similar, figures need to have corresponding linear measures that are in proportion to one another. If these linear measures are not in proportion, the figures are not similar.

Example 1

Are these two figures similar?



If the figures are similar, all ratios for corresponding measures must be the same.

The ratios are:

$$\text{width} = \frac{6}{9} = \frac{2}{3}$$
$$\text{height} = \frac{14}{21} = \frac{2}{3}$$
$$\text{depth} = \frac{8}{12} = \frac{2}{3}$$

Since the three ratios are equal, you can conclude that the figures are similar.

Example 2

Cone A has height 20 *and radius* 5. *Cone B has height* 18 *and radius* 6. *Are the two cones similar?*

If the figures are similar all ratios for corresponding measures must be the same.

The ratios are:

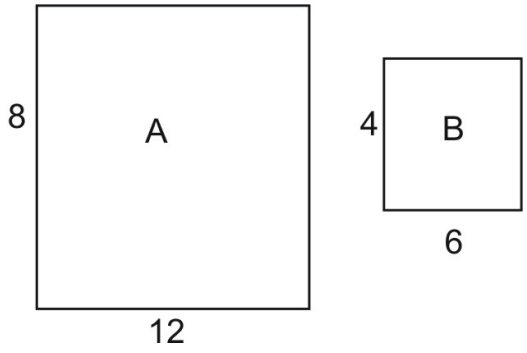
$$\text{height} = \frac{20}{16} = \frac{5}{4}$$
$$\text{radius} = \frac{18}{6} = \frac{3}{1}$$

Since the ratios are different, the two figures are not similar.

Compare Surface Areas and Volumes of Similar Figures

When you compare similar two-dimensional figures, area changes as a function of the square of the ratio of

For example, take a look at the areas of these two similar figures.



The ratio between corresponding sides is:

$$\frac{\text{length } (A)}{\text{length } (B)} = \frac{12}{6} = \frac{2}{1}$$

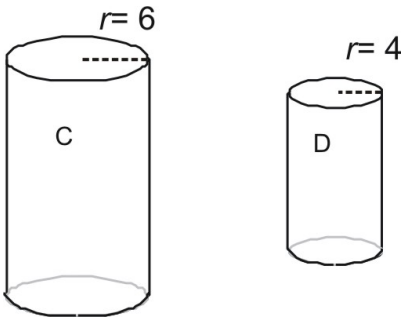
The ratio between the areas of the two figures is the square of the ratio of the linear measurement:

$$\frac{\text{area}(A)}{\text{area}(B)} = \frac{12 \cdot 8}{6 \cdot 4} = \frac{96}{24} = \frac{4}{1} = \frac{2^2}{1}$$

This relationship holds for solid figures as well. The ratio of the areas of two similar figures is equal to the **square** of the ratio between the corresponding linear sides.

Example 3

Find the ratio of the surface area between the two similar figures C and D.



Since the two figures are similar, you can use the ratio between any two corresponding measurements to find the answer. Here, only the radius has been supplied, so:

$$\frac{\text{radius } (C)}{\text{radius } (D)} = \frac{6}{4} = \frac{3}{2}$$

The ratio between the areas of the two figures is the square of the ratio of the linear measurements:

$$\frac{\text{area } (C)}{\text{area } (D)} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Example 4

If the surface area of the small cylinder in the problem above is 80 , what is the surface area of the larger cylinder?

From above we, know that:

$$\frac{\text{area } (C)}{\text{area } (D)} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

So the surface area can be found by setting up equal ratios

$$\frac{9}{4} = \frac{n}{80\pi}$$

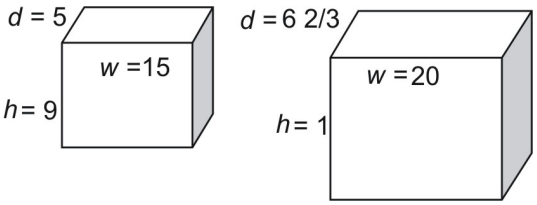
Solve for n.

$$n = 180\pi$$

The ratio of the volumes of two similar figures is equal to the **cube** of the ratio between the corresponding linear sides.

Example 5

Find the ratio of the volume between the two similar figures C and D.



As with surface area, since the two figures are similar you can use the height, depth, or width of the figures to find the linear ratio. In this example we will use the widths of the two figures.

$$\frac{w(\text{small})}{w(\text{large})} = \frac{15}{20} = \frac{3}{4}$$

The ratio between the volumes of the two figures is the **cube** of the ratio of the linear measurements:

$$\frac{\text{volume } (C)}{\text{volume } (D)} = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

Does this cube relationship agree with the actual measurements? Compute the volume of each figure.

$$\frac{\text{volume (small)}}{\text{volume (large)}} = \frac{5 \times 9 \times 15}{62/3 \times 12 \times 20} = \frac{535}{1600} = \frac{27}{64}$$

As you can see, the ratio holds. We can summarize the information in this lesson in the following postulate.

Similar Solids Postulate:

If two solid figures, A and B are similar and the ratio of their linear measurements is $\frac{a}{b}$, then the ratio of their surface areas is:

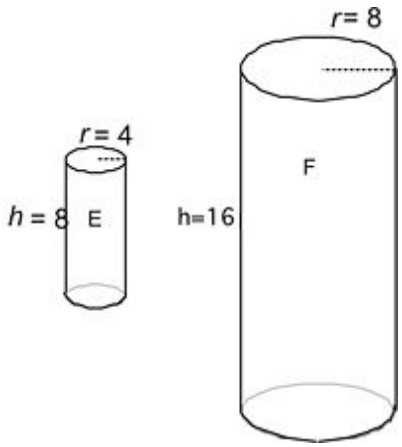
$$\frac{\text{surface area A}}{\text{surface area B}} = \left(\frac{a}{b}\right)^2$$

The ratio of their volumes is:

$$\frac{\text{volume A}}{\text{volume B}} = \left(\frac{a}{b}\right)^3$$

Scale Factors and Models

The ratio of the linear measurements between two similar figures is called the **scaling factor.** For example, we can find the scaling factor for cylinders E and F by finding the ratio of any two corresponding measurements.



Using the heights, we find a scaling factor of:

$$\frac{h(small)}{h(large)} = \frac{8}{16} = \frac{1}{2}.$$

You can use a scaling factor to make a model.

Example 6

Doug is making a model of the Statue of Liberty. The real statue has a height of 111 feet *and a nose that is* 4.5 feet *in length. Doug's model statue has a height of* 3 feet *. How long should the nose on Doug's model be?*

First find the scaling factor.

$$\frac{height(model)}{height(statue)} = \frac{3}{111} = \frac{1}{37} = 0.027$$

To find the length of the nose, simply multiply the height of the model's nose by the scaling factor.

$$\begin{aligned} \text{nose(model)} &= \text{nose(statue)} \cdot \text{(scaling factor)} \\ &= 4.5 \cdot 0.027 \\ &= 0.122 \text{ feet} \end{aligned}$$

In inches, the quantity would be:

$$\begin{aligned}\text{nose(model)} &= 0.122 \text{ feet} \cdot 12 \text{ inches/feet} \\ &= 1.46 \text{ inches}\end{aligned}$$

Example 7

An architect makes a scale model of a building shaped like a rectangular prism. The model measures 1.4 ft *in height,* 0.6 inches *in width, and* 0.2 inches *in depth. The real building will be* 420 feet *tall. How wide will the real building be?*

First find the scaling factor.

$$\frac{height(real)}{height(model)} = \frac{420}{1.4} = \frac{300}{1} = 300$$

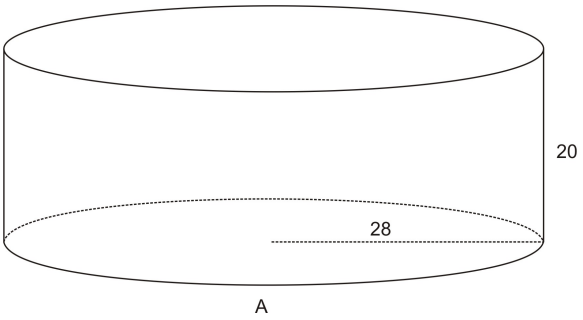
To find the width, simply multiply the width of the model by the scaling factor.

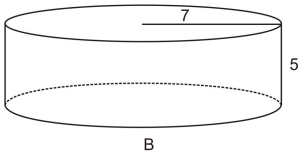
$$\begin{aligned}\text{width(real)} &= \text{width(model)} \cdot \text{(scaling factor)} \\ &= 0.6 \cdot 300 \\ &= 180 \text{ feet}\end{aligned}$$

Review Questions

1. How does the volume of a cube change if the sides of a cube are multiplied by 4? Explain.
2. In a cone if the radius and height are doubled what happens to the volume? Explain.
3. In a rectangular solid, is the sides are doubled what happens to the volume? Explain.
4. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
5. The ratio of the volumes of two similar pyramids is $8 : 27$. What is the ratio of their total surface areas?
6. (a) Are all spheres similar?
   (b) Are all cylinders similar?
   (c) Are all cubes similar? Explain your answers to each of these.
7. The ratio of the volumes of two tetrahedron is $1000 : 1$. The smaller tetrahedron has a side of length 6 centimeters. What is the side length of the larger tetrahedron?

Refer to these two similar cylinders in problems 8 – 10:





8. What is the similarity ratio of cylinder A to cylinder B?
9. What is the ratio of surface area of cylinder A to cylinder B?
10. What is the ratio of the volume of cylinder B to cylinder A?

Review Answers

1. The volume will be 64 times greater. Volume $= s^3$ New volume $= (4s)^3$
2. Volume will be 8 times greater.
3. The volume will be 8 times greater $(2w)(2l)(2h) = 8$ wlh $= 8$ (volume of first rectangular solid)
4. $5^3/9^3$
5. $4/9$
6. All spheres and all cubes are similar since each has only one linear measure. All cylinders are not similar. They can only be similar if the ratio of the radii = the ratio of the heights.
7. 60 cm
8. $20/5 = 4/1$
9. $16/1$
10. $1/4^3$

Chapter 12

Transformations

12.1 Translations

Learning Objectives

- Graph a translation in a coordinate plane.
- Recognize that a translation is an isometry.
- Use vectors to represent a translation.

Introduction

Translations are familiar to you from earlier lessons. In this lesson, we restate our earlier learning in terms of motions in a coordinate plane. We'll use coordinates and vectors to express the results of translations.

Translations

Remember that a translation moves every point a given horizontal distance and/or a given vertical distance. For example, if a translation moves the point $A(3, 7)$ 2 units to the right and 4 units up, to $A'(5, 11)$ then this translation moves every point the same way.

The original point (or figure) is called the ***preimage***, in this case $A(3, 7)$. The translated point (or figure) is called the ***image***, in this case $A'(5, 11)$, and is designated with the prime symbol.

Example 1

The point $A(3, 7)$ *in a translation becomes the point* $A'(2, 4)$. *What is the image of* $B(-6, 1)$

in the same translation?

Point A moved 1 unit to the left and 3 units down to reach A'. B will also move 1 unit to the left and 3 units down.

$$B' = (-6 - 1, 1 - 3) = (-7, -2)$$

$B'(-8, -2)$ is the image of $B(-6, 1)$.

Notice the following:

- $AB = \sqrt{(-6-3)^2 + (1-7)^2} = \sqrt{(-9)^2 + (-6)^2} = \sqrt{117}$
- $A'B' = \sqrt{(-7-2)^2 + (-2-4)^2} = \sqrt{(-9)^2 + (-6)^2} = \sqrt{117}$

Since the endpoints of $\overline{AB}$ and $\overline{A'B'}$ moved the same distance horizontally and vertically, both segments have the same length.

Translation is an Isometry

An **isometry** is a transformation in which distance is "preserved." This means that the distance between any two points is the same as the distance between the images of the points.

Did you notice this in example 1 above?

$AB = A'B'$ (since they are both equal to $\sqrt{117}$)

Would we get the same result for any other point in this translation? Yes. It's clear that for any point X, the distance from X to X' will be $\sqrt{117}$. Every point moves $\sqrt{117}$ units to its image.

This is true in general.

Translation Isometry Theorem

Every translation in the coordinate plane is an isometry.

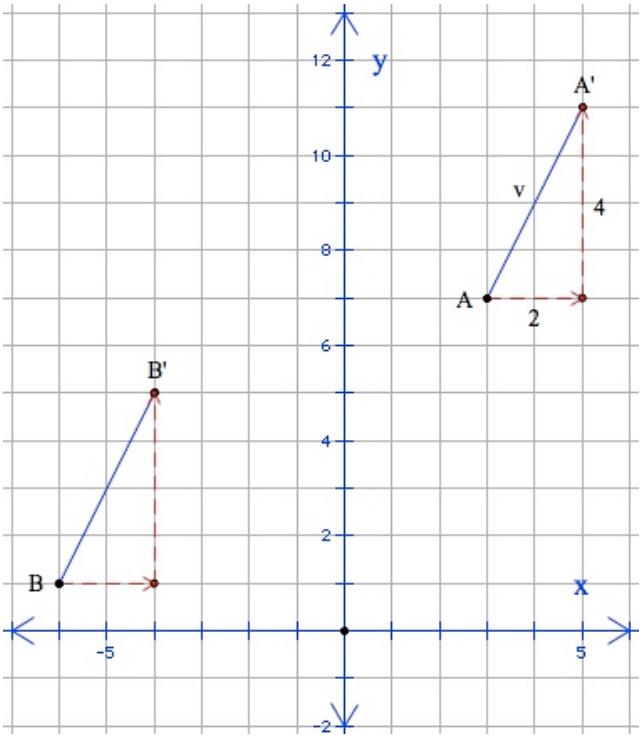
You will prove this theorem in the Lesson Exercises.

Vectors

Let's look at the translation in example 1 in a slightly different way.

Example 2

The point $A(3, 7)$ in a translation is the point $A'(5, 11)$. What is the image of $B(-6, 1)$ in the same translation?



The arrow from A to A' is called a **vector**, because it has a *length* and a *direction*. The horizontal and vertical components of the vector are 2 and 4 respectively.

To find the image of B, we can apply the same transformation vector to point B. The arrowhead of the vector is at $B'(-4, 5)$.

The vector in example 2 is often represented with a boldface single letter v.

- The horizontal component of vector v is 2.
- The vertical component of vector v is 4.
- The vector can also be represented as a number pair made up of the horizontal and vertical components.

The vector for this transformation is $v = (2, 4)$.

Example 3

A triangle has vertices $A(-2, -5), B(0, 2)$, and $C(2, -5)$. The vector for a translation is $v = (0, 5)$. What are the vertices of the image of the triangle?

Add the horizontal and vertical components to the $x-$ and $y-$ coordinates of the vertices.

$$A' = (-2 + 0, -5 + 5) = (-2, 0)$$
$$B' = (0 + 0, 2 + 5) = (0, 7)$$
$$C' = (2 + 0, -5 + 5) = (2, 0)$$

Challenge: Can you describe what this transformation does to the original triangle?

Further Reading

- Vectors are used in physics to represent forces, velocity, and other quantities. Learn more about vectors at: http://en.wikipedia.org/wiki/Vector_(spatial)

Lesson Summary

You can think of a translation as a way to move points in a coordinate plane. And you can be sure that the shape and size of a figure stays the same in a translation. For that reason a translation is called an *isometry*. (Note: Isometry is a compound word with two roots in Greek, "iso" and "metry." You may know other words with these same roots, in addition to "isosceles" and "geometry.")

Vectors provide an alternative way to represent a translation. A vector has a direction and a length—the exact features that are involved in moving a point in a translation.

Points to Consider

Think about some special transformation vectors. Can you picture what each one does to a figure in a coordinate plane?

- $v = (0, 0)$
- $v = (5, 0)$
- $v = (0, -5)$
- $v = (5, 5)$

This lesson was about two-dimensional space represented by a coordinate grid. But we know there are more than two dimensions. The real world is actually multidimensional. Vectors are well suited to describe motion in that world. What would transformation vectors look like there?

Review Questions

1. Prove that any translation in the coordinate plane is an isometry. Given: A translation moves any point h units horizontally and k units vertically.

Let $M(s,t)$ and $N(u,v)$ be points in the coordinate plane.

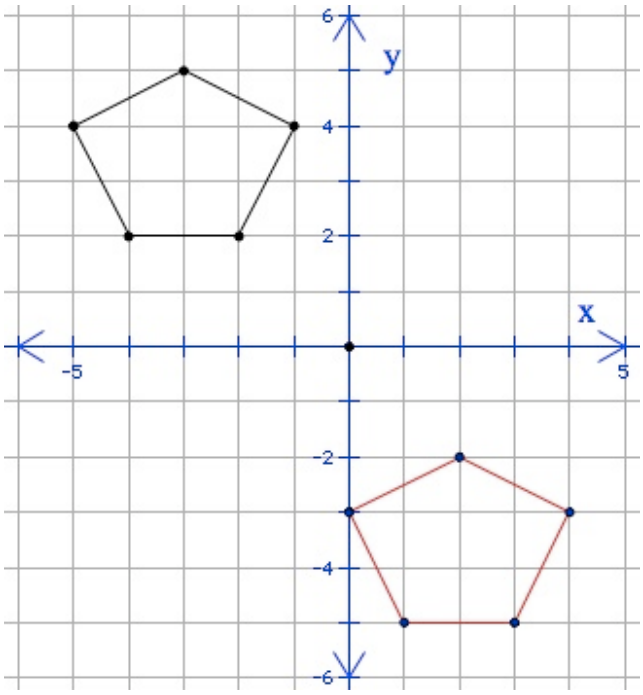
Prove: $MN = M'N'$

[Hint: Express u and v in terms of s, t, h, and k.]

2. A triangle has vertices $A(-2,-5), B(0,2)$, and $C(2,-5)$. The vector for a translation is $v = (0,5)$.

[Example 3] How does this transformation move the original triangle?

3. Write a transformation vector that would move the solid figure to the red figure.



4. The transformation vector $v = (-2,3)$ moves a point to the point $(4,-1)$. What are the coordinates of the original point?
5. Write a transformation vector that would move the point $(-3,8)$ to the origin.
6. How does the translation vector $v = (0,0)$ move a figure?
7. The point $(4,1)$ is moved by the translation vector $v = (-8,6)$. How far does the original point move?
8. Write a translation vector that will move all points 5 units.
9. Point $A(-6,2)$ is moved by the translation vector $v = (3,-4)$. Fill in the blanks. A' (_____,_____).

Review Answers

1. $M' = (s+h, t+k), N' = (u+h, v+k)$
   $MN = \sqrt{(u-s)^2 + (v-t)^2}$
   $M'N' = \sqrt{[(u+h)-(s+h)]^2 + [(v+k)-(t+k)]^2}$
   $= \sqrt{(u-s)^2 + (v-t)^2} = MN$
2. The triangle moves up 5 units.
3. $v = (5, -7)$
4. $(6, -4)$
5. $v = (3, -8)$
6. The figure does not move; it says the same.
7. 10 units along the vector.
8. $v = (\pm 3, \pm 4)$ or $v = (\pm 4, \pm 3)$
9. $(-3, -2)$

12.2 Matrices

Learning Objectives

- Use the language of matrices.
- Add matrices.
- Apply matrices to translations.

Introduction

A **matrix** is a way to express multidimensional data easily and concisely. Matrices (plural of matrix) have their own language. They also have their own arithmetic. You will learn some of the basic matrix terms, as well as how to add two matrices, and how to tell if that is even possible for two matrices!

Matrices have many applications beyond geometry, but in this lesson we'll see how to use matrices to perform a translation in a coordinate plane.

Matrix Basics

Simply stated, a **matrix** is an array (or arrangement) of numbers in rows and columns. Brackets are usually used to indicate a matrix.

$\begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 4 \end{bmatrix}$ is a 2×3 matrix (read "two by three matrix").

- It has 2 *rows* and 3 *columns*. These are the *dimensions* of the matrix.
- The numbers in the matrix are called the *elements*.
- An element is located according to its place in the rows and columns; 4 is the element in row 2, column 3.

Matrices can represent real-world information.

Example 1

A company has two warehouses in their Eastern Region, where they store three models of their product. A 2×3 *matrix can represent the numbers of each model available in each warehouse.*

$$\begin{array}{ll} & \text{Eastern Region} \\ & \text{Model} \\ \text{Ware house} & \begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 4 \end{bmatrix} \end{array}$$

Here the *row* number of an element represents the *warehouse* number, and the *column* number represents the *model* number.

- There are 6 items of Model 3 in Warehouse 1.
- There are 5 items of Model 2 in Warehouse 2.

How many items are in Warehouse 1 *altogether?*

Use row 1. There are 2 Model 1, 3 Model 2, and 6 Model 3 in Warehouse 1.

There are $2 + 3 + 6 = 11$ items in Warehouse 1.

How many Model 3 *items are there altogether?*

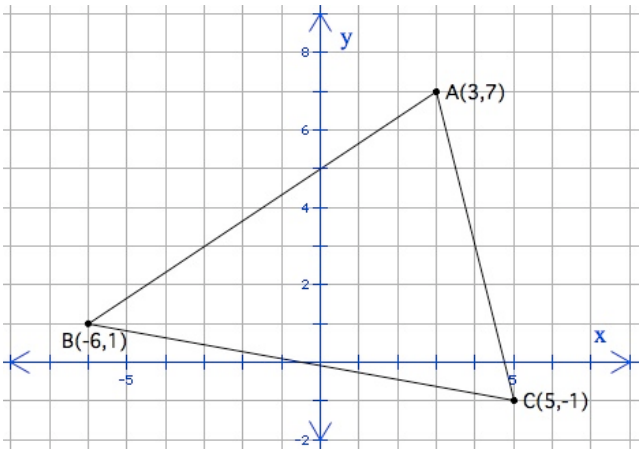
Use column 3. There are 6 in Warehouse 1 and 4 in Warehouse 2.

There are $6 + 4 = 10$ Model 3 items in all.

Matrices in the Coordinate Plane

Matrices can represent points in a coordinate plane.

A matrix can represent the coordinates of the vertices of a polygon.



A is $(3, 7), B$ is $(-6, 1) C$ is $(5, -1)$

Each row represents the coordinates of one of the vertices.

		x	y
Vertex	A	3	7
	B	-6	1
	C	5	-1

Example 2

What are the coordinates of point C in the image above?

The $x-$coordinate of point C is 5; the $y-$coordinate is -1.

Matrix Addition

Matrices have their own version of arithmetic. To add two matrices, add the elements in corresponding positions in the matrices.

- Add the elements in row 1 column 1. Add the elements in row 1 column 2. And so on.
- Place each sum in the corresponding position in a *new matrix*, which is the *sum* of the original two matrices.
- The sum matrix has the same dimensions as the matrices being added.
- To add two matrices, they must have the same dimensions.

Example 3

The company in example 1 has a western region as well as an eastern region. The western region also has two warehouses where they store the three models of their product. The 2×3 matrices represent the numbers of each model available in each warehouse by region.

	Eastern Region
	Model
Ware house	$\begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 4 \end{bmatrix}$

	Western Region
	Model
Ware house	$\begin{bmatrix} 8 & 0 & 2 \\ 2 & 3 & 5 \end{bmatrix}$

To prepare a report, Stuart adds the two matrices. This will give him the combined information for both regions.

$$\begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2+8 & 3+0 & 6+2 \\ 1+2 & 5+3 & 4+5 \end{bmatrix} = \begin{bmatrix} 10 & 3 & 8 \\ 3 & 8 & 9 \end{bmatrix}$$

a) What does the 9 *mean in the sum matrix?*

9 is the total number of Model 3 items in both Warehouse 2 buildings in both regions.

b) How many Model 2 *items are there in all?*

There are 11 Model 2 items: 3 in both Warehouse 1 buildings and 8 in both Warehouse 2 buildings.

c) How many items—any model, any location—are there in all?

41. This is the sum of all the elements in the sum matrix: $10 + 3 + 8 + 3 + 8 + 9 = 41$.

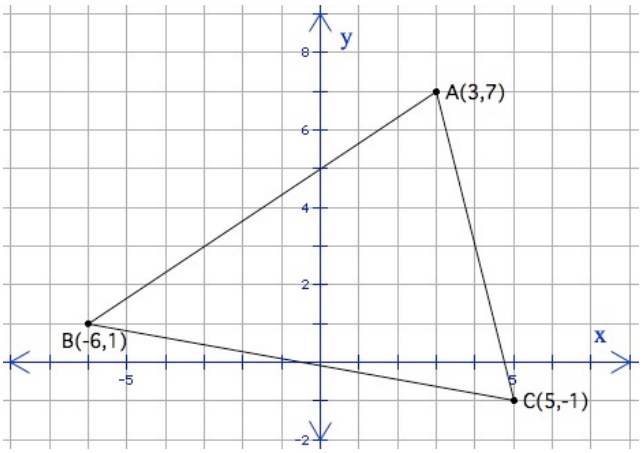
Translations

You worked with translations in the coordinate plane earlier.

- A translation moves each point (x, y) a horizontal distance h and a vertical distance k.
- The image of point (x, y) is the point $(x + h, y + k)$.

Matrix addition is one way to represent a translation.

Recall the triangle in example 2.

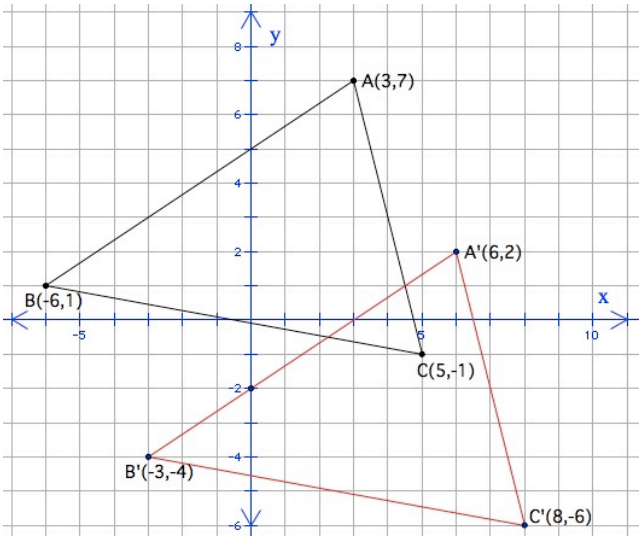


A is $(3, 7)$, B is $(-6, 1)$ C is $(5, -1)$

Each row in the matrix below represents the coordinates of one of the vertices.

Vertex		x	y
A		3	7
B		-6	1
C		5	-1

In the translation, suppose that each point will move 3 units to the right and 5 units down.



A is $(3, 7)$, B is $(-6, 1)$ C is $(5, -1)$.

Each of original points A, B, and C moves 3 units to the right, and 5 units down.

The coordinates of the image of any point (x, y) will be the point $(x + 3, y - 5)$. The translation can be represented as a matrix sum.

$$\begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -5 \\ 3 & -5 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -3 & -4 \\ 8 & -6 \end{bmatrix}$$

Example 4

What is the image of point B in this translation?

The second row represents point B. The image of $B(-6, 1)$ is $B'(-3, -4)$.

Notice:

- The rows of the second matrix are all the same. This is because each point of the triangle, or any point, moves the same distance and direction in this translation.
- If the translation had moved each point 2 units to the left and 7 units up, then the second matrix in the sum would have been:

$$\begin{bmatrix} -2 & 7 \\ -2 & 7 \\ -2 & 7 \end{bmatrix}$$

Lesson Summary

A matrix is an arrangement of numbers in rows and columns. Matrices have their own brand of arithmetic. So far you have learned how to add two matrices.

Matrices have many applications, for example in business and industry. One use of matrices is in working with transformations of points and figures in a coordinate plane. In this lesson you saw that addition of matrices can represent a translation. An unusual feature of a translation matrix is that all the rows are the same.

Points to Consider

In upcoming lessons we'll learn about two kinds of multiplication with matrices. We'll then use multiplication of matrices to represent other types of transformations in the coordinate plane, starting with reflections in the next lesson.

Something you rely on all the time, but probably don't think about very much is the fact that any two real numbers can be added and multiplied and the result is also a real number. Matrices are different from real numbers because there are special conditions for adding and multiplying matrices. For example, not all matrices can be added because in order to add two matrices the addends must have the same dimensions. The conditions on matrix multiplication are even more interesting, as you shall see shortly.

Review Questions

Fill in the blanks.

1. $\left(\begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -5 \\ 3 & -5 \\ 3 & -5 \end{bmatrix}\right) + \underline{?} = \begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix} = \underline{?}$
2. $\begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix} + \underline{?} = \begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix} + \underline{?} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{?}$

Matrix A below represents the vertices of a triangle. How does the triangle move in the translation represented by matrix B?

$$A = \begin{bmatrix} 3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix}$$

4. $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
5. $B = \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 4 & 0 \end{bmatrix}$
6. $B = \begin{bmatrix} -3 & -7 \\ -3 & -7 \\ -3 & -7 \end{bmatrix}$

Matrices A, B, and C are defined below.

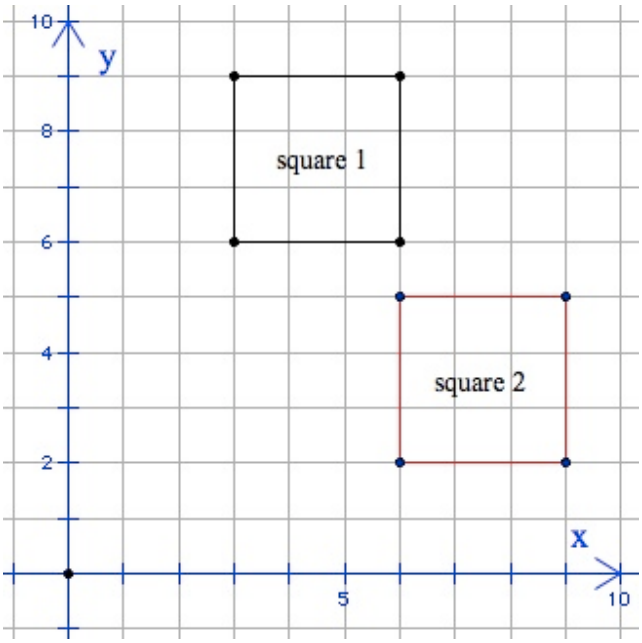
$$A = \begin{bmatrix} -3 & 7 \\ -6 & 1 \\ 5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 0 \\ -4 & 0 \\ -4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 5 \\ 0 & 5 \\ 0 & 5 \end{bmatrix}$$

Matrix A represents the vertices of a triangle. How does the triangle move in the translation represented by the matrix sum?

8. $A + B$
9. $A + C$
10. $(A + B) + C$

11. Write a matrix D so that $A + D = (A + B) + C$.

A translation moves square 1 to square 2.



12. Write a $4 -$ by $- 2$ matrix, A, for the vertices of square 1.
13. Write a $4 -$ by $- 2$ matrix, B, for the vertices of square 2.
14. Write a translation matrix T for which $A + T = B$.

Review Answers

1. $\begin{bmatrix} -3 & 5 \\ -3 & 5 \\ -3 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
3. $\begin{bmatrix} -3 & -7 \\ 6 & -1 \\ -5 & 1 \end{bmatrix}$
4. Does not move
5. 4 units right
6. 3 units left and 7 units down
7. 4 units left
8. 5 units up

9. 4 units left and 5 units up
10. $D = \begin{bmatrix} -4 & 5 \\ -4 & 5 \\ -4 & 5 \end{bmatrix}$
11. $A = \begin{bmatrix} 3 & 6 \\ 6 & 6 \\ 6 & 9 \\ 3 & 9 \end{bmatrix}$
12. $B = \begin{bmatrix} 6 & 2 \\ 9 & 2 \\ 9 & 5 \\ 6 & 5 \end{bmatrix}$
13. $T = \begin{bmatrix} 3 & -4 \\ 3 & -4 \\ 3 & -4 \\ 3 & -4 \end{bmatrix}$

12.3 Reflections

Learning Objectives

- Find the reflection of a point in a line on a coordinate plane.
- Multiply matrices.
- Apply matrix multiplication to reflections.
- Verify that a reflection is an isometry.

Introduction

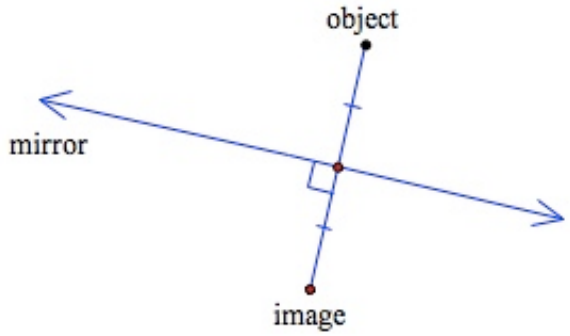
You studied translations earlier, and saw that matrix addition can be used to represent a translation in a coordinate plane. You also learned that a translation is an isometry.

In this lesson, we will analyze reflections in the same way. This time we will use a new operation, matrix multiplication, to represent a reflection in a coordinate plane. We will see that reflections, like translations, are isometries.

You will have an opportunity to discover one surprising—or even shocking!—fact of matrix arithmetic.

Reflection in a Line

A **reflection** in a line is as if the line were a mirror.



An object reflects in the mirror, and we see the image of the object.

- The image is the same distance behind the mirror as the object is in front of the mirror.
- The "line of sight" from the object to the mirror is perpendicular to the mirror itself.
- The "line of sight" from the image to the mirror is also perpendicular to the mirror.

Technology Note - Geometry Software

Use your geometry software to experiment with reflections.

Try this.

- Draw a line.
- Draw a triangle.
- Reflect the triangle in the line.
- Look at your results.
- Repeat with different lines, and figures other than a triangle.

Now try this.

- Draw a line
- Draw a point.
- Reflect the point in the line.
- Connect the point and its reflection with a segment.
- Measure the distance of the original point to the line, and the distance of the reflected point to the line.
- Measure the angle formed by the original line and the segment connecting the original point and its reflection.

Let's put this information in more precise terms.

Reflection of a Point in a Line:

Point P' is the **reflection** of point P in line k if and only if line k is the perpendicular bisector of $\overline{PP'}$.

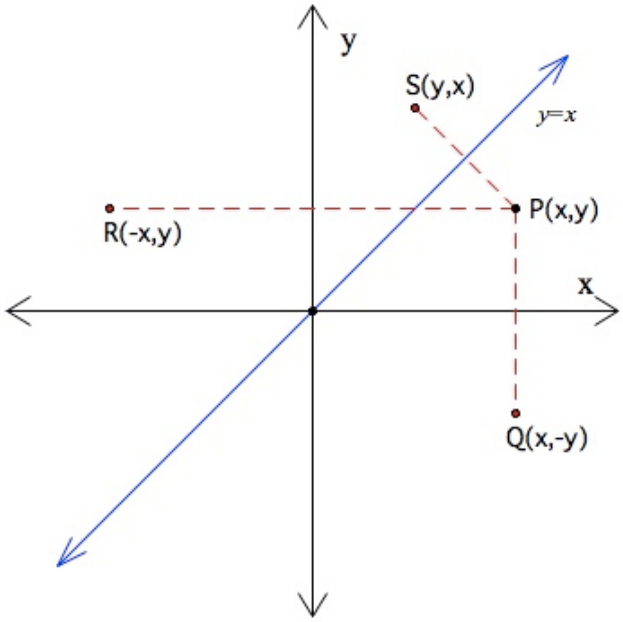
Reflections in Special Lines

In a coordinate plane there are some "special" lines for which it is relatively easy to create reflections.

- the $x-$axis
- the $y-$axis
- the line $y = x$ (this line makes a 45° angle between the $x-$ and $y-$axes)

We can develop simple formulas for reflections in these lines.

Let $P(x, y)$ be a point in the coordinate plane.



We now have these reflections of $P(x, y)$:

- Reflection of P in the $x-$axis is $Q(x, -y)$.

[$x-$coordinate stays the same, $y-$coordinate becomes opposite]

- Reflection of P in the $y-$axis is $R(-x, y)$.

[$x-$coordinate becomes opposite, $y-$coordinate stays the same]

- Reflection of P in the line $y = x$ is $S(y, x)$.

[switch the $x-$ and $y-$coordinates]

A look is enough to convince us of the first two reflections. We'll prove the third one.

Example 1

Prove that the reflection of point $P(h, k)$ in the line $y = x$ is the point $S(k, h)$.

Here is an "outline" proof.

First, we know the slope of $y = x$

- Slope of $y = x$ is $1y = 1x + 0$.

Next, let's assume investigate the slope of $\overline{PS}$.

- Slope of $\overline{PS}$ is $\frac{k-h}{h-k} = \frac{-1(h-k)}{h-k} = -1$.

Therefore, we have just shown that $\overline{PS}$ and $y = x$ are perpendicular.

- $\overline{PS}$ is perpendicular to $y = x$ (product of slopes is -1).

Finally, we can show that $y = x$ is the perpendicular bisector of $\overline{PS}$.

- Midpoint of $\overline{PS}$ is $\left(\frac{h+k}{2}, \frac{h+k}{2}\right)$.
- Midpoint of $\overline{PS}$ is on $y = x$ ($x-$ and $y-$coordinates of $\overline{PS}$ are the same).
- $y = x$ is the perpendicular bisector of $\overline{PS}$.

Conclusion: P and S are reflections in the line $y = x$. ♦

Example 2

Point $P(5, 2)$ is reflected in the line $y = x$. The image is P'. P' is then reflected in the $y-$axis. The image is P''. What are the coordinates of P''?

We find one reflection at a time.

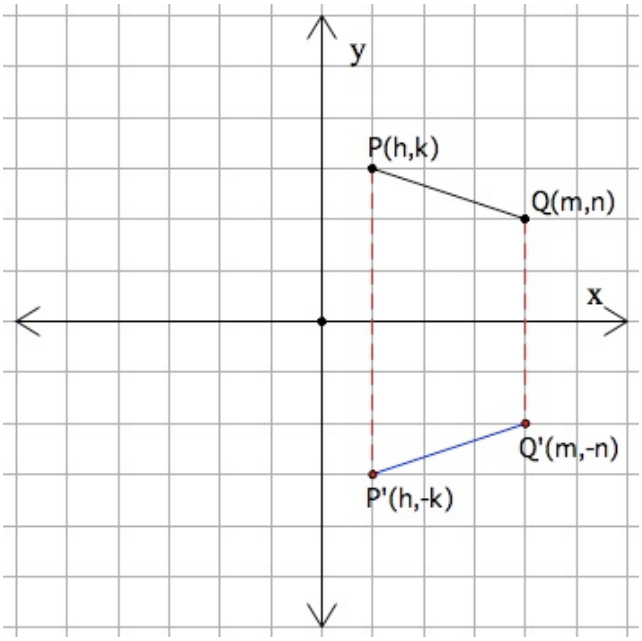
- Reflect P in $y = x$. P' is $(2, 5)$.
- Reflect P' in the $y-$axis. P'' is $(-2, 5)$.

Reflections Are Isometries

A *reflection in a line* is an *isometry.* Distance between points is "preserved" (stays the same).

We will verify the isometry for reflection in the $x-$axis. The story is very similar for reflection in the $y-$axis. You can write a proof that reflection in $y = x$ is an isometry in the Lesson Exercises.

The diagram below shows $\overline{PQ}$ and its reflection in the $x-$axis, $\overline{P'Q'}$.



$$PQ = \sqrt{(m-h)^2 + (n-k)^2}$$
$$P'Q' = \sqrt{(m-h)^2 + (-n-(-k))^2} = \sqrt{(m-h)^3 + (k-n)^2} = \sqrt{(m-h)^2 + (n-k)^2}$$
$$PQ = P'Q'$$

Conclusion: When a segment is reflected in the $x-$axis, the image segment has the same length as the original segment. This is the meaning of isometry. You can see that a similar argument would apply to reflection in any line.

Matrix Multiplication

Multiplying matrices is a little more complicated than addition. Matrix multiplication is sometimes called a "row-by-column" operation. Let's begin with examples.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$, and $C = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} = \begin{bmatrix} 1 \times 7 + 2 \times 9 & 1 \times 8 + 2 \times 10 \\ 3 \times 7 + 4 \times 9 & 3 \times 8 + 4 \times 10 \\ 5 \times 7 + 6 \times 9 & 5 \times 8 + 6 \times 10 \end{bmatrix} = \begin{bmatrix} 25 & 28 \\ 57 & 64 \\ 89 & 100 \end{bmatrix}$$

$3 - \text{by} - 2 \qquad 2 - \text{by} - 2 \qquad 3 - \text{by} - 2$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \times 11 + 2 \times 12 \\ 3 \times 11 + 4 \times 12 \\ 5 \times 11 + 6 \times 12 \end{bmatrix} = \begin{bmatrix} 35 \\ 81 \\ 127 \end{bmatrix}$$

$3 - \text{by} - 2 \qquad 2 - \text{by} - 1 \qquad 3 - \text{by} - 1$

$$BC = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 7 \times 11 + 8 \times 12 \\ 9 \times 11 + 10 \times 12 \end{bmatrix} = \begin{bmatrix} 173 \\ 219 \end{bmatrix}$$

$2 - \text{by} - 2 \qquad 2 - \text{by} - 1 \qquad 2 - \text{by} - 1$

Notice:

- The product is a matrix.
- The number of rows in the product matrix is the same as the number of rows in the left matrix being multiplied.
- The number of columns in the product matrix is the same as the number of columns in the right matrix being multiplied.
- The number of columns in the left matrix is the same as the number of rows in the right matrix.
- To compute a given element of the product matrix, we multiply each element of that row in the left matrix by the corresponding element in that column in the right matrix, and add these products.

Some of this information can be stated easily in symbols.

- If A is an $m-\text{by}-n$ matrix, then B must be an $n-\text{by}-p$ matrix in order to find the product AB.
- AB is an $m-\text{by}-p$ matrix.

Let's look again at matrices A and B above:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \; B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$$

We found AB. Is $BA = AB$ true? Surprisingly, we cannot even *calculate* BA. This would have us multiplying a left matrix that is $2-\text{by}-2$ times a right matrix that is $3-\text{by}-2$. This does not satisfy the requirements stated above. It's not that BA does not *equal* $AB-$ the fact is, BA does not even exist! Conclusion: Multiplication of matrices is not commutative.

Translated loosely, some matrices you can't even multiply, and for some matrices that you can multiply, the operation is not commutative.

Example 3

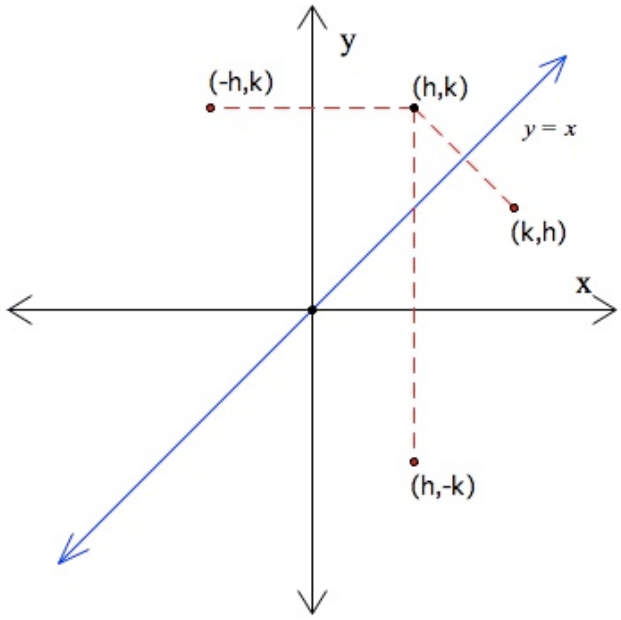
Do the following operation: $\begin{bmatrix} 3 & -5 \\ -4 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & -5 \\ -4 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3+-5 & 0+0 \\ -4+2 & 0+0 \\ -2+3 & 0+0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2 & 0 \\ 1 & 0 \end{bmatrix}$$

Notice: This multiplication in effect adds the elements of each row of the left matrix for the first element in the product matrix, and inserts a 0 for the second element in each row of the product matrix.

Matrix Multiplication and Reflections

We know from earlier work how reflections in the $x-$axis, the $y-$axis, and the line $y = x$ affect the coordinates of a point. Those results are summarized in the following diagram.



Now we can use matrix arithmetic to express reflections.

Given a point (h, k) in the coordinate plane, we will use matrix multiplication to reflect the point. Note: In all the matrix multiplications that follow, we multiply (h, k) on the **left** by a reflection matrix on the **right**. Remember (from above), left and right placement matter!

- Reflection in the $x-$axis: Multiply any point or polygon matrix by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof. $\begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} h.1 + k.0 & h.0 + k(-1) \end{bmatrix} = \begin{bmatrix} h & -k \end{bmatrix}$

- Reflection in the $y-$axis: Multiply any point or polygon matrix by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. $\begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} h.(-1) + k.0 & h.0 + k.1 \end{bmatrix} = \begin{bmatrix} -h & k \end{bmatrix}$

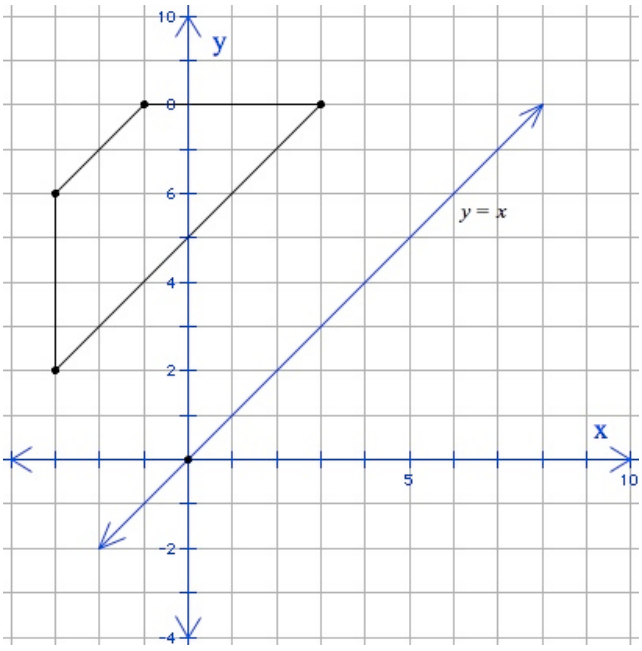
- Reflection in $y = x$: Multiply any point or polygon matrix by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof. The proof is available in the Lesson Exercises.

Example 4

The trapezoid below is reflected in the line $y = x$.



What are the coordinates of the vertices of the image of the trapezoid?

1. Write a polygon matrix for the coordinates of the vertices of the trapezoid.

$$\begin{bmatrix} -3 & 2 \\ -3 & 6 \\ -1 & 8 \\ 3 & 8 \end{bmatrix}$$

2. Multiply the polygon matrix by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (on the right).

$$\begin{bmatrix} -3 & 2 \\ -3 & 6 \\ -1 & 8 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 6 & -3 \\ 8 & -1 \\ 8 & 3 \end{bmatrix}$$

3. Interpret the product matrix.

The vertices of the image of the trapezoid are $(2, -3), (6, -3), (8, -1)$, and $(8, 3)$.

Lesson Summary

A point or set of points, such as a polygon, can be reflected in a line. In this lesson we focused on reflections in three important lines: the $x-$axis, the $y-$axis, and the line $y = x$.

Matrices can be multiplied. Matrix multiplication is a row-by-column operation. Matrix multiplication is not commutative, and not all matrices can even be multiplied.

Multiplication of a polygon matrix by one of the special matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ will reflect the polygon in the $x-$axis, the $y-$axis, or the line $y = x$ respectively.

Points to Consider

You saw in this lesson that reflections correspond to multiplication by a particular matrix. You might be interested to investigate how multiplication on the right of a polygon matrix by one of the following matrices changes the original matrix.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

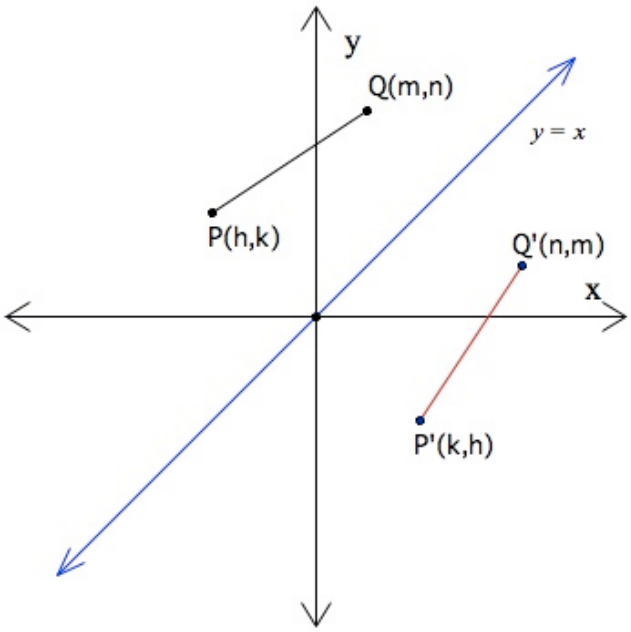
As if matrix arithmetic is not "different" enough, in an upcoming lesson we'll see that there is another kind of multiplication called *scalar* multiplication. Scalar multiplication will enable us to use matrices to represent dilations in the coordinate plane.

Review Questions

1. Prove that reflection in the line $y = x$ is an isometry.

Given: $\overline{PQ}$ and $\overline{P'Q'}$, its reflection in $y = x$

Prove: $PQ = P'Q'$



$PQ = \sqrt{(m-h)^2 + (n-k)^2}$

Let $A = \begin{bmatrix} 2 & -5 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$.

Write the matrix for each product.

2. AB
3. BC
4. A^2

Let A be a polygon matrix. Fill in the blank(s).

5. If $AB = A$, then $B =$___?___.
6. If $AB = BA$, and A is a $5 - \text{by} - 2$ matrix, then AB is a _____-by_____ matrix.

If $\begin{bmatrix} 2 & -5 \\ 1 & 4 \end{bmatrix} X = \begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}$, then $X =$____?____

7. If A is an $n - \text{by} - 2$ matrix, then $\left(A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \underline{?}$

Review Answers

1. $P'Q' = \sqrt{(n-k)^2 + (m-h)^2}$
   $PQ = P'Q'$

2. $\begin{bmatrix} 19 \\ -10 \end{bmatrix}$
3. $\begin{bmatrix} 2 & 7 & -7 \end{bmatrix}$
4. $\begin{bmatrix} -1 & -30 \\ 6 & 11 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
6. 2 by 5
7. A

12.4 Rotation

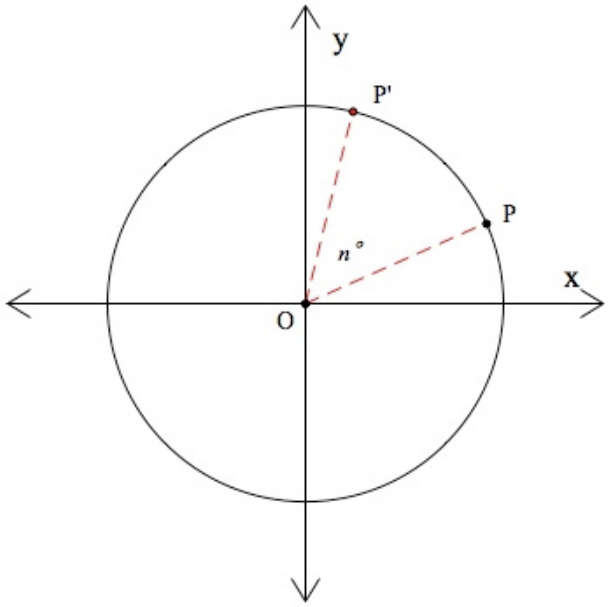
Learning Objectives

- Find the image of a point in a rotation in a coordinate plane.
- Recognize that a rotation is an isometry.
- Apply matrix multiplication to rotations.

Sample Rotations

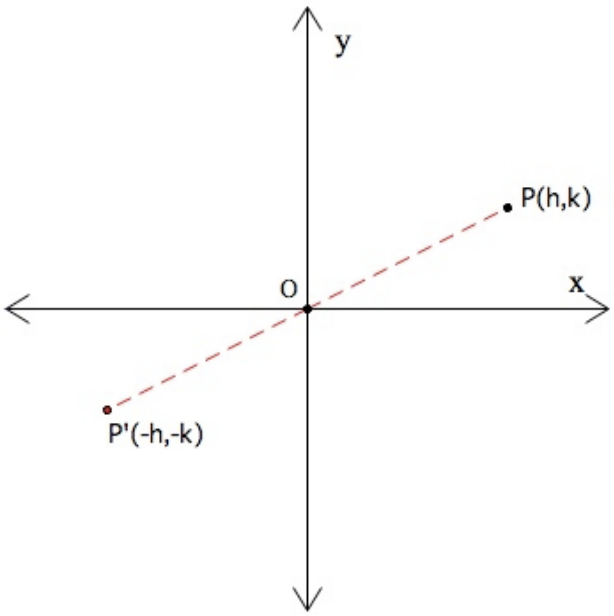
In this lesson we limit our study to rotations centered at the origin of a coordinate plane. We begin with some specific examples of rotations. Later we'll see how these rotations fit into a general formula.

Remember how a rotation is defined. In a rotation centered at the origin with an angle of rotation of n°, a point moves counterclockwise along an arc of a circle. The central angle of the circle measures n°. The original point is one endpoint of the arc, and the image of the original point is the other endpoint of the arc.



180° Rotation

Our first example is rotation through an angle of 180° .

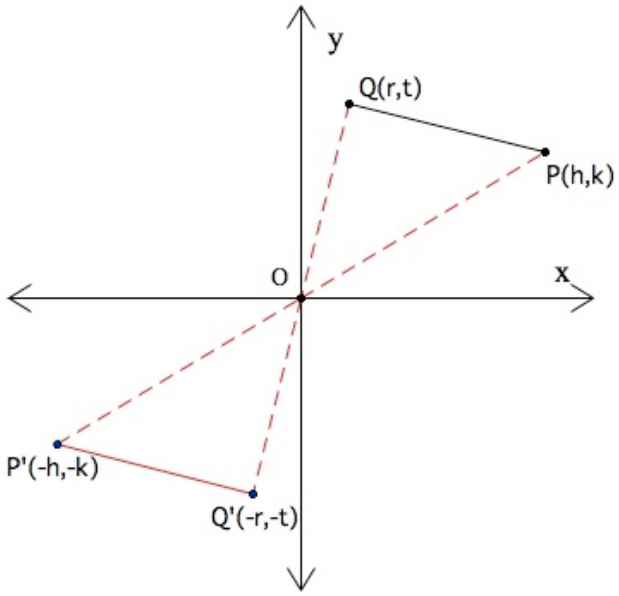


In a 180° rotation, the image of $P(h, k)$ is the point $P'(-h, -k)$.

Notice:

- P and P' are the endpoints of a diameter.
- The rotation is the same as a "reflection in the origin."

A 180° rotation is an isometry. The image of a segment is a congruent segment.

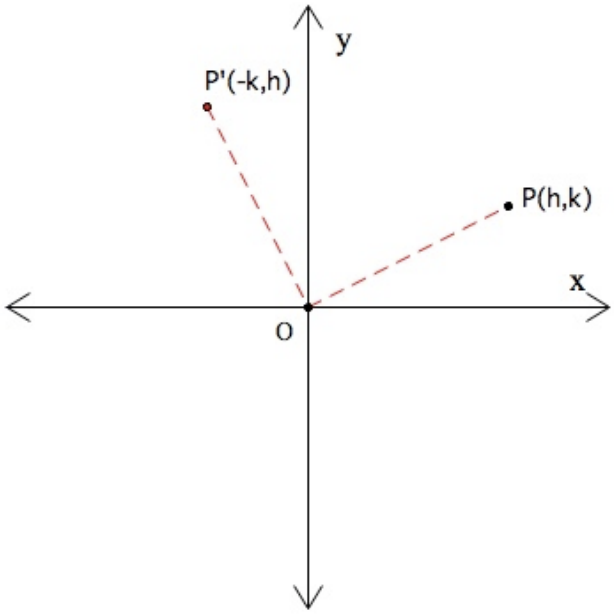


$$\begin{aligned} PQ &= \sqrt{(k-t)^2+(h-r)^2} \\ P'Q' &= \sqrt{(-k--t)^2+(-h--r)^2} = \sqrt{(-k+t)^2+(-h+r)^2} \\ &= \sqrt{(t-k)^2+(r-h)^2} = \sqrt{(k-t)^2+(h-r)^2} \\ PQ &= P'Q' \end{aligned}$$

If M is a polygon matrix, then the matrix for the image of the polygon in a 180° rotation is the product $M \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. The Lesson Exercises include exploration of this matrix for a 180° rotation.

90° Rotation

The next example is a rotation through an angle of 90° .

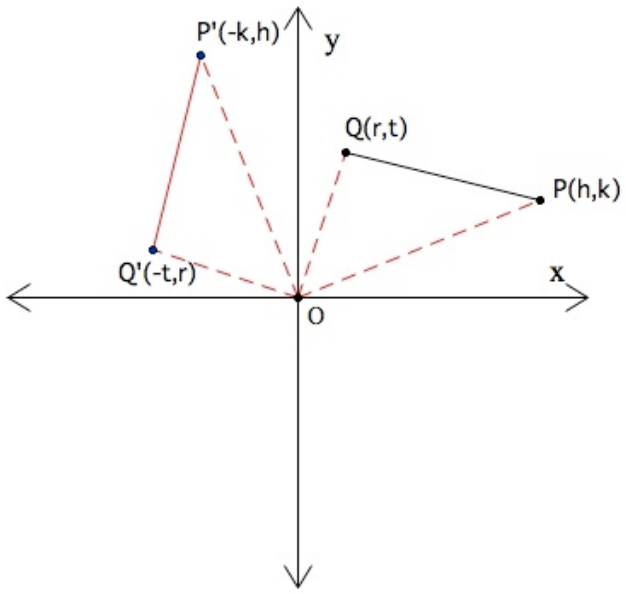


In a 90° rotation, the image of $P(h, k)$ is the point $P'(-k, h)$.

Notice:

- $\overline{PO}$ and $\overline{P'O}$ are radii of the same circle, so $PO = P'O$.
- $\angle POP'$ is a right angle.
- The acute angle formed by $\overline{PO}$ and the $x-$axis and the acute angle formed by $\overline{P'O}$ and the $x-$axis are complementary angles.

A 90° rotation is an isometry. The image of a segment is a congruent segment.

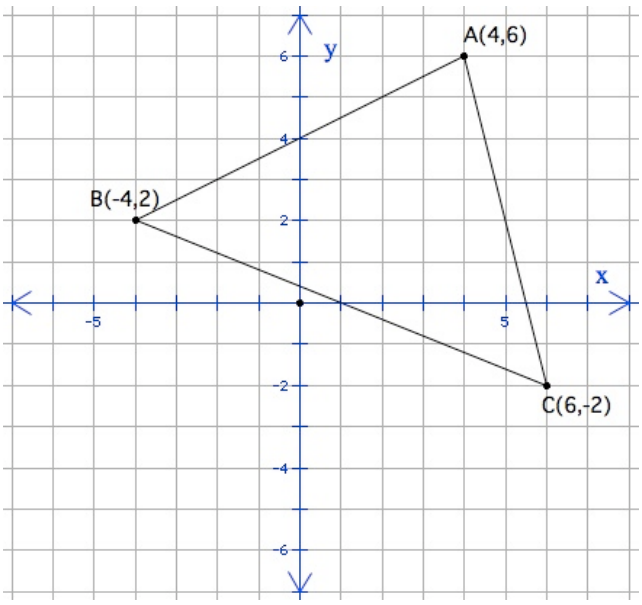


$$\begin{aligned} PQ &= \sqrt{(k-t)^2+(h-r)^2} \\ P'Q' &= \sqrt{(h-r)^2+(-k--t)^2} = \sqrt{(h-r)^2+(t-k)^2} \\ &= \sqrt{(k-t)^2+(h-r)^2} \\ PQ &= P'Q' \end{aligned}$$

If M is a polygon matrix, then the matrix for the image of the polygon in a 90° rotation is the product $M' = M \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The Lesson Exercises include exploration of this matrix for a 90° rotation.

Example 1

What are the coordinates of the vertices of $\triangle ABC$ *in a rotation of* 90°?



Mark axes by 1 s A is $(4, 6)$, B is $(-4, 2)$, C is $(6, -2)$.

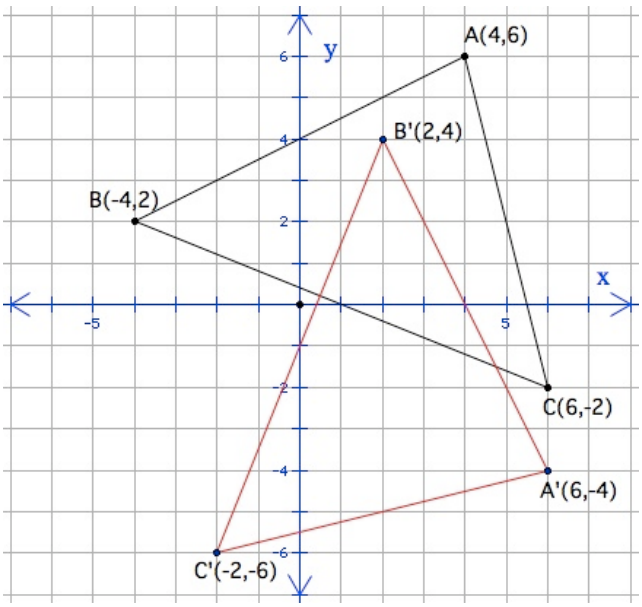
The matrix below represents the vertices of the triangle.

$$\begin{bmatrix} 4 & 6 \\ -4 & 2 \\ 6 & -2 \end{bmatrix}$$

The matrix for the image of $\triangle ABC$ is the product:

$$\begin{bmatrix} 4 & 6 \\ -4 & 2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 \times 0 + 6 \times 1 & 4 \times (-1) + 6 \times 0 \\ (-4) \times 0 + 2 \times 1 & (-4) \times (-1) + 6 \times 0 \\ 6 \times 0 + (-2) \times 1 & 6 \times (-1) + (-2) \times 0 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 2 & 4 \\ -2 & -6 \end{bmatrix}$$

The vertices of $\triangle A'B'C'$ are $(6, -4), (2, 4)$, and $(-2, -6)$



Rotations in General

Let M be a polygon matrix. The matrix for the image of the polygon in a rotation of θ degrees is the product M', where

$$M' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Example 2

Verify that the matrix product for a $90°$ *rotation is a special case of this formula for a general rotation.*

We know that $\sin 90° = 1$ and $\cos 90° = 0$.

For $\theta = 90°$, the general matrix product is

$$M \begin{bmatrix} \cos 90° & \sin 90° \\ -\sin 90° & \cos 90° \end{bmatrix} = M \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Note that this is the matrix product for a $90°$ rotation. (See above.)

Example 3

The point $P(4, 4)$ *is rotated* $45°$. *What are the coordinates of* P'?

$$\begin{bmatrix}4 & 4\end{bmatrix}\begin{bmatrix}\cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ\end{bmatrix} = \begin{bmatrix}4 & 4\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}0 & 4\sqrt{2}\end{bmatrix}$$

The coordinates of P' are $\left(0, 4\sqrt{2}\right)$.

Note: The distance from the origin to P is $(0, 4\sqrt{2})$. When P rotates 45°, its image is on the $y-$axis, the same distance from the origin as P.

Lesson Summary

To find M', the image of polygon matrix M rotated about the origin:

1. 180° Rotation

$$M' = M\begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix}$$

2. 90° Rotation

$$M' = M\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}$$

3. θ° Rotation

$$M' = M\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$

Points to Consider

You've now studied several transformations that are isometries: translations, reflections, and rotations. Yet to come is one more basic transformation that is *not* an isometry, which is the dilation.

The Lesson Summary above listed a few formulas for rotations. Suppose you only had the first two formulas. Would you be able to find the coordinates of the image of a polygon that rotates 270°, or −90°, or 810°? For that matter, would formula 2 in the summary be enough to find the image of a polygon that rotates 180°? These rotations can be solved using *compositions* of other rotations, a topic coming up in a later lesson.

Review Questions

Let P be the point with coordinates $(-3, 8)$.

1. Write a matrix M to represent the coordinates of P.
2. Write the matrix for the product $M \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
3. What are the coordinates of the point Q represented by the product?
4. Prove that P, the origin O, and Q are collinear.
5. Prove that $\overline{PO} \cong \overline{OQ}$.

A line in the coordinate plane has the equation $y = 3x - 6$.

6. Where does the line intersect the $y-$axis?
7. What is the slope of the line?

The line is rotated $45°$.

8. Where does the rotated line intersect the $y-$axis?
9. What is the slope of the rotated line?
10. What is the equation of the rotated line?

The endpoints of a segment are $P(6, -2)$ and $Q(-3, 1)$. The segment is rotated $90°$.

11. What are the coordinates of P' and Q'?
12. What is the slope of $\overline{PQ}$?
13. What is the slope of $\overline{P'Q'}$?
14. Let $M = \begin{bmatrix} 6 & -2 \end{bmatrix}$. Explain how the product $M \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2$ would move point P.

Review Answers

1. $M = \begin{bmatrix} -3 & 8 \end{bmatrix}$
2. $M = \begin{bmatrix} 3 & -8 \end{bmatrix}$
3. $Q(3, -8)$
4. slope of $\overline{PQ} = \frac{-8-8}{3-(-3)} = -\frac{16}{6} = -\frac{8}{3}$
   slope of $\overline{PO} = \frac{8-0}{-3-0} = -\frac{8}{3}$
   slope of $\overline{QO} = \frac{-8-0}{3-0} = -\frac{8}{3}$

5. $PO = \sqrt{(-3-0)^2 + (8-0)^2} = \sqrt{73}$
   $OQ = \sqrt{(3-0)^2 + (-8-0)^2} = \sqrt{73}$
   $PO = OQ$
   $\overline{PO} \cong \overline{OQ}$
6. $(0, -6)$
7. 3
8. $(0, 3\sqrt{2})$
9. -2
10. $y = -2x + 3\sqrt{2}$
11. $P'(2, 6), Q'(-1, -3)$
12. $-\frac{1}{3}$
13. 3
14. P would not move. The product is equivalent to two 180° rotations, which is equivalent to a 360° rotation, which is equivalent to no rotation.

12.5 Composition

Learning Objectives

- Understand the meaning of composition.
- Plot the image of a point in a composite transformation.
- Describe the effect of a composition on a point or polygon.
- Supply a single transformation that is equivalent to a composite of two transformations.

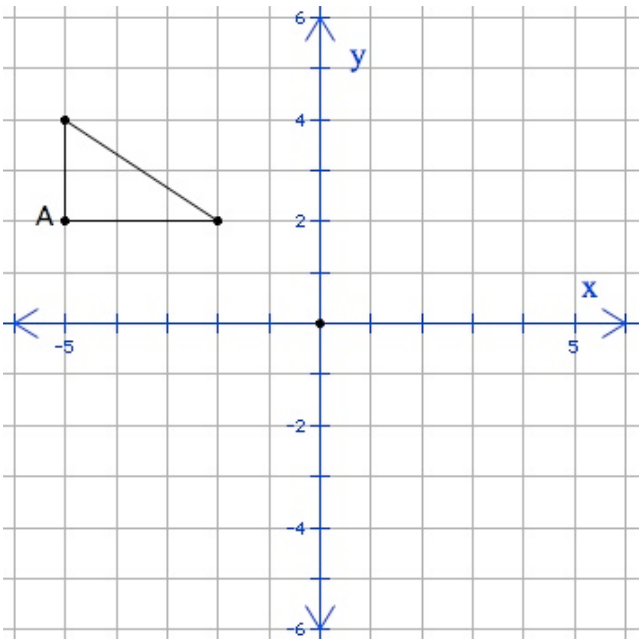
Introduction

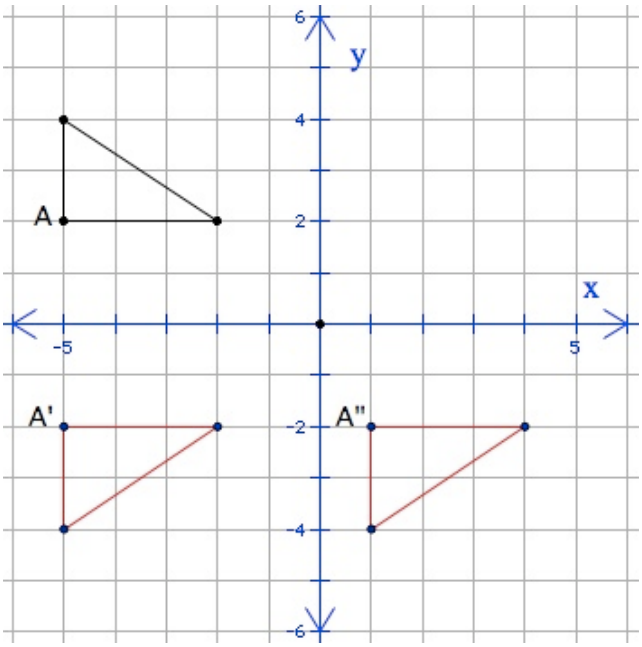
The word *composition* comes from Latin roots meaning *together*, *com-*, and *to put*, *-position*. In this lesson we will "put together" some of the basic isometry transformations: translations, reflections, and rotations. Compositions of these transformations are themselves isometry transformations.

Glide Reflection

A **glide reflection** is a composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

The shape below is moved with a glide reflection. It is reflected in the $x-$axis, and the image is then translated 6 units to the right.





In the diagram, one point is followed to show how it moves. First $A(-5, 2)$ is reflected in the $x-$axis. Its image is $A'(-5, -2)$. Then the image is translated 6 units to the right. The final image is $A''(1, -2)$.

Example 1

a) *What is the image of* $P(10 - 8)$ *if it follows the same glide reflection as above?*

P is reflected in the $x-$axis to $P'(10, 8)$. $P'(10, 8)$ is translated 6 units to the right to

$P''(16, 8)$. The final image of $P(10, -8)$ is $P''(16, 8)$.

b) *What is the image of* $P(h, k)$*?*

(h, k) is reflected in the $x-$axis to $(h, -k)$. $(h, -k)$ is translated 6 units to the right to $(h + 6, -k)$. The final image of $P(h, k)$ is $(h + 6, -k)$.

Notice that the image of (h, k) can be found using matrices.

$$\begin{bmatrix} h+6 & -k \end{bmatrix} = \left(\begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) + \begin{bmatrix} 6 & 0 \end{bmatrix}$$

reflection in $x-$axis translation 6 units right

Example 2

How can a rotation of 270° *be expressed as a composition?*

A 270° rotation is the same as a 180° rotation followed by a 90° rotation. If M is the matrix for a polygon, then

$$\left(m \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

180°rotation 90°rotation

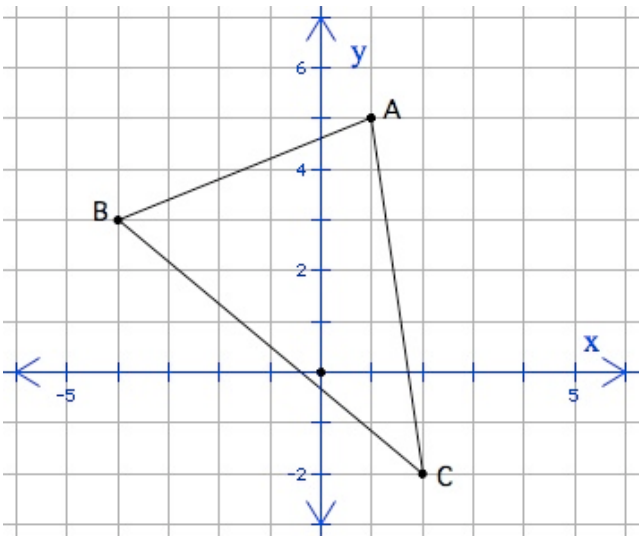
is the matrix for the image of M in a 270° rotation.

Note that

$$\left(M \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(90° rotation followed by 180° rotation) is also the image of M in a 270° rotation.

The triangle below is rotated 270°.



Vertices are $A(1, 5), B(-4, 3)$, and $C(2, -2)$.

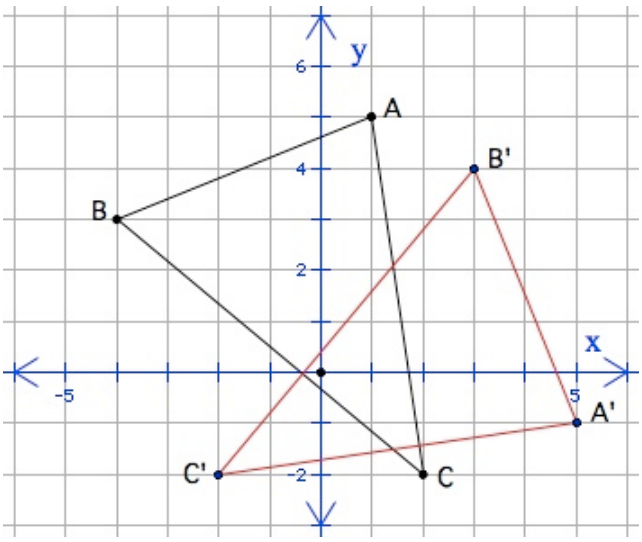
What are the coordinates of the vertices of the image triangle?

The matrix for the triangle is $M = \begin{bmatrix} 1 & 5 \\ -4 & 3 \\ 2 & -2 \end{bmatrix}$.

$$M' = \left(M \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \left(\begin{bmatrix} 1 & 5 \\ -4 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 \\ 4 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 4 \\ -2 & -2 \end{bmatrix}$$

The image of the original triangle is shown as the dashed triangle below. The vertices are $A'(5, -1), B'(3, 4)$ and $C'(-2, -2)$.



Vertices of left hand triangle are $(1, 5), (-4, 3)$, and $(2, -2)$ vertices of the other triangle are $(5, -1), (3, 4)$, and $(-2, -2)$.

Reflections in Two Lines

The Technology Note below gives a preview of how to reflect in two lines.

Technology Note - Geometer's Sketchpad

The following animations show a reflection in two parallel lines, and a reflection in two intersecting lines, in a step-by-step view. **Note:** Geometer's Sketchpad software is required to view these files.

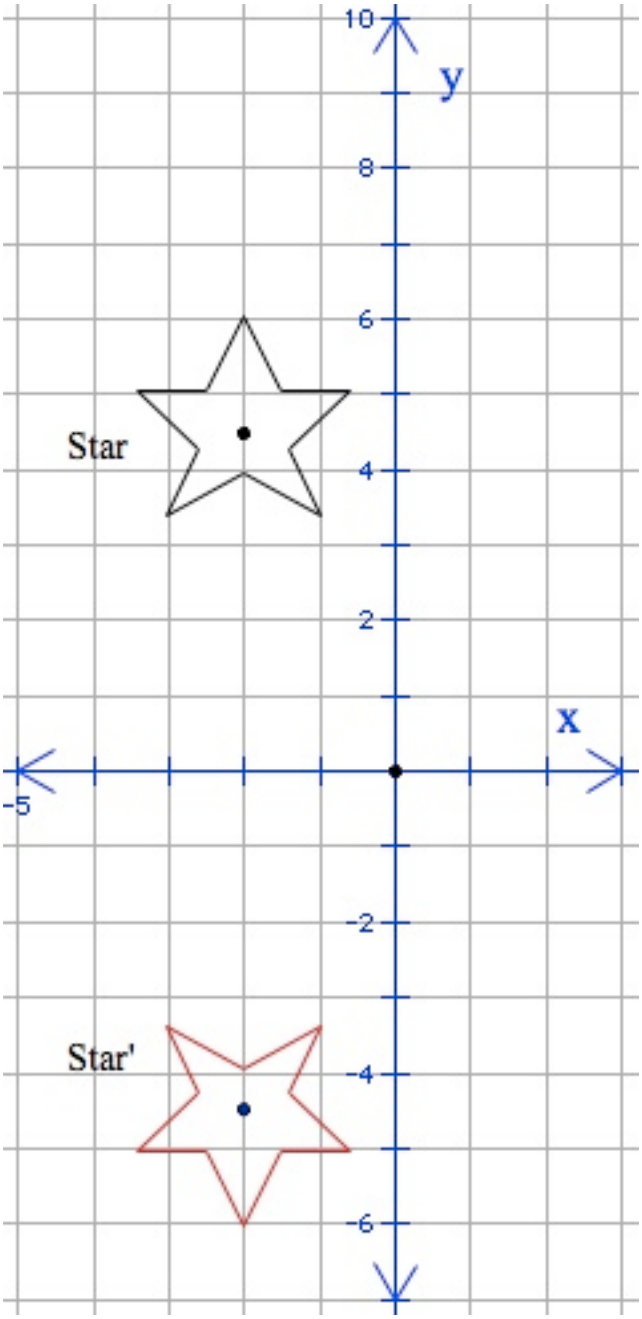
At http://tttc.org/find/wpShow.cgi?wpID=1096, scroll down to Downloads. Find:

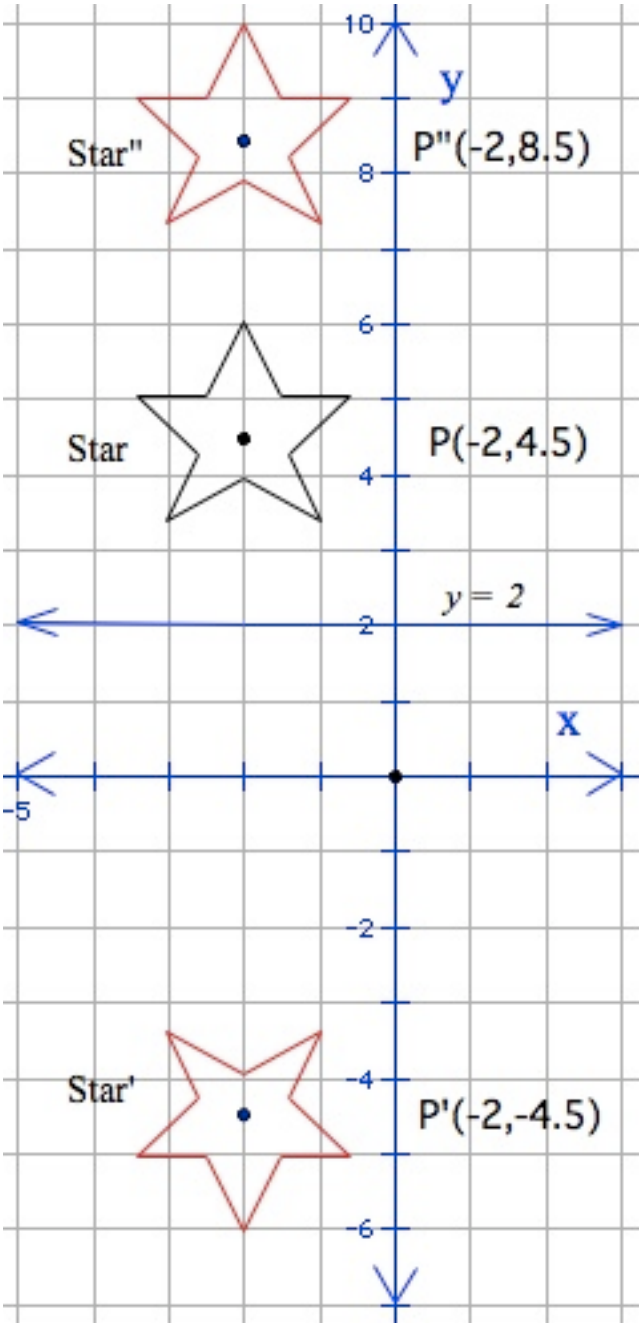
- http://tttc.org/find/wpFile.cgi?id=17534 Composite Reflection Parallel Lines
- http://tttc.org/find/wpFile.cgi?id=17532 Composite Reflection Intersecting Lines

Example 3

The star is reflected in the $x-$axis. The image of the reflection in the $x-$axis is **Star**'.

Then the image is reflected in the line $y = 2$. The image of the reflection of Star' in the line $y = 2$ is **Star**".


10
y
8
6
Star
4
2
x
-5
-2
Star'
-4
-6



One point on the original star, $(-2, 4.5)$, is tracked as it is moved by the two reflections.

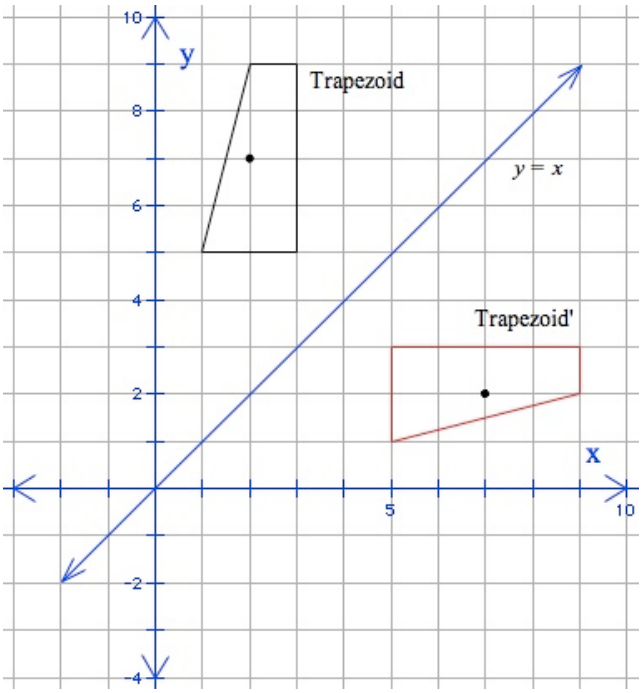
Note that:

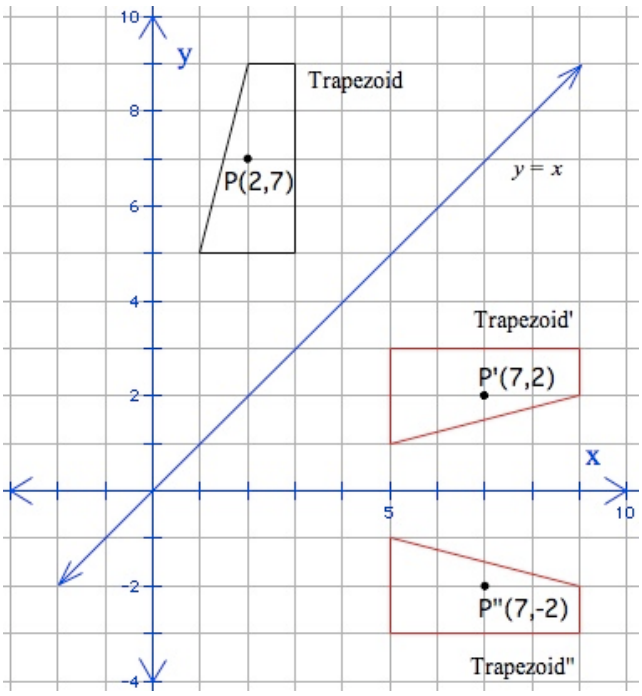
- Star is right side up, **Star** is upside down," and **Star** is right side up.
- P is 4.5 units above the $x-$axis. P' is 4.5 units below the $x-$axis.
- P' is 6.5 units below the line $y = 2$. P'' is 6.5 units above the line $y = 2$.

Example 4

The trapezoid is reflected in the line $y = x$. The image of the reflection of Trapezoid in $y = x$ is **Trapezoid'**.

Then the image is reflected in the $x-$axis. The image of the reflection of **Trapezoid'** in the $x-$axis is **Trapezoid"**.





One point on the original arrow box, $P(2, 7)$, is tracked as it is moved by the two reflections.

Note that:

- Trapezoid is rotated $-90°$ to produce **Trapezoid'**.
- P' is 2 units above the $x-$axis. P'' is 2 units below the $x-$axis.

Lesson Summary

A composition is a combination of two (or more) transformations that are applied in a specific order. You saw examples of several kinds of compositions.

- Glide reflection
- Two rotations
- Reflection in parallel lines
- Reflection in intersecting lines

These compositions are combinations of transformations that are isometries. The compositions are themselves isometries.

For some of the compositions in this lesson you saw a matrix operation that can be used to find the image of a point or polygon. Also, you saw that in some cases there is a simple basic transformation that is equivalent to a composition.

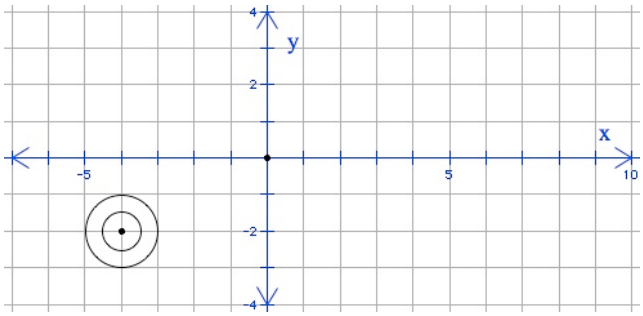
Points to Consider

In this lesson you studied compositions of isometries, which are also isometries. We know that there are other transformations that are not isometries. The prime example is the dilation. We return to dilations in a later lesson, where a second type of multiplication of a matrix is introduced.

Does geometry have a place in art and design? Most people would guess that they do. We'll get a chance to start to see how when we examine tessellations and symmetry in future lessons.

Review Questions

1. Explain why the composition of two or more isometries must also be an isometry.
2. Recall the glide reflection in example 1. Suppose this glide reflection is applied to a triangle, and then applied again to the image of the triangle. Describe how the final image compares to the original triangle.
3. What one basic transformation is equivalent to a reflection in two parallel lines?
4. A point is reflected in line k. The image is reflected in line m. $k : m$, and the two lines are 5 units apart. What is the distance from the original point to the final image?
5. Point P is reflected in two parallel lines. Does it matter in which line P is reflected first? Explain.
6. What one basic transformation is equivalent to a reflection in two perpendicular lines?
7. Point P is reflected in the two axes. Does it matter in which axis P is reflected first? Explain.
8. Prove: Reflection in $y = x$, followed by reflection in $y = -x$, is equivalent to a 180° rotation.
9. The glide reflection in example 1 is applied to the "donut" below. It is reflected over the $x-$ axis, and the image is then translated 6 units to the right. The same glide reflection is applied again to the image of the first glide reflection. What are the coordinates of the center of the donut in the final image?



10. Describe how the final image is related to the original donut.

Review Answers

1. Images in the first isometry are congruent to the original figure. The same is true of the second isometry. If P is a polygon, P' the image after the first isometry, and P'' the image of P' after the second isometry, we know that $P \cong P'$ and $P' \cong P''$. So $P \cong P''$, i.e., the final image is congruent to the original.
2. The original triangle has moved 12 units to the right.
3. A translation
4. 10
5. Yes. Both final images will be equivalent to translations, but the final images can be different distances from the original point P.
6. A 180° rotation
7. No. The final image is the same regardless of the order of the two reflections.
8. Reflection in $y = x$ maps (x, y) to (y, x). Reflection in $Y = -x$ maps (x, y) to $(-y, -x)$. Point (h, k) goes to (k, h) in the first reflection, then that point goes to $(-h, -k)$ in the second reflection. It was established earlier that $(h, k) \rightarrow (-h, -k)$ is equivalent to a 180° rotation.
9. $(8, -2)$
10. The original donut has been moved 12 units to the right. It's the same as a translation 12 units to the right.

12.6 Tessellations

Learning Objectives

- Understand the meaning of tessellation.
- Determine whether or not a given shape will tessellate.
- Identify the regular polygons that will tessellate.
- Draw your own tessellation.

Introduction

You've seen *tessellations* before, even if you didn't call them that!

- a tile floor
- a brick or block wall
- a checker or chess board
- a fabric pattern

Here are examples of tessellations, which show two ways that kites can be tessellated:

Figure 12.1

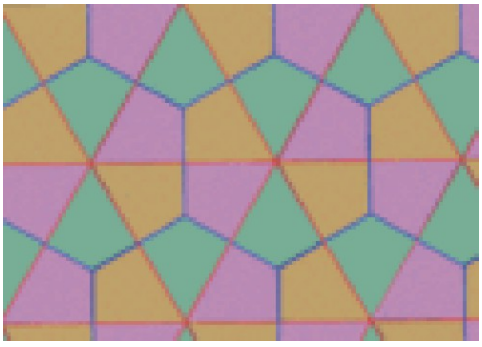


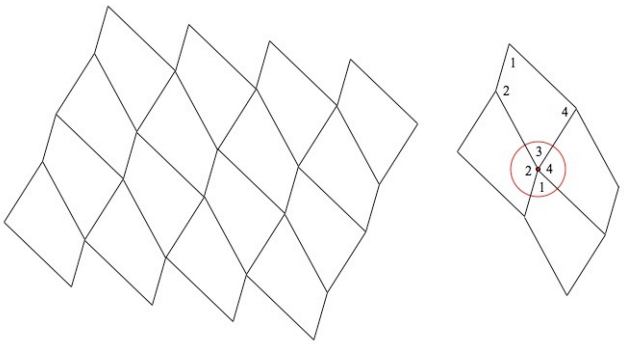
Figure 12.2

What is a tessellation? What does it mean to say that a given shape tessellates?

To **tessellate** with a given shape means that the copies of that shape can cover a plane.

- There will be no uncovered gaps.
- There will be no overlapping shapes.
- The entire plane will be covered, in all directions.

Look at an example.



A quadrilateral fits together just right. If we keep adding more of them, they will entirely cover the plane with no gaps or overlaps. Now the tessellation pattern could be colored creatively to make interesting and/or attractive patterns.

Note: To tessellate, a shape must be able to exactly "surround" a point.

- What is the sum of the measures of the angles?

360° is the sum of the angle measures in any quadrilateral.

- Describe which angles come together at a given point to "surround" it.

One of each of the angles of the quadrilateral fit together at each "surrounded" point.

Will all quadrilaterals tessellate? Yes, for the reasons above.

Tessellating with a Regular Polygon

Technology Note- "Virtual" Pattern Blocks-TM

At the National Library of Virtual Manipulatives you can experiment with Pattern Blocks ^{TM} and other shapes. Try to put congruent rhombi, trapezoids, and other quadrilaterals

together to make tessellations. The National Library of Virtual Manipulatives can be found at: http://nlvm.usu.edu/en/nav/siteinfo.html.

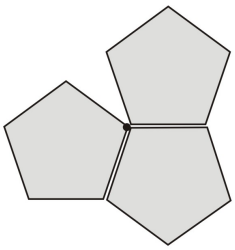
A good tessellation site is http://nlvm.usu.edu/en/nav/frames_asid_163_g_4_t_3.html?open=activities&from=applets/controller/query/query.htm?qt=tessellations&lang=en.

A square will tessellate. This is obvious, if you've ever seen a chessboard or a graph paper grid. A square is a regular polygon. Will all regular polygons tessellate?

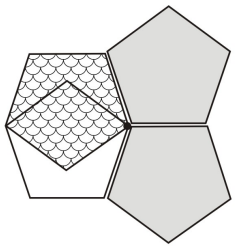
Example 2

Can a regular pentagon tessellate?

Here's what happens if we try to "surround" a point with congruent regular pentagons.



3 regular pentagons leave a gap



4 regular pentagons
4th pentagon "overlaps"

Regular pentagons can't surround a point. Three aren't enough, and four are too many. Remember that each angle of a regular pentagon measures 108°. Angles of 108° do not combine to equal the 360° that it takes to surround a point.

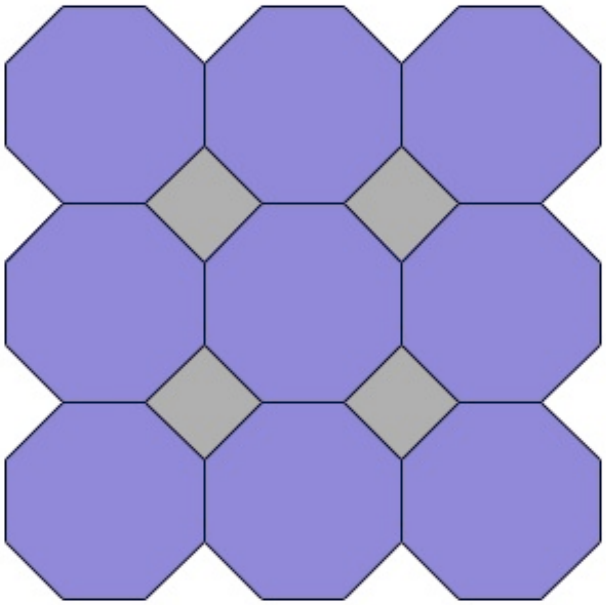
Apparently some regular polygons will tessellate and some won't. You can explore this more in the Lesson Exercises.

Tessellating with Two Regular Polygons

You saw that some regular polygons can tessellate by themselves. If we relax our requirements and allow two regular polygons, more tessellations can be drawn.

Example 3

Here is a tessellation made from squares and regular octagons.



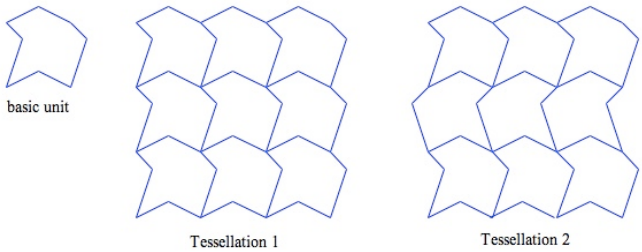
Note the measures of the angles where two octagons and a square come together to surround a point. They are $135°, 135°$, and $90°$, and $135° + 135° + 90° = 360°$.

One way to look at this tessellation is as a translation. Each octagon is repeatedly translated to the right and down. Or the tessellation could be seen as a reflection. Reflect each octagon in its vertical and horizontal edges. The tessellation can even be viewed as a glide reflection. Translate an octagon to the right, then reflect it vertically.

A tessellation based on more than one regular polygon is called a *semi-regular tessellation*. One semi-regular tessellation is shown in the example 3. You'll have the opportunity to create others in the Lesson Exercises.

Tessellation DIY ("Do It Yourself")

All people can create their own tessellations. Here is an example.



Note that the same basic "unit" can be used to make different tessellations.

Further Reading

M. C. Escher was a famous twentieth-century graphic artist who specialized in extremely original, provocative tessellations. You can read about him and see many examples of his art in *M.C. Escher: His Life and Complete Graphic Work*. New York: H.N. Abrams

Lesson Summary

Tessellations are at the intersection of geometry and design. Many—but not all—of the most common polygons will tessellate; some will not. Some of the regular polygons will tessellate by themselves. Semi-regular tessellations are made up of two (or more) of the regular polygons. There is no need to limit tessellations to regular polygons or even to polygons. Anyone can draw a tessellation, using whatever shape desired as long as it will, in fact, tessellate.

The repetitive patterns that make tessellations are related to transformations. For example, a tessellation may consist of a basic unit that is repeatedly translated or reflected.

Points to Consider

All tessellations show some kind of symmetry. Why? Because that is a natural result of creating a pattern through reflection or translation. We will examine symmetry more thoroughly in an upcoming lesson.

Tessellations can also be created through rotations. Just as we have seen composite transformations, there are also composite tessellations that use two or more transformations.

Look around in your daily life. Where do you see tessellations?

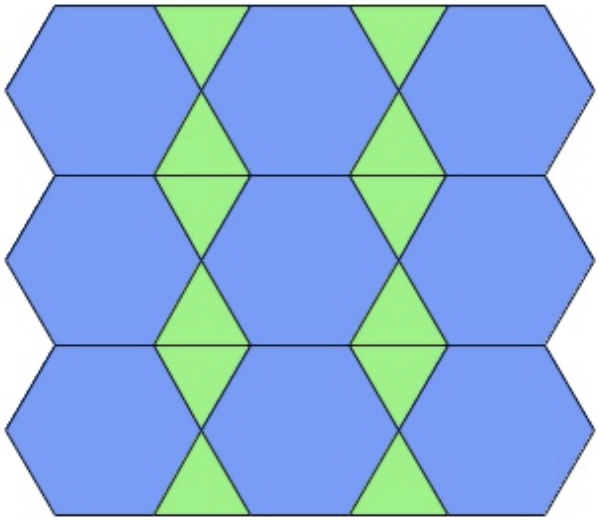
Review Questions

Will the given shape tessellate? If the answer is yes, make a drawing on grid paper to show the tessellation. (D1)

1. A square
2. A rectangle
3. A rhombus
4. A parallelogram
5. A trapezoid
6. A kite
7. A completely irregular quadrilateral
8. Which regular polygons will tessellate?
9. Use equilateral triangles and regular hexagons to draw a semiregular tessellation.

Review Answers

1. Yes
2. Yes
3. Yes
4. Yes
5. Yes
6. Yes
7. Yes
8. Equilateral triangle, square, regular hexagon
9. One example is shown here



These are regular hexagons with equilateral triangles that fit in to exactly fill the space between the hexagons.

12.7 Symmetry

Learning Objectives

- Understand the meaning of symmetry.
- Determine all the symmetries for a given plane figure.
- Draw or complete a figure with a given symmetry.
- Identify planes of symmetry for three-dimensional figures.

Introduction

You know a lot about symmetry, even if you haven't studied it before. Symmetry is found throughout our world—both the natural world and the human-made world that we live in. You may have studied symmetry in math classes, or even in other classes such as biology and art, where symmetry is a basic principle.

The transformations we developed in earlier work have counterparts in symmetry. We will focus here on three plane symmetries and a three-dimensional symmetry.

- line symmetry
- rotational symmetry
- point symmetry
- planes of symmetry

A plane of symmetry may be a new concept, as it applies to three-dimensional objects.

Line Symmetry

Line symmetry is very familiar. It could be called "left-right" symmetry.

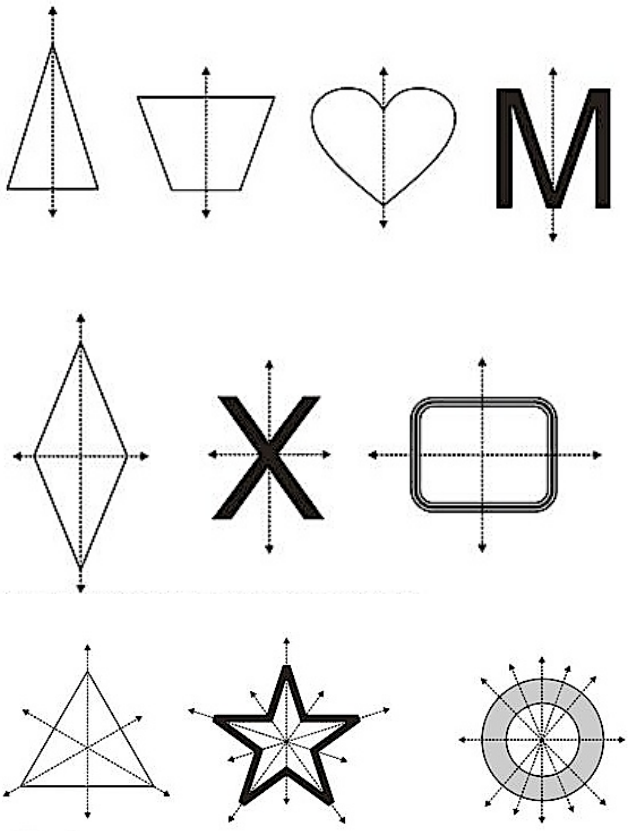
A plane (two-dimensional) figure has a **line of symmetry** if the figure can be reflected over the line and the image of every point of the figure is a point on the original figure.

In effect, this says that the reflection is the original figure itself.

Another way to express this is to say that:

- the line of symmetry divides the figure into two congruent halves
- each half can be flipped (reflected) over the line
- and when it is flipped each half is identical to the other half

Many figures have line symmetry, but some do not have line symmetry. Some figures have more than one line of symmetry.



In biology line symmetry is called *bilateral symmetry*. The plane representation of a leaf, for example, may have bilateral symmetry—it can be split down the middle into two halves that are reflections of each other.

Rotational Symmetry

A plane (two-dimensional) figure has **rotational symmetry** if the figure can be rotated and the image of every point of the figure is a point on the original figure.

In effect, this says that the rotated image is the original figure itself.

Another way to express this is to say: After being rotated, the figure looks exactly as it did before the rotation.

Note that the center of rotation is the "center" of the figure.

Leaf with vertical axis of symmetry

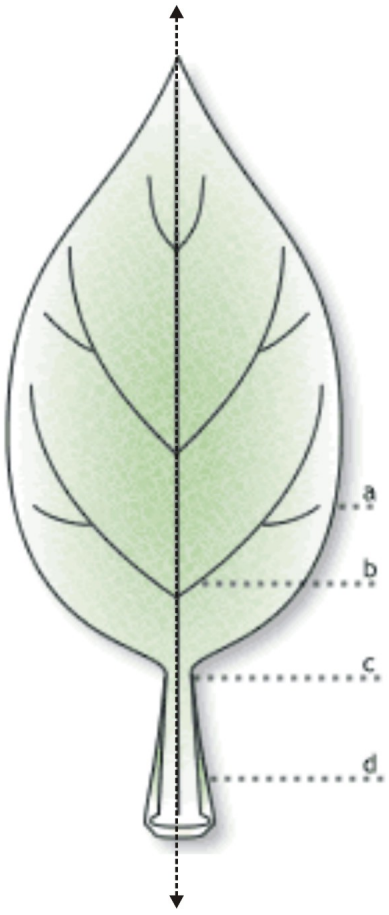
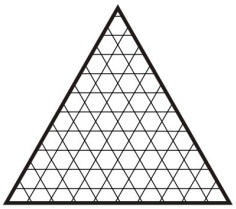


Figure 12.3

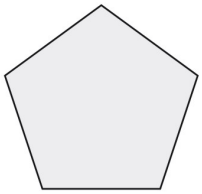


Rotate 120° or 240°

Rotate 90°,180°, or 240°

Rotate 180°

Rotate 180°



Rotate 72°,144°,216°, or 288°

In biology rotational symmetry is called *radial symmetry*. The plane representation of a starfish, for example, may have radial symmetry—it can be turned (rotated) and it will look the same before, and after, being turned. The photographs below show how sea stars (commonly called starfish) demonstrate 5-fold radial symmetry.

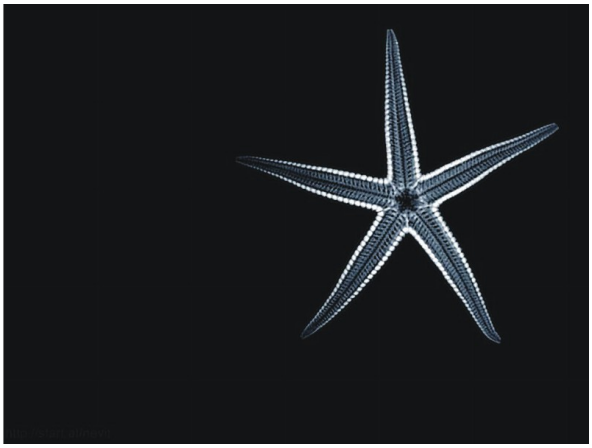
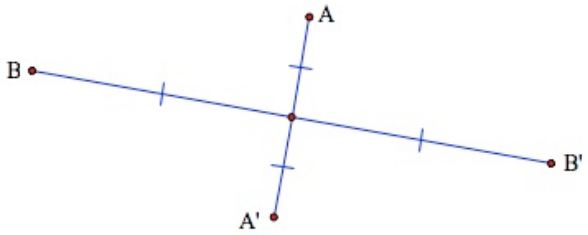


Figure 12.4

Point Symmetry

We need to define some terms before point symmetry can be defined.

Reflection in a point: Points X and Y are reflections of each other in point Z if X, Y, and Z are collinear and $XZ = YZ$.



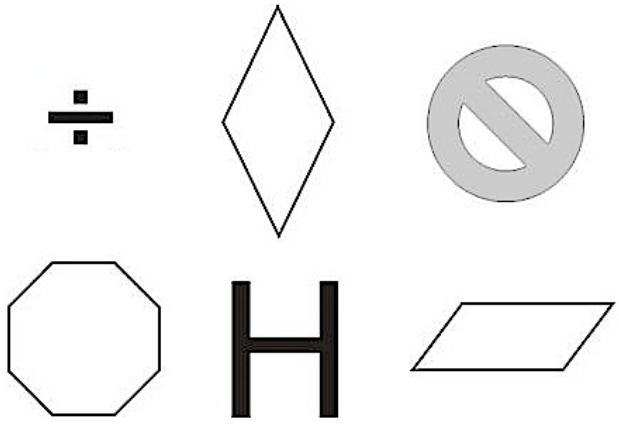
In the diagram:

- A' is the reflection of A in point P (and vice versa).
- B' is the reflection of B in point P (and vice versa).

A plane (two-dimensional) figure has **point symmetry** if the reflection (in the center) of every point on the figure is also a point on the figure.

A figure with point symmetry looks the same right side up and upside down; it looks the same from the left and from the right.

The figures below have point symmetry.



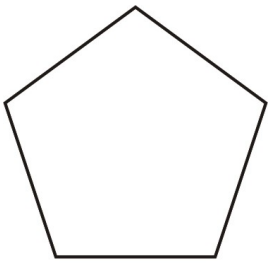
Note that all segments connecting a point of the figure to its image intersect at a common point called the *center*.

Point symmetry is a special case of rotational symmetry.

- If a figure has point symmetry it has rotational symmetry.
- The converse is not true. If a figure has rotational symmetry it may, or may not, have point symmetry.

Many flowers have petals that are arranged in point symmetry. (Keep in mind that some flowers have 5 petals. They do *not* have point symmetry. See the next example.)

Here is a figure that has rotational symmetry but not point symmetry.



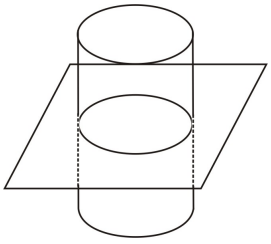
regular pentagon

Planes of Symmetry

Three-dimensional (3-D) figures also have symmetry. They can have line or point symmetry, just as two-dimensional figures can.

A 3-D figure can also have one or more *planes of symmetry*.

A **plane of symmetry** divides a 3-D figure into two parts that are reflections of each other in the plane.



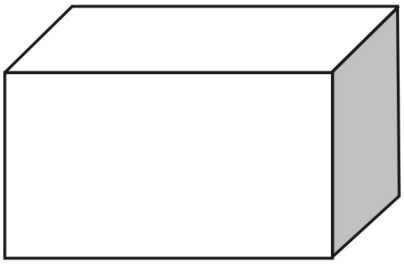
The plane cuts through the cylinder exactly halfway up the cylinder. It is a plane of symmetry for the cylinder.

Notice that this cylinder has many more planes of symmetry. Every plane that is perpendicular to the top base of the cylinder and contains the center of the base is a plane of symmetry.

The plane in the diagram above is the *only* plane of symmetry of the cylinder, that is parallel to the base.

Example 1

How many planes of symmetry does the rectangular prism below have?



rectangular prism

There are three planes of symmetry: one parallel to and halfway between each pair of parallel faces.

Lesson Summary

In this lesson we brought together our earlier concepts of transformations and our knowledge about different kinds of shapes and figures. These were combined to enable us to describe the symmetry of an object.

For two-dimensional figures we worked with:

- line symmetry
- rotational symmetry
- point symmetry

For three-dimensional figures we defined one additional symmetry, which is a plane of symmetry.

Points to Consider

Symmetry seems to be the preferred format for objects in the real world. Think about animals and plants, and about microbes and planets. There is a reason why nearly all built objects are symmetric too. Think about buildings, and tires, and light bulbs, and much more.

As you go through your daily life, be alert and aware of the symmetry you encounter.

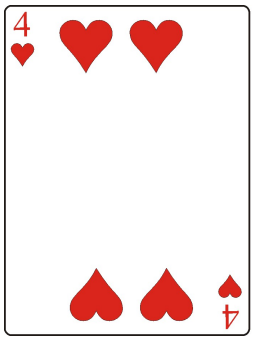
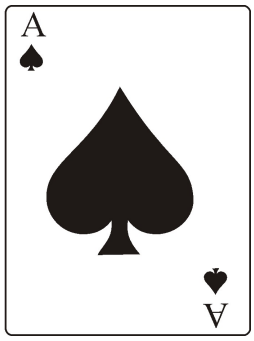
Review Question

True or false?

1. Every triangle has line symmetry.
2. Some triangles have line symmetry.
3. Every rectangle has line symmetry.
4. Every rectangle has exactly two lines of symmetry.
5. Every parallelogram has line symmetry.
6. Some parallelograms have line symmetry.
7. No rhombus has more than two lines of symmetry.
8. No right triangle has a line of symmetry.
9. Every regular polygon has more than two lines of symmetry.
10. Every sector of a circle has a line of symmetry.
11. Every parallelogram has rotational symmetry.
12. Every pentagon has rotational symmetry.
13. No pentagon has point symmetry.
14. Every plane that contains the center is a plane of symmetry of a sphere.
15. A football shape has a line of symmetry.
16. Add a line of symmetry the drawing.

Figure 12.5

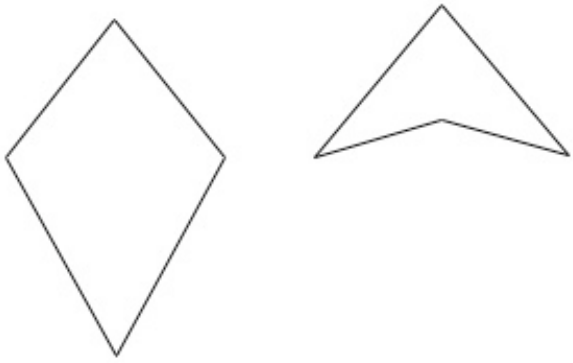
17. Draw a quadrilateral that has two pairs of congruent sides and exactly one line of symmetry.
18. Which of the following pictures has point symmetry?

19. How many planes of symmetry does a cube have?

Review Answers

1. False
2. True
3. True
4. False
5. False
6. True
7. False
8. False
9. True
10. True
11. True
12. False
13. True
14. True
15. True
16.
17. Any kite that is not a rhombus. Two examples are shown.



18. The four of hearts

Figure 12.6

19. Nine

12.8 Dilations

Learning Objectives

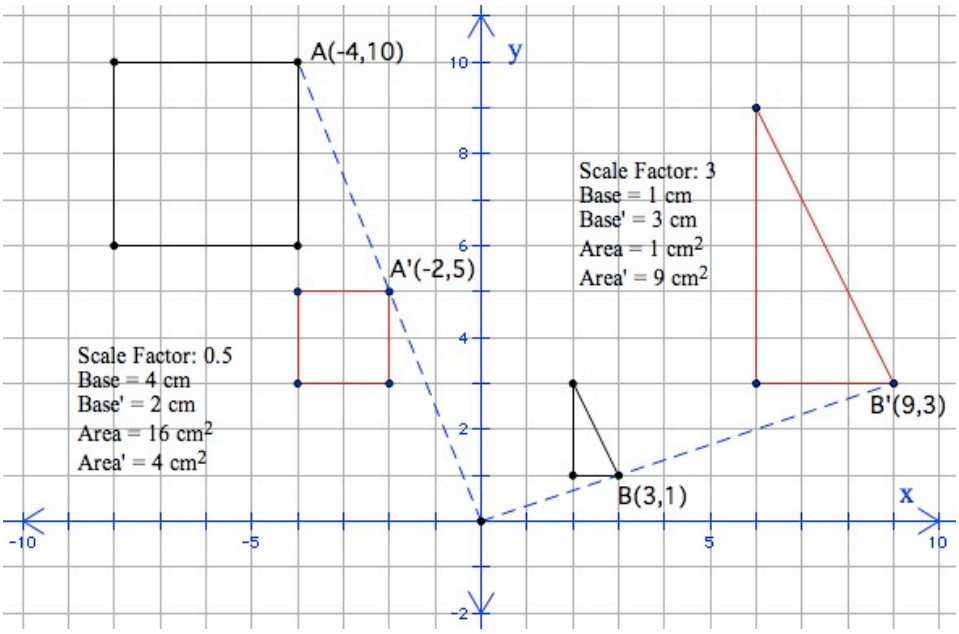
- Use the language of dilations.
- Calculate and apply scalar products.
- Use scalar products to represent dilations.

Introduction

We begin the lesson with a review of dilations, which were introduced in an earlier chapter. Like the other transformations, **dilations** can be expressed using matrices. Before we can do that, though, you will learn about a second kind of multiplication with matrices called **scalar** multiplication.

Dilation Refresher

The image of point (h, k) in a dilation centered at the origin, with a scale factor r, is the point (rh, rk).



For $r > 1$, the dilation is an enlargement.

For $r < 1$, the dilation is a reduction.

Any linear feature of an *image* is r times as long as the length in the original figure.

Areas in the *image* are r^2 times the corresponding area in the original figure.

Scalar Multiplication

In an earlier lesson you learned about **matrix multiplication:** multiplication of one matrix by another matrix. Scalar multiplication is the multiplication of a matrix by a real number. The product in scalar multiplication is a matrix. Each element of the original matrix is multiplied by the scalar (the real number) to produce the corresponding element in the scalar product.

Note that in scalar multiplication:

- any matrix can be multiplied by any real number
- the product is a matrix
- the product has the same dimensions as the original matrix

Example 1

Let $t = 4$, *and* $M = \begin{bmatrix} -3 & 2 \\ -1 & 5 \\ 4 & -2 \end{bmatrix}$

What is the scalar product tM?

$$tM = 4 \begin{bmatrix} -3 & 2 \\ -1 & 5 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 4(-3) & 4(2) \\ 4(-1) & 4(5) \\ 4(4) & 4(-2) \end{bmatrix} = \begin{bmatrix} -12 & 8 \\ -4 & 20 \\ 16 & -8 \end{bmatrix}$$

Let's continue an example from an earlier lesson.

Example 2

A company has two warehouses in their eastern region, where they store three models of their product. A 2×3 *matrix can represent the numbers of each model available in each warehouse.*

$$\begin{array}{cc} & \text{Eastern Region} \\ & \text{Model} \\ \text{Ware house} & \begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 4 \end{bmatrix} \end{array}$$

Here the row number of an element represents the *warehouse* number, and the *column* number represents the *model* number.

Suppose that managers decide to increase the number of each model in each warehouse by 300%. *After the increase there will be three times as many of each model in each warehouse.*

A scalar product can represent this situation very precisely. Let S be the matrix that represents the distribution of items by model and warehouse *after* the increase.

$$S = 3 \begin{bmatrix} 2 & 3 & 6 \\ 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 18 \\ 3 & 15 & 12 \end{bmatrix}$$

For instance, after the increase there will be 18 Model 3 items in Warehouse 1.

Scalar Products for Dilations

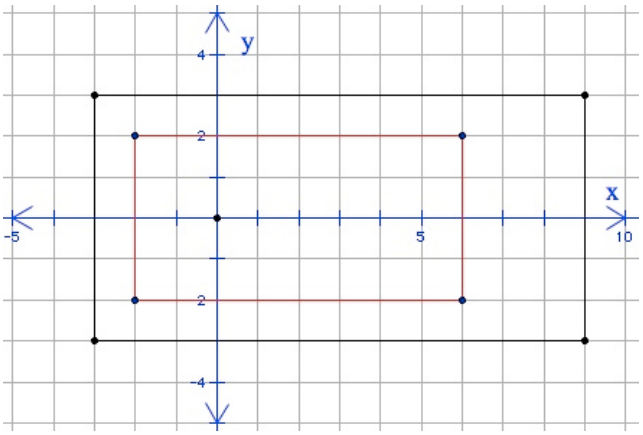
Recall from the Dilation Refresher above:

The image of point (h, k) in a dilation centered at the origin, with a scale factor r, is the point (rh, rk).

This is exactly the tool we need in order to use matrices for dilations.

Example 3

The following rectangle is dilated with a scale factor of $\frac{2}{3}$.



a) What is the polygon matrix for the rectangle?

$$\begin{bmatrix} -3 & -3 \\ -3 & 3 \\ 9 & 3 \\ 9 & -3 \end{bmatrix}$$

b) Write a scalar product for the dilation.

$$\frac{2}{3}\begin{bmatrix} -3 & -3 \\ -3 & 3 \\ 9 & 3 \\ 9 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 2 \\ 6 & 2 \\ 6 & -2 \end{bmatrix}$$

c) What are the coordinates of the vertices of the image rectangle?

$$(-2, -2), (-2, 2), (6, 2), (6, -2)$$

d) What is the perimeter of the image?

Perimeter of the original rectangle is $6 + 12 + 6 + 12 = 36$.

Perimeter of the image is $\frac{2}{3} \times 36 = 24$.

e) What is the area of the image?

Area of the original rectangle is $6 \times 12 = 72$.

Area of the image is $\left(\frac{2}{3}\right)^2 \times 72 = 32$.

Compositions with Dilations

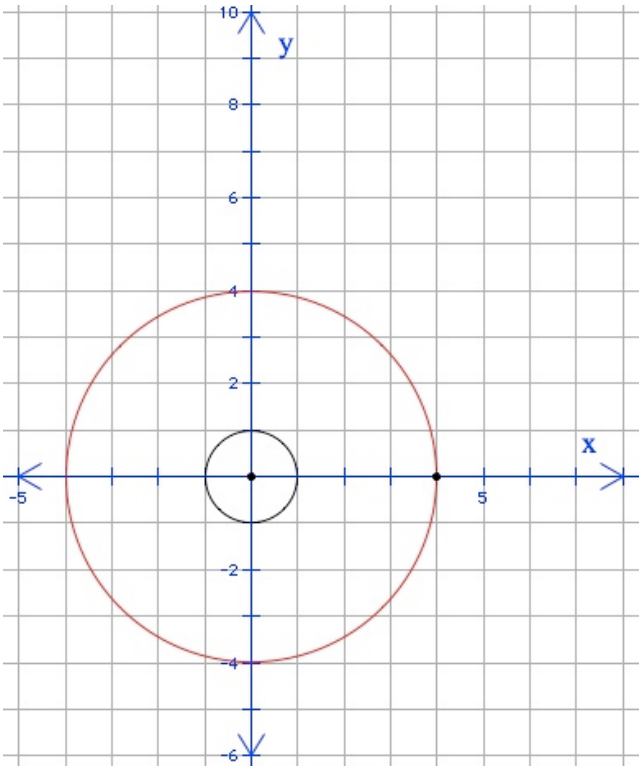
Dilations can be one of the transformations in a composition, just as translations, reflections, and rotations can.

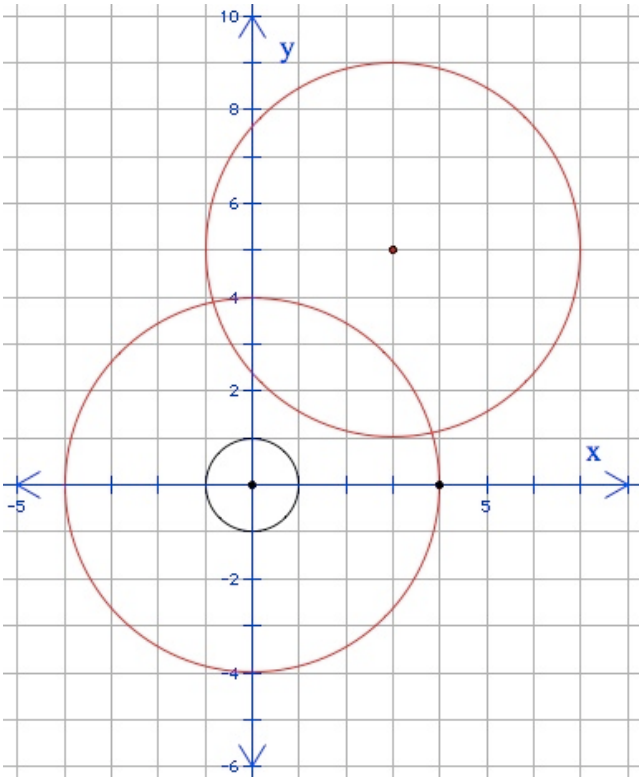
Example 4

We will use two transformations to move the circle below.

First we will dilate the circle with scale factor 4. *Then, we will translate the new image* 3 units *right and* 5 units *up.*

We can call this a translation-dilation.





a) What are the coordinates of the center of the final image circle?

$(3, 5)$

b) What is the radius of the final image?

4

c) What is the circumference of the final image?

8π

d) What is the area of the original circle?

π

e) What is the area of the final image circle?

16π

If P is a polygon matrix for a set of points in a coordinate plane, we could use matrix arithmetic to find P', the matrix of the image of the polygon after the translation-dilation of this example 4.

Let's use this translation-dilation to move the rectangle in example 3.

$$\begin{bmatrix} -3 & -3 \\ -3 & 3 \\ 9 & 3 \\ 9 & -3 \end{bmatrix}$$

Dilation scalar is 4

Translation matrix is $\begin{bmatrix} 3 & 5 \\ 3 & 5 \\ 3 & 5 \\ 3 & 5 \end{bmatrix}$

$$P' = \left(4 \begin{bmatrix} -3 & -3 \\ -3 & 3 \\ 9 & 3 \\ 9 & -3 \end{bmatrix} \right) + \begin{bmatrix} 3 & 5 \\ 3 & 5 \\ 3 & 5 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -12 & -12 \\ -12 & 12 \\ 36 & 12 \\ 36 & -12 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 5 \\ 3 & 5 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -9 & -7 \\ -9 & 17 \\ 39 & 17 \\ 39 & -7 \end{bmatrix}$$

The final image is the rectangle with vertices at $(-9, -7), (-9, 17), (39, 17)$, and $(39, -7)$.

Lesson Summary

In this lesson we completed our study of transformations. Dilations complete the collection of transformations we have now learned about: translations, reflections, rotations, and dilations.

Scalar multiplication was defined. Differences of scalar multiplication compared to matrix multiplication were observed: any scalar can multiply any matrix, and the dimensions of a scalar product are the same as the dimensions of the matrix being multiplied.

Compositions involving dilations gave us another way to change and move polygons. All sorts of matrix operations—scalar multiplication, matrix multiplication, and matrix addition—can be used to find the image of a polygon in these compositions.

Points to Consider

All of our work with the matrices that represent polygons and translations in two-dimensional space (a coordinate plane) has rather obvious parallels in three dimensions.

A matrix that represents n points would have n rows and 3 columns rather than 2.

A dilation is still a scalar product.

A translation matrix for n points would have n rows and 3 columns in which the rows are all the same.

And, of course, there seem to be many more than three dimensions!

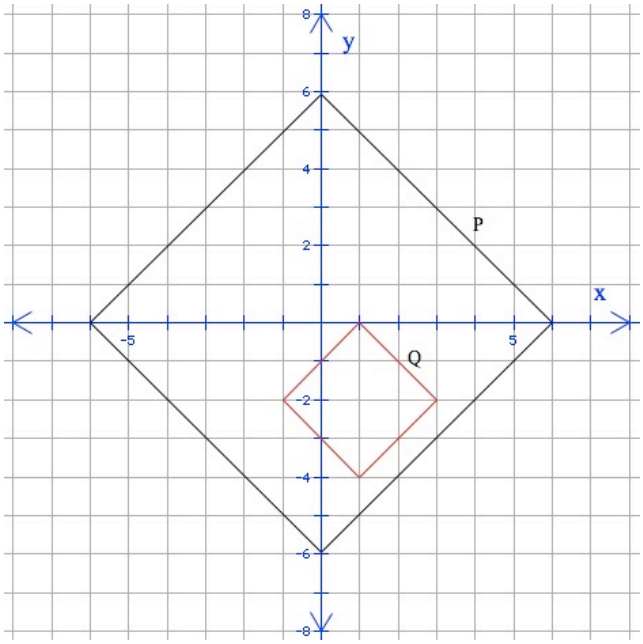
Review Questions

1. A dilation has a scale factor of 1. How does the image of a polygon compare to the original polygon in this dilation?
2. The matrix $D = \begin{bmatrix} 22 & 30 & 36 & 40 \end{bmatrix}$ represents the prices Marci's company charges for deliveries in four zones. Explain what the scalar product 1.20D could represent.

The matrices for three triangles are:

$$P = \begin{bmatrix} -1 & 5 \\ -2 & -6 \\ 2 & 4 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -5 \\ 2 & 6 \\ -2 & -4 \end{bmatrix} \quad R = \begin{bmatrix} -2 & 10 \\ -4 & -12 \\ 4 & 8 \end{bmatrix}$$

3. Describe how the triangles represented by P and Q are related to each other.
4. Describe how the triangles represented by P and R are related to each other.
5. Write the product 3P in matrix form.
6. Describe how the triangle represented by the product (3P) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is related to the triangle represented by P.
7. In example 4 above, the circle is first dilated and then translated. Describe how to achieve the same result with a translation first and then a dilation.
8. Describe a dilation-translation that will move polygon P to polygon Q.



Review Answers

1. The image is the same—there is no change.
2. Answers will vary. For example, the product could represent the delivery charges in the four zones after the prices for all the zones were raised 20%.
3. Same except one is the reflection in the origin of the other.
4. R is the enlargement of P with scale factor 2.
5. $\begin{bmatrix} -3 & 15 \\ -6 & -18 \\ 6 & 12 \end{bmatrix}$
6. The triangle is enlarged (dilated) with a scale factor of 3. The enlarged triangle is then reflected in the $x-$axis.
7. Translate 3 right and 5 up. Then dilate with a scale factor of 4 but making $(3, 5)$ the center of the dilation.
8. A dilation with scale factor $\frac{1}{3}$ followed by a translation 1 unit right and 2 units down.

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