Calculus Teacher's Edition - Teaching Tips

CK-12 Foundation

December 10, 2009

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1.1 Calculus TE Teaching Tips

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Lesson 1: Equations and Graphs

It is almost cliché how math courses start out with a review of material from previous years. Students are out of practice and never seem to have either been taught, or don't remember what has happened in previous classes (and will always claim to have not been taught it if they don't remember). There are two considerations here as the calculus course starts. First, a complete calculus course is a full years worth of university material. This means that the course is conducted at a faster pace than high school students are used to. Compounding the problem for many classes is the even shorter year with the AP examination. Therefore, it is dangerous to get bogged down in the preliminaries.

However, a strong case can be made that not much can be accomplished in a calculus class without a firm grounding in the fundamentals presented here. To have a conceptual understanding of functions and graphs is essential to gaining mastery of the basis for the limit, derivative and integral. In case of limited time, the key idea that needs to be driven home is how the relationship between the two variables creates a graph, and what the line means. The way that limits, derivatives and integrals are presented in a first course of calculus is all graphical. If students do not understand what they are looking at when the text later talks about zooming in on an area, strictly increasing or looking at activity at a minimum or maximum, to name a few examples, the key concepts will be lost.

Graphing calculators can be valuable tools at this point, especially as they allow for fast manipulation of accurate graphs. There is some danger in relying too much on the graphing calculator, however. I have observed students who have done all of their graphing since linear function on graphing calculators and they end up with some peculiar habits. The most noticeable of which is losing track of the activity of a function outside of the domain graphed, lack of understanding of what happens near vertical asymptotes (the calculator often shows a continuous line), and an over reliance on guess and check methods, especially when the student gets to the chapter on extrema. Use the graphing tool to illustrate some key concepts quickly, check work done by hand, and use some of the calculation tools that may be useful on the university examination of choice, but make sure everything could theoretically be done by hand.

Lesson 2: Relations and Functions

While it may seem like an issue of semantics, I encourage my students to use, and try to use exact terminology when talking about mathematical relationships. The terms "expression", "equation", and "function" all have specific meaning. Students will often confuse them, or believe they can be used interchangeably. Knowing the difference pays off later in sections on inverse and transcendental functions. It is also useful when it comes to writing clear solutions, especially those with prose attached, because they author can then be absolutely clear, presenting work in an easy to follow manner.

There is some inconsistency in the way students are taught to express intervals; the topic is pertinent here in expressing the domain and range of functions. The text uses mostly the inequality notation to state which numbers act as endpoints for each variable. Another option is to use the strict set notation with the parenthesis for not inclusive intervals and brackets for inclusive intervals, with the union set operator to join discreet intervals. Example:

$$D = \{-3 < x \leq 0, 1 \leq x < 2\} = (-3, 0] \cup [1, 2)$$

There are also the standard sets that have defined bold-face letters: R = Real numbers, Q = Rational numbers, Z = Integers, N = Natural numbers. None of this is important to drive home to students except for the fact that a textbook, or instructor, often chooses one notation method and sticks to it. Different texts and classes may have different notations so students should be at least aware of the different choices.

Speaking of notation, the different forms for writing the operation of composition for functions is a source of potential confusion for many students. The operator: $(f \circ g)(x)$ tends to cause all kinds of problems. First, it looks like even more of a product than a single function. Second, we do everything left to right, but the action here is more right to left, made even worse by the fact that composition is one of the few non communicative operations that students have yet come across. Please use, and have students convert to, the nested notation, where the previously mentioned operation is equivalent to $f(g(x))$. This is clearer because the function $g(x)$ is placed into the function f as if it were the variable, just like the composition is written.

Lesson 3: Models and Data

One of the tough things for students to do at this point is to have a sense for function behavior given a set of data points. The best tool is experience, of which the students are at a disadvantage. There are a few rules of thumb to help them out.

- Population and monetary (interest, investment) data sets are almost always modeled with exponential functions.
- Repeating data sets, like measurements taken every hour for a day, every month for a year etc., are almost always modeled by periodic functions.
- Look at the difference in endpoints for suspected linear functions. The change in values on each extreme end will be the same for linear functions and no others.

The text recommends plotting the point in either a calculator or by hand to choose a model based on the shape of the graph. This is often a useful task, but one with a chance to be misleading. The scaling of each axis can determine the shape of the graph sometimes more than the data points themselves. There is no clear rule for determining the correct scaling, other than choose endpoints far enough to show all the data points, so again experience and trial and error are the best tools. It is useful to use different scaling to see if it appears to change the shape of the graph. Linear functions will always appear to be linear, regardless of scaling (unless the data points vary substantially and you are zoomed in very "close"), where other functions may appear to be linear at some scales, but their curves will appear at others. Also, filling the screen as best you can will often help.

Something to remember is that the functions are not meant to be perfect reflections of observed phenomena, but useable models for a defined range. Negative time may not make sense, and the quadratic function that models a falling object fails to model correctly after the time at which the object comes to rest after hitting the ground. Students should always keep in mind that models are just that, and restrictions are useful to note.

Lesson 4: The Calculus

I sometimes joke with my students that calculus is an hour and a half of content that we manage to stretch out over two or three years. There is a nugget of truth to it—the central concepts are not complicated. The chapter presented here illustrates the basic concepts and alleviates some of the chicken-egg situations that sometimes happen.

Calculus is the science of "close enough". Before presenting the words derivative, integral and limit, it can be a fun and useful activity to look at some of the everyday situations where smaller and smaller iterations are used for measurement. Things like mapping the ocean floor, finding volumes for figures, and using data points to make a smooth curve all give insight to the basic concepts presented here.

This is also a great opportunity to use some of the features of calculators and other computer math systems. There is no harm in teaching the concepts and solving problems numerically with the calculator performing the "magic". Some teachers and classes have the philosophy that you need to be able to do everything by hand before using a computer's assistance. I don't agree for the following reasons. First, there is no "hiding" technology from the students these days. Second, there are plenty of problems where everything but the most advanced computers systems have no chance of solving. Finally, it is good to have the students used to using calculators now for every problem where it makes sense. There are calculator mandatory sections on the AP exam, and it makes no sense not to use a calculator for some of the problems.

Lesson 5: Limits

The chapter starts out with evaluating limits using a calculator for assistance. There is no reason not to do this; it is a very efficient way of evaluating some numerical limits. The most common trouble is when an exact irrational number is needed, the calculator will only return a decimal and the student may or may not know what that number is. Another problem that I have seen is that students over use the close number technique with the calculators. It is good to always have a backup in case of total confusion, but going to the calculator every time is time consuming, and will not be allowed on calculator illegal test sections. All of the same applies in using the zoom rather than the table or iterating evaluations.

A decision needs to be made about how strict of a definition for limits will be presented. Limits as a concept are relatively easy to understand, but involve a tricky definition. A first year student typically will have a hard time understanding "small enough" and "large enough" comparisons that seem arbitrarily made up. The definition is never really used in a first year class, so a strict definition is rarely presented in texts, as is true here. An advanced class, however, may need to see the formal definition, or have a little more interaction with the definition that is presented in this text. There are a select few functions and situations that are run into where there is no limit where it seems like there should be one, and the only way to show it is with the formal definition.

Lesson 6: Evaluating limits

The most common thing for students to want to do at this point is to apply the techniques used to illustrate the derivative and limit conceptually. While there is real value in using the calculator to show the concept behind limits, for some reason students seem to latch onto the zoom over and over, or table technique when they run into any difficulty. It is not a bad thing to always have an "out" in complicated situations, as finding an answer is always better than not finding one. The problem is in accuracy, if the answer is expected to be in exact form for an irrational number, and time. Time is the big one here, as students are likely entering the first of some years of tests where every level of student is likely to be under stress to finish within the time limit. The calculator techniques frequently take extra time, and can really cause trouble for the overall score on the test.

If there is a technique to focus on, it is finding the limits of rational functions. There are two reasons for this. First, they are common problems on standard examinations, like the AP exam. They also tend to be some of the "easier" problems, but like any problem, are only easy if you are confident in the method of solution. Where students may lack some confidence is in the high powered algebraic manipulation needed for some problems to find factors for each polynomial to cancel. Students should be given ample time to practice, and should have a safe environment to ask questions, as many will be afraid, remembering that many of the answers will be from an Algebra I class. Second, the techniques used for finding limits of rational functions are often the very same techniques that will be used later in finding derivatives using the limit definition. If students have the confidence to tackle these problems, it will make teaching this later chapter much easier, as the focus will be more on specific application and concepts.

Lesson 7: Continuity

There is sometimes a habit to brush off one sided limits. They are taught at this time, but seem to then be forgotten about for a long period of time. Later topics do revisit them, but often times in proofs and justifications for rules that students do not often directly interact with. Another problem with one sided limits is that many of the techniques used for evaluating limits already learned are not applicable for one sided limits (unless the one sided limit matches the two sided limit, of course). Sometimes this means that more brute force methods, or computers and calculators, are used which many instructors feel is less important or desirable than the analytic techniques. They are important, and they should be understood, but at the same time, without context, they may not stick and are best considered here in the context of continuity.

In teaching, it is sometimes useful to have a library of functions that have different kinds of discontinuities. Here is a primer on how to write examples of each:

Piecewise discontinuities: These are probably the easiest to write, and the easiest to identify. Any type of function that is defined differently for different intervals often has discontinuities. An interesting thing about piecewise functions is that a favorite question on the standard exams is to identify a coefficient that makes a piecewise function continuous. Example:

$$f(x) = \left\{ \begin{matrix} x^2 \ for \ x < 3 \\ -2x + c \ for \ x \geq 3 \end{matrix} \right\}$$

Where the students will be asked to find the c that makes the functions "match". An added level of complexity is to have the function given undefined at the endpoint necessitating the use of a limit.

Functions with vertical asymptotes: These are going to occur most frequently in rational functions, but happen anytime the denominator of a function equal to zero. (there is an exception, see the next example)

Rational expressions with removable discontinuities: If the denominator is approaching zero at the same rate that there is a factor of the numerator approaching zero then no asymptote can occur, as there isn't the chance for values to become very large by being divided by a very small number. Therefore, if there are matching factors top and bottom, there will be a point discontinuity, but no asymptote (this is why factoring and canceling for limits works).

Special functions: The most common one to look at here is the integer step function, notated $[x]$, which takes the decimal truncated value of x, making it complete "steps" as x increases. Another common one for calculus is Dirichelt's function, which take the value zero for irrational numbers, and the value of the rational number for each one. This function is only continuous at $x = 0$. Most of the special functions, however, are fairly trivial at this point and are more useful for showing concepts than being used for anything in particular.

Lesson 8: Infinite Limits

There is a lot of mathematical language that is typically used for infinite limits. It will be of use to introduce students to the terminology you, and texts, are going to use that students have not yet heard.

Some key vocabulary:

- End Behavior: The activity of a function way, way out in either direction. The temptation will be to establish a certain number that is large, or small, enough but some limits converge very slowly, so it is important to stress that clues about end behavior can be found with very large numbers, but actual end behavior is an analytic concept.
- Dominates: When we have rational expressions we tend to look at where the variables are and how the numerator and denominator act. In the simple case $\lim_{x\to\infty} \frac{1}{x}$ we can see that the only thing changing is that the denominator is getting very large, and is dominating, and therefore sending the limit to zero.
- Indeterminate Form: If one breaks down the first word it is clear, but it is worth stressing that indeterminate forms are the expressions where no clue is given to the behavior of the function. Typical indeterminate forms are anything divided by zero, infinity plus or minus infinity, and infinity divided by infinity. Due to the conceptual nature of infinity (it's not a number!) none of these can say exactly what is going on with the answer.
- Gets Large, Gets Small: Again these are conceptual descriptions of what is happening to numbers. The tendency is to think of infinite limits as a sequence of increasing (or decreasing) variable values. The behavior of this informal sequence is often described as getting large, or getting small.

There are a number of conceptual analytic themes for students to understand at this point. They should get a sense that large values in denominators tend to zero, and large values in numerators diverge. There is no hurry to teach many of the specific techniques for evaluating limits at this time as they are covered in later sections. The only tool I might teach at this time is the polynomial rule, where the large exponent dominates, or if the degree of both top and bottom functions are the same the limit is the fraction of leading coefficients. This is covered later, and is clearest as a consequence of l'Hopitals rule, but is a handy and easy tool to start using immediately.

Lesson 9: Tangent lines and rates of change

The most important concept is to understand that the derivative is the slope function. A nice aspect of differential calculus is the relationship between all the concepts, and some ideas from algebra in years past.

Before students begin to develop the formula for the slope of the tangent line they need to have a strong understanding of what the tangent line to a curve is. This is accomplished quickly with physical items. Students can have fun taking straightedges to various curved surfaces in the classroom like sports balls, balloons or any other curved surface. If available, taking a large pole outside to a hill or other curve can be fun. Simply saying that the line touches the curve at a single point is not sufficient (try giving that definition to a classroom of new calculus students and have them sketch a tangent line.) It is also worth noting that strictly speaking, it is possible to draw a tangent line that crosses a graph at multiple points, and a non-tangent line that only intersects at that single point. The idea of a line resting on the curve, staying on a single side and not intersecting the curve at any point near by (except in the case of points of inflection) may take some time for students to understand.

It is not unreasonable for students to come up with the tangent slope function on their own. Given a linear function, students should have no trouble calculating the slope. Now give them a curve and ask them to find the slope at a point. Some may try to sketch a tangent line and find the slope of that line. Not a bad idea, but they should know that this is circular, that is, the goal is to find the slope to properly draw the line. The first thing students should realize is that two points are needed for a slope, so two points must be chosen. They can do so to find an estimation, and then some students can be selected to show how students who chose closer points appear to have better approximations. While they may not come up with exactly the standard form used for the tangent line slope, combining the new concept of the limit with the "closer points" concept just figured out, students should have a pretty good definition for the derivative. Then the next step is only in attaching the standard notation, usually using the standard diagram.

Something to stress throughout the course is that Rate = Slope = Derivative. This especially helps later for related rate and other applied problems.

Lesson 10: The Derivative

Teaching the definition and the conceptual rule is a little bit strange. So much of the first year involves taking derivatives, yet students seem to run through this section, and then forget about it as soon as the specialized techniques are presented. As it is, other than a couple of exercises, there are few instances where they will use it. However, those instances are important. The classic exercise is to use the definition a few times for very simple polynomials, like x^3. Make sure students clearly show each and every step when working these problems. They will all use a very similar process of expanding out and canceling the numerator. They can then extend this process to the general form for the power rule. It will include some undetermined terms in the middle, but students should recognize how those will cancel.

Another common use of the definition of the derivative is in finding limits that look like derivatives, and using the derivative function to evaluate those limits. Here is an example:

$$\lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{2}+x\right)-\sin\left(\frac{\pi}{2}\right)}{x}$$

This could be a fairly involved limit, but if you can see that this is the definition of the derivative, we can actually write the limit as follows:

$$\lim_{x\to\infty} \frac{\sin(u+x)-\sin(u)}{x} = \frac{d}{dx}(\sin(u))$$

If the student then takes the derivative of $\sin(u)$ (knowing that learning this derivative is presented in a later chapter) and then evaluates the derivative at pi over 2, the limit will be found. This type of problem is nearly guaranteed to show on the AP examination.

Lesson 11: Techniques of Differentiation

The first technique presented, the derivative of a constant, may seem trivial to students, but it is a place where students make mistakes. The problem is not with simple examples using familiar number, but rather what I call "sneaky" numbers. Often times in physics, or other applied problems, there will be many constants that have either letter names, like c for the speed of light, or quantities that can change from problem to problem, but are not variables. An example is the formula for conservation of momentum of a ballistic pendulum (a projectile colliding with a stationary weight at the end of an arm).

$$u = \frac{(m+M)\sqrt{2gh}}{m}$$

Where u is the velocity of the projectile, m is the mass of the projectile, M is the mass of the pendulum weight, g is the gravitational constant and h is the height above the center of mass at rest of the pendulum. Some of these can be treated as variables, depending of what is observed and what is being asked. The constant g is always going to be the same value on earth, so it is always a number. So in any problem, there are two variables here, and 3 numbers masquerading as variables. Slightly more common is the trouble encountered with π and e which are again constants unless being acted on by a variable.

Students should see the utility of the power rule immediately. Because it is easy to use, if any algebra can be done to use it more often, then it should be done. The most common algebraic changes to make are changing expressions in the denominator to negative exponents and using fractional exponents for square roots. I have observed students making mistakes applying the quotient rule at a much higher rate than when applying the product rule. If it can be done, change the rational expression to a product using negative exponents.

One of the things that can be done to help students with remembering the quotient rule is forcing students to learn, and apply, the product rule in a particular manner. While it doesn't make a difference for the product rule which order the derivatives is take, the subtraction in the quotient rule makes it so that the terms can't be switched. Forcing students to think of the product rule as "derivative of the first times the second, plus the first times derivative of the second" then it's just a simple change for the quotient rule by replacing the plus with a minus and dividing by the denominator squared.

Lesson 12: Derivatives of Trig Functions

The trig functions do require a certain degree of memorization. It is up to the instructor, and the students, to decide what they wish to memorize, and what to work out. In my personal experience, I have found it easier, and more useful, to memorize trig identities, double and half angle formulae, and how all of the other standard trig functions can be expressed in terms of sine and cosine. I never did, and still don't have the derivatives of any of the trig functions memorized beyond sine and cosine. If I need to take the derivative of tangent, I convert it to sine over cosine and apply the quotient rule. The advantage of this method is that it involves less overall memorization, the information memorized is applicable to more types of problems than only derivatives and is probably more flexible for solving new problems that don't conform to any of the standard derivatives.

A couple of disadvantages are that I will often end up doing more work than someone who knows the standard derivatives. Another is that knowing all of the standard trig derivatives helps when it comes time to find anti-derivatives, as it will often be helpful to quickly identify functions that have easy anti-derivatives. Another advantage to memorizing all of the standard trig functions is that the current section will be easier to teach, and probably faster for students to learn. In my classroom I endeavor to teach both, and allow students to choose.

Getting solutions in exact terms for trig functions is a challenge for many students. Most students will seek to use their calculators to evaluate nearly any numerical answer, which in the case of trig functions often leads to non-exact answers, sometimes in a different form than requested. Students will likely need a refresher on the standard unit circle values for the trig functions, and it may be useful for the students to have a ready reference. Another thing for students to start to recognize is when exact answers are needed. On multiple choice tests the answer can give clues on what needs to be done. If the answers are all with decimal approximations, then there is no need to worry about exact answers and calculators should be utilized to the fullest extent.

Lesson 13: The Chain Rule

Compositions are sometimes the least familiar method of combining functions to students. The other operations are more familiar from having used them with numbers. Many functions that students have worked with in the past can be deconstructed as a compositions of two, or many, functions, even ones that seem fairly simple. Since only the most basic functions have known derivatives, the chain rule gets applied very frequently. Combine this with the lack of familiarity with compositions and students have many little struggles.

First is in identifying that the chain rule needs to be applied. There are a few clues: parenthesis, radicals, and exponents are the usual places to look. If there is anything more than a simple variable, then the chain rule will need to be applied. Another way to look at it is that the chain rule can always be applied. This is also useful to start setting up for implicit differentiation, as it sets up why the differential term gets chained out from each variable. This way, if the derivative of the variable is anything more than a $\frac{dx}{dx}$ then the chain rule will need to be applied.

Next is in understanding what the two functions involved are. Many textbook examples are not particularly helpful for understanding the mechanics of the chain rule as they keep referring back to the composition notation that students are not particularly comfortable with. Sometimes the idea of "inside" and "outside" functions can be used. This is probably the clearest way to think about functions involving parenthesis, or inside of radicals, trig and log functions. This can get confusing when the composed function is in the exponent. Sometimes it helps to think of "little" and "big" functions, where the little function is inside of the big one. This is maybe not as clear for parenthesis, but can be helpful for exponents.

It is always a good idea to try to get the class to use clear language when asking questions, or presenting solutions. If students are all using clear descriptions of what the composed functions are, even if it is not a single standard among the whole class, it will help all students understand how to identify where and how to apply the chain rule.

Lesson 14: Implicit Differentiation

This seems to be a stumbling point for many students. Technically speaking there is nothing new going on here. Implicit differentiation is really just an instance of the chain rule applied to each variable, where one variable is not defined explicitly. The problem is, and this is not unique to this section, that to make things easier to teach and learn in the earlier sections, not everything is exactly written out in full technical form. I'm not advocating doing so, as it would turn simple problems into massive undertakings.

It is always useful to see if the function can, in any way, be solved explicitly for one of the variables. Once getting an explicit function is ruled out, it's useful to make sure all of the various rules are identified that will need to be invoked. The chain rule is a given, and will frequently be required multiple times for each term. Quotient and product rules are also often needed. All of these will be combined, so keeping track of each will be quite a task.

This is a useful section to spend lots of time with class wide examples. Start out by solving problems with students following along, and progress towards having the class work problems with step by step check-ins to make sure everyone is getting fast feedback on the example problems. Students should begin to see there is a sort of a rhythm to the problems. The process of applying the rules, collecting the terms with a $\frac{dy}{dx}$ in them on one side, and the other terms on the other side of the equals sign and then dividing to get the derivative is going to be very similar among all of these problems.

Another thing for students to understand is that these problems require a point, rather than an $x-$value, if a numerical derivative is to be computed. This is especially true as the implicit expressions are not always going to be functions, and may have multiple y values for each x. Sometimes the original function will need to be revisited in order to get the point if some information is not given in the problem.

Lesson 15: Linearization and Newton's Method

The description at the start of the text of "zooming-in" is a great idea for how to illustrate the idea for the class. Use a graphing calculator or a computer program to show that nearly any function "looks" like a straight line if you get in close enough. (As a counter example, it can be useful to show some non-differentiable functions to show that differentiability is a necessary requirement for a linear approximation. Some functions to look at would be the absolute value function, which will always have a sharp point at 0, and $x^2 \sin \frac{1}{x}$ which will just about always look the same about the origin because it increases in frequency.) After showing that the function is nearly linear after zooming in, find the value of the derivative and use the point to also graph the linear function that approximates the original function at that point. Students can then use the trace function, with the up and down buttons, to switch between the two functions to see how close they really are.

Students may make the mistake of thinking that the approximations they find are good everywhere. It is important to stress that linear approximations get worse the further away they get from the point chosen. There will be better approximation techniques, and encourage students to try to think of how they might do so.

In the age of calculators everywhere this chapter may be a tough sell. There isn't any trouble in finding quantities like $\sqrt{5}$ these days. The key here, and it is useful to let the students in on this, is that using calculus to approximate functions and values is a recurring theme. This is not a chapter to be ignored, as the ideas here will be expanded on later in more complicated problems.

Lesson 16: Related Rates

Students can be guaranteed to encounter a couple of these problems on the AP examination. Often, there is a related rate question in the free response section. Related rate problems have many steps, involving many variables and can be a little bit of a logistical challenge for students. There are a couple of things you can do to help out.

First, many related rate problems require area, distance and volume formulae. Some of these students will have memorized, like the pythagorean theorem. Others will not be so familiar, like the volume of a pyramid or the surface area of a sphere. Having a poster or individual cheat sheets of common formulae will help students learn and recognize what they will need to solve the problems. Second, this is a great opportunity for "we do" instruction where the class is working on the same problem together with check-ins after every step. This will help students learn the mechanics of the problems without getting stuck.

There are not many different kinds of problems that are typically seen in a first year calculus class. Make sure the students have seen a couple of examples of each. Those are:

- Tracking a straight moving object at an angle (or the sliding ladder)
- Inflating spherical balloons
- Filling various geometric containers
- Temperature change in a steady environment

If a few examples are seen of each then identifying how to solve each problem will become easier later.

Lesson 17: Extrema and the Mean Value Theorem

It is often a dilemma for teachers of a first year calculus course to decide how much of the classic analytic proofs to present. None of the proofs are necessary to understand how to apply each of the theorae. Furthermore, there are only a select few students who will be moving on to levels of mathematics where the mechanics of the proofs are helpful. However, as an applied mathematician, not all of calculus is applied and the ideas and mechanics of the proofs are valuable. How much of each proof to present is up to the instructor; it depends on the ability level of the class and where it looks like the students may go in to the future.

If proof are presented to the class, the traditional process has been to present them in lecture format with students taking notes and following along. The idea is that the proofs are challenging and just the exposure will rub off on the students. I don't believe this is helpful. Students tend to tune out long lecture type presentations. Some sort of interaction is needed with the material to make it worth the time. Students are capable of writing the proofs themselves at this point. They may not be perfectly rigorous, but using previous theorae, they can be done. Some ways of helping students along is with starting them off with the first couple of lines, telling them some of the theorae they will need to apply, giving the students the trickiest part, or proving a similar result and having the students complete a corollary.

Applications of theorae need to be presented whether the proofs are taught or not. Most of the proofs here are existence proofs, showing the existence of certain properties of functions. This is especially true of the mean value theorem, which will probably be the most applied in this section.

Lesson 18: The First Derivative Test

The first derivative test is a huge foundation for the analysis of functions to come. Also, there are a couple of things that can be confusing if there is not a good foundation for the concept presented here. The students will learn that the first derivative can be used to find maxima and minima. More important is to understand that the possible critical points are where the slope of the function, and therefore the derivative of the function, is zero. This helps with the confusion that sometimes happens when looking at functions that have a slope of zero, but do not possess a maximum or minimum at that point. For instance x^5 at $x = 0$. The derivative at zero is zero, but there is no extrema at that point.

One tricky thing for many students is to interpret the graph of the derivative. Given a plot of a function, it is not hard for students to identify increasing and decreasing intervals. What does seem to be tough is to reflect the slopes on a new graph. There seems to be a mental block in drawing a new line that is positive and decreasing at a point where the original function has negative values and is increasing. Be sure to spend some structured time helping students to fully understand how to create and understand these graphs. The concepts that they will learn will help with this section, and others down the road.

Lesson 19: Second Derivative Test

Here it is absolutely critical that students have some sort of organization technique for all of the information for each interval. After taking the first and second derivative tests students will know where there are maxima, minima, intervals where the function is decreasing, increasing, and points of inflection. The problem solving guide has a recommendation for how to keep track of all the information using tables.

Again, this is sometimes considered to be an antiquated chapter. With graphing calculators readily available, students do not need all of the support to graph a function, unless they are strictly doing it by hand. For these reasons, I somewhat devalue the sketching aspect for the derivative tests. One things that is very important to understand is that the derivatives test will sometimes show information that might not show up on calculators due to the domain used, or the scaling. It is not uncommon for test to specifically choose functions that may not show all extrema on a standard graphing screen. Also, a little bit of work can help in determining the best scaling for the window to show the graph on a calculator or computer grapher.

Lesson 20: Limits at Infinity

L'Hopital's rule is a student favorite. Limits are often tedious to find by hand, involving either complicated comparison rules or non-trivial amounts of algebra. L'Hopital's rule is a relatively quick and easy way to compute limits. The issue is going not going to be getting students comfortable with the rule, but rather holding them to only use it when it is permitted. Stressing the fact that the rule can only be applied for $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and that it is not allowed for finding the derivative using the limit definition is important.

A very nice application, and then shortcut for infinite limits, of l'Hopital's rule is looking at rational expressions. Give the students some examples, like:

$$\lim_{x\to\infty}\frac{3x^4+2x^3-1}{7x^5-4x^2+8} \qquad \lim_{x\to\infty}\frac{6x^5-9x^3+5}{8x^5+4x^2-3} \qquad \lim_{x\to\infty}\frac{x^6+3x^2-5}{2x^5+x^4+1}$$

All of these are indeterminate and can be found using l'Hopital's rule. More importantly is finding the pattern, which students should be able to do after not too many questions similar to these. The rational function shortcut is that only the leading term is important. Since the behavior out at very, very large or very, very small numbers means that the largest degree will make so much of an impact that none of the other terms will have any effect. Along those lines, the larger degree in either the top or bottom will dominate the whole expression. If the polynomial on top has a larger degree, the limit diverges. If the polynomial on the bottom has the greater degree, the limit converges to zero. If the degree of each is the same, then the fraction of the leading coefficient is the value of the limit. This is verified with l'Hopital's rule as the coefficients are the only thing left after multiple iterations of the rule. A nice challenge for an advanced student is to show this rule works in general terms.

Lesson 21: Analyzing the Graphs of a Function

The table that is presented in the text should provide some help in getting students started on this section. Some students may find it tedious, and it is true that there is no standard format to present the information from text to text, class to class and test to test. There will frequently be free response questions that ask for some parts of these tables. Therefore, it is good practice to be able to fill in the table. I would not get too worried about the form students present the intervals in. There are a number of standard ways to do so, and teaching a new notation is probably not the best use of time at this point. Clarity is the key, not standardization.

The text restricts the domain in the subsection regarding trig functions. There are two considerations about restricting domains. Frequently the question students will need to answer is related to an applied problem where the a minimum, or interval of increasing values, for instance, is asked for. In applied problems there will often be limits placed on the domain from physical (negative time, etc.) or logical (the race ends at 400 m) that will be reflected in the domain. With all models, the function will return values outside of this interval, but they are meaningless, as the model only holds for the interval specified.

The consideration in the text is slightly different. Periodic functions will have infinite local max, min and intervals increasing and decreasing. Therefore, with minor exception, if one or two periods are analyzed the whole function is described, as it will be a repeat of the same values or behavior. One trouble is in making sure that the domain restriction allows for at least one period. After examining the function on this period, it is possible to construct the critical information for the entire graph. It requires the use of describing critical points as a sequence, and may add an unnecessary level of complexity for students at this point. If you have students who are likely to continue with math, engineering or physics, it is a very useful exercise to have them complete.

Lesson 22: Optimization

Optimization problems, along with related rate questions, are the heart of the applied questions in the first year of calculus. In many cases, the two are interrelated; if the question is asking for the optimization of a rate, then there will often be a related rate question embedded. Students should be made aware of this, and should refresh their memories regarding related rate questions. I have in the past held back on the related rate chapter and presented it immediately before the optimization chapter more or less showing them as one unit. The advantage is that students are doing the same types of problems in a logical progression. The down side is that these are two sections that first year students need above average effort and time to master. It can be mentally taxing and frustrating.

Like related rate questions, there are a few classical problems that should each be covered so students have confidence when encountering them. They are:

- Optimizing volume to surface area, or perimeter to area (the most classic question, and a fun one, is minimizing surface area for a 355 ml soda can it can be shown that the standard size is the optimal dimensions)
- Optimizing the pathway with different rates (the walking problem, the road problem)
- Chemistry or Pharmacy problems (rates of reactions, yields)
- Cost/profit problems (minimizing warehousing costs, maximizing profit)

There are, of course, many problems possible. These are the ones I have seen most frequently, in descending order of frequency. Students should be, at the least, very familiar with the first two on the list. It is nearly possible to guarantee one, or both, problems on a standard exam.

Clear organization of facts is very important. Because optimization problem inevitably include substantial information about the problem, and much of it is not actual content. Also in the cases of volume, area and perimeter the functions are frequently not included in the problem. Pulling the information of what shape is needed and then connecting to the correct formula is a skill that needs to be developed.

Lesson 23: Approximation Errors

The text here relies mostly on techniques with the graphing calculator. Sometimes teaching with the TI calculators can be a bit of a challenge. The menus and buttons can be a challenge to negotiate for students who are not completely familiar with the calculators. It is advisable to get students more comfortable any way possible as the calculator is a necessary tool for many of the standard exams. Guiding the class along is partially dependent on what sort of technology you have. The overhead projector adapter is very helpful in keeping the class together. Another option is using an emulator on a computer attached to a projector. There are some serious issues with this, as the ROM for the calculators is protected property. There are assorted guides available online, but this is not for the technologically faint of heart. A buddy system works very well, where a student who is very comfortable with the menus of the calculator with a student who may need some help in finding all of the functions.

The presentation of taylor polynomials here is truly optional. It is not a topic on the AB AP exam. Furthermore, some teachers (and as a consequence, their students) are not comfortable with using tools and theorae that are not fully presented. It is, however, good to show that there is a world beyond linear approximations, and that truly there is nothing needed for taylor polynomials beyond being able to take a derivative. Therefore, it is good to present here, but do be careful and honest about the "hand waving" that needs to take place here.

Lesson 24: Indefinite Integrals

Sometimes teachers can get carried away with using too precise of language. An early example is in introducing a chapter on functions in algebra 1. The teacher swears they are important (which they are) and the students can't understand how $f(x) =$ is different from the $y =$ equations they have been using the while time. I see a similar deal going on with antiderivatives and integrals. Is there really a difference between the two? Technically, yes. In a first year calculus class, no. Sometimes I will have students refer to individual parts of the integration process as antidiferentiation. For instance, if parts is needed to take the integral, part of the process is taking the antiderivative of one of the parts. Most classes, however, will use the two terms interchangeably, and there really isn't a problem with that.

I would recommend using the lower case, upper case notation before introducing the integral symbol. It is important for students to have a level of comfort in taking the antiderivatives before the added notation complexities. Also, if you choose to introduce basic differential equations at this time, this is the notation that will make it more clear. It also helps with some of the notation issues for the fundamental theorem. A notation issue you have to be strict about is the dx term at the end of each integral. This is important for substitution and parts later.

Lesson 25: The Initial Value Problem

Something students are going to have trouble with is when to put the initial values in to compute the constant left from integration. Even more challenging is when there are multiple sets of information, like for differential equations starting from acceleration and getting both the velocity and the position functions. Another common problem is where there are multiple constants and we combine them to create a new constant, that looks exactly like the old one. The thing to remind student of here is that these constants are a sort of hybrid variable. They are a yet undetermined quantity, but we will have a chance to find their value later. A good example, building on prior experience, is to have them work is finding the equation for a line using the slope intercept form. The $y-$intercept is just like the constant of integration and we will find each of them exactly the same way.

There is a great deal of importance of initial value information. In physics it is very common to use easy to measure quantities and models to find information that would be tough to observe. A big part of this process is setting up the initial conditions for the experiment. It is also worth noting that while it is most common to use the true initial values, usually time is 0, any observable point will work to find the constant of integration.

Lesson 26: The Area Problem

The tough part for students is usually understanding where the height of the rectangle is coming from. There sometimes seems to be a disconnection from their easy understanding that the area will be base times height, and where those quantities are coming from in the rectangle area process. I often observe students hoping for, and then exclusively using, more of a formula than develop a true understanding of the problem. I would try to spend enough time to make sure that the students understand that the function value is the height, and how the left, right, and middle methods change where the function is evaluated.

The toughest part about introducing sigma notation for sums is the indexing. Without significant experience, students get confused about how different terms, with different subscripts, are related, why there seem to be more than one variable and how it all fits together. Scaffolding work with sequences and their subscripts will keep it so that it is only a single new notation issue at a time. It will also be helpful to work with series without the subscript notation for each of the endpoints for the rectangles. The fewer variables at a time students have to work with the easier time they will have. Working with indices, variables and new symbols can be overwhelming.

Something to consider is how much to focus on sums. There are frequently a few questions on exams that ask for the estimation of areas using different approximations. None of these, however, require an understanding or use of the summation notation. The notation, along with the rules for finding values of infinite sums, is really used to establish a definition for the integral. It can be presented as such. Make sure, however, that whether they use the summation notation or not, the students are comfortable with finding area approximations using different shapes.

Lesson 27: Definite Integrals

What is a Riemann sum and why do they get such a name? At least that I what I wondered as a first year student. Typically if a theorem or a rule is important, we will give it a title, like Mean Value Theorem. If it is really important it gets a name, like Fermat's last, or the Pythagorean theorem. Riemann is one of the giants of math, so these sums must be really important, right? Well, not really, and it may be useful to let students know that this is the case. It's not that Riemann sums are not important, it's just that there really isn't anything unique about them. It is a fancy name for a summation of rectangular areas to approximate the area under the curve. Really that's it. Students in calculus have learned some of the decoding techniques to identify important information and discard others, which is a very good skill and shows high academic literacy. This happens to go against the rule and you can save your students some trouble by alerting them to the fact that this term is worth knowing, and it will be referred to at various times, but it is of small utility for this course. Especially because understanding of infinite series is needed to find all but the most basic integrals using Riemann sums they should not be stressed at this time.

In some ways it is easier for students to think of definite integrals as an instance of indefinite integrals, even though from a completely analytical perspective this is a bit backwards. Since the easiest way to evaluate definite integrals is to take the anti-derivative and evaluate at the endpoints, there is no reason to evaluate definite integrals in another fashion at this time. There are other sections that focus on definite approximations using different rules.

Something that is mentioned in the text, but is worthy of reiteration, is that the definite integral does not give total area between the curve and the axis. You can think of it as net area, and total area, or area under the curve, or a number of other ways of saying it, but it is important that students are aware of this distinction and take a second to make sure they are clear on what the question is asking for. Not every question wants total area, so there is no blanket statement that can be made, just warnings to be careful and detailed about answering the question.

Lesson 28: Evaluating Definite Integrals

There is a little bit of a divergence here between the strictly pure and the applied crowds. The applied group, with the focus on answer the question presented in the simplest possible way, will teach that definite integrals are just like indefinite integrals with the extra step of evaluating the endpoints and subtracting. The pure crowd, with the focus on theory, will insist that definite integrals and numerical methods are what came first, and that the fundamental theorae are what connect this process to indefinite integrals. I tend to the former for a first year class, even though my background is decidedly pure. Few students will be continuing in that path, and if they do they will have the chance to learn everything in the strict manner in their real analysis class, and get to curse their high school teacher who clearly didn't know any better. With that said, likely the best thing for your class is to present definite integrals as an instance of integration.

Definite integrals do have a nice side effect of being able to be evaluated easily by nearly all graphing calculators and computer solvers. Which brings up an important tip for the AP examination. If there is a problem in the calculator legal section of the test calling for a definite integral, by all means use the calculator to solve it. There are no bonus points for doing extra work. The only problem could arise when all of the options are exact answers, but even still students can get a decimal approximation for each answer and compare to the solution found for the definite integral on the calculator.

Lesson 29: Integration by Substitution

This is one of the key chapters in a first year class. Very few integrals can be evaluated without substitution. Furthermore, the practice of variable substitution is a useful skill for solving all kinds of problems, even outside of calculus. It is a skill that many student are not comfortable with, so that is a good place to start from. Even outside of calculus problems, students should practice changing variables to find solutions.

Students can run into organization problems with substitution. With complicated composite functions and their derivatives space on the page can be at a premium. There are a couple of things that can help. First, develop a style and stick with it. Use lots of space. If a routine is developed, and there is enough room to easily read the work, then any single piece of information is easily found. Furthermore, if trouble is encountered, then mistakes and problems can easily be solved. Another important thing is to use all of the correct notation. Don't cut corners, otherwise the derivative terms or other variables can be lost and substitution can go haywire.

A tricky point of notation can be when to substitute the limits of integration. It is not technically correct to leave the limits of integration for the original variable in after substitution. However, sometimes it is easier to use the original limits of integration and substitute back the original variable after integration. Other times it would be easier to change the limits of integration. The choice is when to change the limits, and when not to. If I choose not to change the limits, I leave them blank until after substituting the original variable back in. I would encourage students to use some sort of consistent practice to avoid confusion.

Having a consistent approach and technique is even more important for integration by parts. Because of the extra anti-derivative and then putting the whole thing back together, parts can get to be a mess very fast. That's not yet even considering the circular functions that require parts multiple times and wrapping back around to the front. Aside from helping students understand how to make good choices about which functions to choose for substitution and parts, organization of work is the best thing to do for them.

Lesson 30: Numerical Integration

While there will always be formulae for each method for numerical approximation, it is always preferable to have students understand that each of the formulae are simply areas of common shapes. Have students develop the formulae themselves as an exercise or activity. There are two benefits to this. First, they will have the confidence to find solutions even if they don't recall the formula immediately. Second, this helps with later sections on volumes of revolution and the process of using iterations of areas of common shapes. While I don't stress the formula for any of the rectangular approximation methods, I do have my students learn the formula for the trapezoid method. Because so many terms can be combined and canceled, the formula here provides a real advantage over doing it from scratch each time.

Simpson's rule is a little bit different. Here we are beginning ot see some non-linear approximation methods come into play for the first time. As these are no longer simple geometric figures that easy areas to find, students should be aware that there is no simple way to develop this formula, nor is it as free from restrictions. Ideally, students should get a sense, or be made aware of the fact, that Simpson's rule is somewhat analagous to using Taylor polynomials. A common theme is that linear methods of approximation are very good sometimes, and not so much at others. However, non-linear methods are frequently very good approximations for all circumstances. It also should be shown as an activity why there must be an even number of subdivisions.

After introducing the rules and working some simple examples, it should be known that nobody really uses these by hand. Part of the power of the numerical methods is that they are handled very well by computers. There is frequently a program already in calculators for the rectangle and trapeziod rule. There is an activity outlined in the problem solving flexbook that has students programming their own Simpson's rule program. I would recommend having these, as there is no limit as to what program can be used on the standard tests in the calculator legal sections.

Lesson 31: Area Between Two Curves

I have observed that, in some ways, students find it easier to take the area between to curves than under one curve. Having a clear comparison between two functions can be clearer than having a single function and an axis. There are two major problems students are likely to run into. First is the trouble with the negative getting distributed correctly. Again, the best tool to fight against such mistakes is to have clear work, with every step detailed. Also, there is no harm in ever using more parenthesis (correctly of course). The second problem happens when the two lines intersect and the total area is needed. The top function must be the first listed, so the intersection point has to be found and two separate integrals taken . Sometimes finding this point of intersection can be difficult, or if a graph is not provided, intersections can sneak up on students. There is an activity detailing this process in the problem solving flexbook for this section.

A harder thing for students to do is work problems where integration with respect to x may not be the best choice. The rule of thumb is that you want to avoid using more than one integral if at all possible. This may mean changing the functions around to get a simpler single integral rather than having to split up the interval into different parts. Also, I have no problem with students simply changing the variables around if it makes them feel more comfortable. As long as they record the changes they made so they can change them back at the end, there is no harm in making the problem set up in a more familiar manner.

Lesson 32: Volumes

Students will have more success with this section if they have a strong understanding of the conceptual development of the work done in past sections on finding the area under figures. There are many formulae to remember here if students don't know that they only need to remember how to find a few basic areas, and how they add together to make volumes. A good place to start with this is a basic cylinder. The volume of a cylinder is known, and is easiest to find as iterating the area of the base through the height of the object. The volume can also be found by taking a area of the surface and adding it with all of the concentric "surfaces" to the middle, just like the shell method. After working with a generic cylinder, then it can be developed as a simple solid of revolution, and students can see that it works exactly the same way now with functions and calculus. In fact, all of the problems work like this and it is far better to think of the volume as the area of the bases all added rather than a strict formula. I have observed students who are strict users of formulae do fine with volumes of revolution, but then get into trouble when they are asked to find the volume of an object with a base defined by a function or two and a geometric profile, like semi-circles. For a similar reason, I also avoid pushing "discs" and "washers" and refer only to "circles" as those are what we are going to be using for area in all revolution problems, whether we are using a single circle, or subtracting a second one out of a larger one.

A key part of solving all of these problems is being able to create a clear diagram. Frequently the shape will be found by rotating a region that is bounded by a couple of functions or lines. Students will need to determine what the limits of integrations will be and what function is going to be responsible for the radius of each circular slice. Further complicating problems is when there is an axis of rotation other than the $x-$axis. Having a well graphed figure that is large enough to label will keep much of the usual confusion from happening.

Lesson 33: The Length of a Plane Curve

This section can be a little bit frustrating. I always try to come up with many kinds of interesting or rich problem for students to work, but the form of the integral for arc length does not lend itself to being solved by hand. The only easy problems to work are those that either have a root in the derivative that will be squared out, or those that have an identity that allow the inside to be solved. For this reason, there are only a few types of questions that students are likely to see. The problems that are presented as exercises in the text are a good sampling of the common types of problems. The problem is that if you take a function's derivative and then square it, there will be few substitutions that will allow the integral to be taken with the radical there.

This topic only appears on the BC examination, and then usually only as a single question or two. Therefore, it can be a chapter that is de-emphasized in most classes, and is a good topic to cover in an AB class in the days after the exam.

Unfortunately, there is no way to find this formula in the course of first year calculus. Therefore this is a formula that needs to be memorized if the students are to use it on an exam. Once the formula is memorized, the process is plug-and-chug, where every problem works in the same manner.

Lesson 34: Area of a Surface of Revolution

This is the analogue of volumes of revolution for area under a curve. By iterating the arc length over the surface of the rotation, the surface area is found. Many of the same rules and troubles are the same. The integral is still a very challenging quantity to take, with a significantly limited list of functions that can be found by hand.

Lesson 35: Applications for Physics, Engineering and Statistics

The best calculus class, in my opinion, is an applied science class. Nearly all of the problems in calculus were motivated originally by physical observations needing models and solutions. The problems presented in the text provide an excellent cross sampling of the types of problems from science and economics that first year students have the tools to solve.

A big part of these problems is that they require students to sometimes select the proper technique without any other guidance. Because there are so many skills that students need to learn calculus tends to get compartmentalized. Students then know what technique they need to apply by seeing what chapter they are in. This changes when applied problems are presented, as there usually aren't clues as to how the problem needs to be solved. Another challenge is that the quantities are frequently not as "nice" as the prepared problems where the numbers often come out nicely. For these reasons, students will need some support in solving these extended problems. The key to many of these problems is setting up the proper calculus problem from the words. Drawing pictures, listing key information and the other common word problem techniques apply here also.

It is always a good idea to throw a couple of applied problems in with every assignment. Since free-response questions are a huge part of any standard exam, having practice is key, and it helps especially if they can be graded on the same rubric. These problems can be somewhat difficult to find. The College Board posts free response questions on their website from many years past. Most high school science textbooks are not going to be helpful, as they will not require calculus, but university physics, economics, statistics and probability texts will frequently have quality questions to use in the calculus classroom.

Lesson 36: Inverse Functions

Students tend to struggle mightily with inverse functions. In first year calculus classes a formal definition of functions and the sets the relate is not presented, so some of the language about injections and surjections is also not applicable making the formal definition of inverse functions not really possible in a first year class. Fortunately, only a basic understanding is needed.

There are two types of questions to focus on. First, it is more important that student can read a graph of a function and answer questions about inverses than to answer many questions about inverse functions from a rule. This also allows the focus to be on abnormal behavior, like discontinuities, and other abnormalities that are sometimes hard to reproduce with a function rule, but are easy to plot. The second type of question is finding the derivative of an inverse. There will be at least one question on the AP exam of this type. The problem is rather challenging if students don't have the formula memorized. Therefore, memorize the formula! It is an easy point or two if the time is taken to commit this to memory.

Lesson 37: Differentiation and Integration of Logarithmic and Exponential Functions

A consideration needs to be made whether to focus at all on logarithms that are not base e. Every once in a while students will encounter a problem that requires a derivative for an exponential that is not base e, but I have rarely encounter any integration problems that are anything else. While it is true that other bases have use in some applied sciences, base e has become so prevalent that most computer solving systems have base e set up as log and require a modifier to use any other base.

The importance of the log rules, especially those involving exponents, products and quotients, can't be understated. One of the challenges of the tougher integration problems is setting things up so that simple anti-derivatives can be found. These rules, like their analogues for trig functions, make it so that simpler problems can be created algebraically.

Students should be made aware of the fact that they are likely to encounter a ton of exponential problems. They are a favorite of problem writers because they have integrals and derivatives that work out nicely. They are common for applied problems because so many problems are exponential in nature and probably modeled with e as a base.

Lesson 38: Exponential Growth and Decay

The most important thing from this chapter is to get as many applications as possible. There are so many problems from physics and the social sciences that are modeled with exponentials that rich applied problems should not be hard to find. For me, this is the fun part of calculus. Earlier years of mathematics problems need to be deeply "sanitized", or cleaned up with contrived situations and unrealistic numbers to make the problems workable under the skillset of the students at the time. No such restriction is needed anymore at the calculus level and all of the preparation for students can pay off.

It is also worth spending some time with the number e. Pi gets all of the publicity, as far as transcendental numbers go, largely because people have lots of experience with circles, but the definition of e requires some calculus to find. It is, however, every bit as important as pi. As a fun thing to boggle the minds of students, a large pool of physicists, engineers and mathematicians were polled to find the "most important equation" a while ago by a major newspaper. Maxwell's equations, Newton's laws, the pythagorean theorem, the triangle inequality (not an equation... for shame) and $E = mc^2$ all received votes, but the winner was: $e^{\pi i} - 1 = 0$. Have students think about that: a transcendental number, to the power of another transcendental times an imaginary, added to an integer is... nothing. There are some very fine mathematicians that still don't really understand exactly how that works, but it is a nice relation for all the most important numbers in math.

Lesson 39: Derivatives and Integrals Involving Trigonometric Functions

Trig functions can be easy or a nightmare. The good is that sine and cosine have easy derivatives and integrals, and even the combination of these functions with others are still relatively easy problems to solve. The bad news is the seemingly completely unrelated nature of trig substitutions and inverse trig functions.

Some advanced classes may be able to have an understanding of why the inverse trig function integrals are the way they are. However, most first year classes will be lost, and are better served by attempting to memorize a couple of the functions. The derivatives of arctan, arcsine and arccosine are the three to memorize. There are usually a couple of questions that require the knowledge of these derivatives, or anti-derivatives, on the AP AB examination. They are often easy points to get if the form is known, so there is not much benefit in working problems that are very challenging. There are two parts to the process for students. First is knowing the forms, and the second is recognizing when to use them. There is an added level of difficulty in that there are problems that look very alike, but are solved in completely different ways. Example:

$$\int \frac{1}{\sqrt{4-x}}dx \qquad\qquad \int \frac{1}{\sqrt{4-x^2}}dx$$

The first integral is solvable by a simple substitution, but the second can't, and is an inverse trig function. While this may seem clear to an experienced mathematician, first year students will need many examples of problems that look similar, but some are solvable with substitution and others with inverse trig functions.

Lesson 40: L'Hopitals Rule

It will help to have students be able to quickly identify indeterminate forms that are solvable by l'Hopital's rule. First year students don't usually have the experience to tell the different forms apart. If students can identify the appropriate forms, then l'Hopital's rule is an easy one to apply. To wit, I see the rule being correctly applied in incorrect times more often than I see the rule incorrectly applied. More often than not the instructor will have to reign in student's use of the rule, as they are more than happy to apply it for nearly any limit they come across. It is rare, but I also have seen students confuse l'Hopital's rule with the quotient rule. It should be made clear that l'Hopital's rule does not involve taking the derivative of the whole function, but separates the numerator and denominator and treats each separately.

Another thing to point out is that the limit still needs to be evaluated after taking the derivative of the top and bottom functions. This means that while sometimes direct substitution will provide the solution, frequently more steps are needed to evaluate this new limit, including the continued use of l'Hoptial's rule.

Frequently l'Hopitals rule is presented after local linearity so that the existence can be justified. In a first year course I teach it right away after students are comfortable with derivatives. It is such a powerful and easy tool, there is no reason not to other than somewhat antiquated prove it before you use it habits.

Lesson 41: Integration by Substitution

Technically speaking substitution is the analogue of the chain rule. I never present it as such. It may be useful to let some student know that fact if they are having a bit of trouble, but for whatever reason, my students have all kinds of problems with understanding composition of functions. Therefore, I go with the "inside, outside" idea of which functions to substitute for. Another little thing to help out is making sure that students do not get lazy with their notation. Keeping track of all of the derivative terms will help with making the substitution work out correctly.

This is one of the few chapters where drill-and-kill is somewhat necessary. Substitution is the most common integration technique, so it's a section worth waiting, and reviewing, and practicing until the entire class feels comfortable with the skill. Since students will pick up the skill at very different rates, students with a quick understanding can move on to take on more challenging substitutions. There are a few listed in the problem solving guide, but these often involve making a substitution, then solving for a variable, or multiple substitutions. Such advanced problems are not necessary for a vast majority of first year students, but are helpful for students who may go on.

Lesson 42: Integration by Parts

Integration by parts is tricky for most students. The first barrier to overcome simply knowing the formula. This is one that has to be memorized, no way around it. Each student may have their own preferred way, just as each teacher has their own techniques. Some people like to sing songs (I avoid this like the plague) as a mnemonic device, you can have students write a little song or poem, make posters, have pop quizzes or other assignments testing knowledge. It's worth spending the time to have students know the formula.

The next thing to do is to have students develop a method for attacking parts problems. There is lots going on, with two variables being used for substitution, an anti-derivative and a derivative taken, and frequently additional steps after that. A small table with 4 spaces is really useful:

u	dv
du	v

Or some other orientation if it makes more sense. Having a routine will really help when the problems get tough.

There are a few classic problems that students are likely to encounter. The text outlines the problems of the exponential and another function type and the common technique. Another very common problem is the exponential and trig function problem that requires the "wrap-around" of the original integral to complete the problem. Students need to at the very least see and try this problem with guidance in class once.

Many instructors do not teach the tabular method, including myself. I find that the mechanics of it take too long for the limited benefit that it provides. However, you may find that some students really take to it, and it can be a useful tool.

Lesson 43: Integration by Partial Fractions

Partial fractions are not strictly a calculus topic, but are often introduced for the first time in calculus. The reason being that separating rational expressions into different termed fractions is really only used as a method for being able to take integrals. There are a couple of things that can make the method easier. First, students need to understand that the process of breaking up the rational expression does not have anything to do with integration or calculus. Yes, it is a needed step to get an expression that you can integrate, but it is a separate algebraic step, like applying trig identities or other substitutions of equivalent expressions.

I find that it helps to write every term and coefficient, even if the coefficient is zero. Setting up each of the equations becomes much more clear if students do not have to guess what the variable coefficients are equal to. Another thing that helps with little mistakes is using matrices. For two term, two variable systems there is no need, but for anything more students make far fewer mistakes using matrices and the calculator to find each of the constants.

In the grand scheme of things, partial fractions can eat up a lot of class time for a topic that is not as important as other integration techniques in the long run. If students don't understand this topic completely, it may be ok to move on. Substitution, and parts are much more important techniques.

Lesson 44: Trigonometric Integrals

A lot of trig integration, as well as all trig problems, involve the trig identities, half angle and double angle formulas. The volume of all of the identities can be overwhelming for students, so it is useful to target a few important ones and make sure students can utilize them. Here is my hierarchy of utility:

1. Know all the trig functions in terms of sine and cosine. This allows for less memorization in other areas, and will also help with not having to memorize all kinds of derivatives and anti-derivatives for the other trig functions.

2. $\sin^2 x + \cos^2 x = 1$. This is the grand daddy of them all.
3. The angle sum formulae, like $\sin(x + y) = \sin x \cos y + \sin y \cos x$. It is very difficult to take integrals with terms in the trig function, separating out into products helps.
4. The double angle formulae, like $\sin(2x) = 2 \sin x \cos x$
5. The power reduction formulae, like $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$. Notice that this is easily shown by a combination of formulae above. Hence, the low placement on this list.

There are certainly others, but again, most can be found through a combination of the simpler identities.

The book does not address hyperbolic trig functions. This is typical of calculus taught in the United States. The hyperbolic functions have quite a bit of utility in very complicated integrals, and are therefore taught more frequently in calculus classes in Asian countries. For practical purposes, they are only useful to the very top end students, and even then more for math competitions than classes. However, if you do have a student, or students, who are competing in integration bees or other math competitions, it is a useful thing to know.

Lesson 45: Trig Substitution

Trig substitution is not a key topic for integration in a first year class that is aligned with the AP standards. The only real requirement is to know the basic trig derivatives, and by consequence anti-derivatives, involving arctan, arcsine and arccosine. However, there is a great deal of good mathematical utility in solving problems with trig substitution. They are frequently very challenging problems that are tough to identify at first. Even with identification, making the correct substitution is still difficult.

I hold this section back until after the exam in an AP course. There are a couple of months that would otherwise be wasted, but there are some great calculus topics and problems that are not covered on the exam that fill the last month to month and a half nicely.

The key for students is pattern identification. These are not problems that can be easily figured out "on the fly", but rather need a good deal of supported practice. There are two dangers with such problems and methods. First is the spectator sport trap. Students begin to feel comfortable watching example after example being worked, feeling like the correct substitution is always obvious. They then have no clue how to tackle a problem on their own. The other problem is the follow the leader trap. This is where students take a verbal, or written example and can only pattern their work after the example they have. In each case, students need help not in understanding what is going on, but rather in transitioning to being self-sufficient in working these problems. There are many ways to accomplish this, but all of them involve making students work and think, and then giving them immediate feedback. This is not always easy to do, as students will do anything to avoid having to work out of their comfort zone. One of the finest tools I have seen in use is to have each student possess a small whiteboard and a dry erase marker. Students are then asked to work a problem up to a certain point and then prop up their solution on their whiteboard. From this vantage it is easy for the instructor to check everyone's work (and to make sure everyone is doing the work) and either move forward or stop and reinforce. In any case, students need to interact with the challenge of the open ended nature of problems like those integrals requiring trig substitution.

Lesson 46: Improper Integrals

Teaching improper integrals can be a little be awkward at this time. The reason being is that the ideas of convergence or divergence are more related to series, a later chapter. However, there isn't much need to understand the nature of what is going on. Many students will question why they need to go through the formality of replacing the infinity and using a limit, rather than just treating the infinity like a limit and interpreting it later. I had a student work a problem like this:

$$\int_1^\infty \frac{1}{x^2} = dx = \int_1^\infty x^{-2}dx = -x^{-1}|_1^\infty = \frac{-1}{\infty} - \frac{-1}{1} = 0 - -1 = 1$$

Which ends up being numerically correct, but mathematically all wrong. However, the harm done is...? It is up to each instructor to decide how much of a focus to put on having completely rigorous reasoning and notation. It depends substantially the level of the class, and what the future interests of those students are. It's painful to look at, but I don't have a major problem with the student's work listed above if it's from a high school first year class with students who are not the strongest math students in the school. I do have a huge problem if that's from a university math, physics or engineering student. The main point of all of this, however, is that improper integrals are not a major new idea for most students, and involve a simple extra step.

Lesson 47: Ordinary Differential equations

The important part of ODE in a first year class is not the mechanics of solving ODE, but the idea behind them. Engineering and physics depend on modeling systems through differential equations from observed rates. In fact, one of the most cited, but most misunderstood principles can be discussed with the class in this section. The Butterfly Effect is sometimes (probably due to the first Jurassic Park movie) linked to chaos theory, which is a stretch. The idea is that a butterfly flaps it's wings in West Africa and a hurricane traveling over Florida is born. The problem is really one about differential equations and initial conditions. The problem with very complex differential equations is not that our observations are bad (although sometimes they aren't good enough), nor is it that the math involved is poor, but rather that a solution depends on having initial conditions, and we either can't observe them, or can't do it accurately enough. A problem today involves the topology of the universe. With oversimplification, the shape of the universe can be determined by how much mass is in the universe. Unfortunately, with dark matter, and other quantities being yet unobservable, we can't solve the exact shape of the universe. This is a great topic to talk about, or have as a research project, after introducing ODE and especially slope fields.

Students can easily get confused about the treatment of differentials in separable equations. Tell them to suck it up. I'm only slightly kidding. I remember getting the sensation in first year calculus through PDE of "Wait, you can do that?" There is some justification for the madness in the problem solving guide, but in general the solution methods for differential equations are better treated as techniques to learn, practice and use, rather than to think deeply about why they work.

Lesson 48: Sequences

Students' greatest challenge here is notation. There are a whole bunch of new items, or familiar looking symbols and objects used in new ways. The two things I see students struggling with most are indices and showing infinitive behavior.

The indices problem is probably nothing new. Students struggle from the first day they see subscript numbers attached to variables. Indices can also mean different things in different situations. Maybe students are familiar with the most common presentation of the slope formula:

$$\delta = \frac{y_2 - y_1}{x_2 - x_1}$$

Where (x_1, y_1) and (x_2, y_2) are different points. Here the subscripts mean that there are two different pieces of information taken from the same variable in a function. This is not exactly the same as the indexing in sequences: $a_n = a_0, a_1, a_2, a_3, a_4, \ldots$. Here the index is referring to a place, and can store a number or an expression. Further confusing the matter, often the index will play a role in finding the value of that entry. For this reason, indexing is not always going to start in the same place, or work in the same manner. Another complication is that subsequences are often notated by a second subscript, a_{i_j} where sometimes there will be a dual indexed sequence, $a_{i,j}$ which is treated completely differently. Although both of those examples are less likely to occur in a first year course, they do give students fits the first time they see them. Not a whole lot can be done to help prepare the students in advance for the difficulties of indices. Therefore the best thing to do is to not take anything for granted and being explicit about the meaning every time sequences are being talked about until you are confident that students are on the same page.

Lesson 49: Infinite Series

Students are less likely to have the assignment of finding solutions to strange infinite series at this point. Therefore a conceptual treatment is not always the best use of time for this section. Practically, most students will be best served to be trained to recognize and use the particular forms for infinite series that are presented.

Geometric series are the most common, and the form and formula for convergence is on the "must-know" list. A tool that students will need to know to make geometric series work in less than perfect situations is the change of index tool. While on the surface this is an easy change to make, because it deals with a topic students tend to struggle with, I always complete, and insist my students complete, a check to make sure the first few terms end up being the same after the change of index. The more physical examples you can employ in this section the better the students will grasp the topic.

Students may take a little bit of extra time to understand sequences of partial sums. The way the topics are presented don't help, as we go from sequences, then add them to make series, and now go back to a sequence of those answers. To try to make things as clear as possible, I try to stay consistent with the "variables" I use for partial sums or sequences, I write way more than I would in my own work, rewriting to make things clear, and take extra time to make sure everyone understands where each number is coming from.

Since we are starting to develop a library of tests, I have some students start a long poster on a sheet of butcher paper. Whenever we encounter a new test, like the nth root test, we add it to the poster. By having this visual "crutch" for the duration of the unit takes some of the stress out of all of the information coming, and has the students focus on identifying and applying tests correctly. Honestly, this is probably around where the class really begins to get lost. There is no shame, or harm, in trying to make it as simple as possible.

Lesson 50: Series Without Negative Terms

This unit presents some of the most common series students are likely to encounter. Again, my strong recommendation is to have each student, or the class, or both, keeping a reference page/poster of each test as they come across them. They will seem very easy to apply when an exercise is set up with the correct test listed, but much harder when a choice has to be made by the student. Something to help with this is to keep looping back to an earlier problem or two without telling the students so that they can begin to develop some pattern recognition and keep practicing already learned tests.

Presenting this material can be less than exciting. There aren't very many application that can be shown at this time, as the goal is really to develop the toolbox of convergence tests. Also, there isn't really the possibility for having students developing rules or tests as the content is likely at the very limit of what students will feel comfortable with at best. Hang with it, and try to make sure students are getting lots of supported practice. Their enthusiasm when they "get it" is the sustaining energy through these chapters.

Lesson 51: Series With Odd or Even Negative Terms

The additional test listed in this section should be added to the previous tests. It is debatable whether to teach these as a separate unit, but any choice has its merits. I would recommend teaching the whole unit on series and tests as a single unit. A reason to break it up would be to provide practice for students in a timely fashion. I feel like the traditional process of lectures and then practice after is not the best choice for this unit. It is a very practical unit; it does not possess many conceptual or pure problems, but has many, many tools and skills to practice. Therefore students need lots and lots of practice, hopefully with a significant amount of support. This means that times where the teacher is presenting information directly should be kept to a minimum, and every effort should be made to have students directly involved with the problem solving process as soon as possible. It is also very important to have problems from the previous sections mixed in. A huge part of determining convergence of series is being able to recognize the form to apply the correct rule. The only way to help students develop this pattern recognition skill is to have frequent practice with unstructured sections, meaning that there are mixed techniques even in a chapter that is outlining a couple of specific tests.

Lesson 52: Ratio Test, Root Test and Summary of Tests

Students finally have all of the tests they will know for the first year of calculus. The table summary is perfect, and should be used by students extensively. The table not only gives a brief description of each test, but also is ordered in a hierarchy that will allow for the least amount of work if a student does not know what test to apply immediately. This is the key to any hierarchical process. Just like for integration students were encouraged to try a substitution, and then parts, and continue moving down the list, the list presented here is what students should follow. The idea is that you want to start out with the easiest and/or most accurate tools. Not only will this allow students to have success with the easiest problems, but also moves with the likely confidence levels of the students. That is to say, a student will probably not feel stuck or unsure with the easier tests, and will then likely have a greater chance of success overall. This is what the author means with "inexpensive". There isn't a huge investment, or risk, with applying many of the earlier tests, so it is beneficial to start there and progress to more complicated or obscure tests.

Lesson 53: Power Series

Power series are the first of what are the two key series, and the reasons for teaching all of the sequences and series in a first year class. There are a number of tricky points about power series, most specifically that they can be centered about a point that is not zero, but most of the time, it's zero. For that reason the most common description is the first one listed in this section, but the completely accurate description is the next one that is listed. Another unique concept to the power series is the radius of convergence. Students need to understand that the inclusion of a variable makes things a little bit different for power series, and that the special rules that are included here are because of that.

The key point to get across and practice is finding the radius of convergence. Virtually any question a first year students is likely to encounter about power series is going to be about the radius of convergence. Fortunately, even students who are have a tough time understanding exactly what is going on with power series can follow the somewhat algorithmic process.

Lesson 54: Taylor and MacLaurin Series

Congratulations if you are getting here with your first year class. I say it somewhat in jest, but it seems like in the great tradition of running out of time in the school year, and how most US history classes get to WWII and then run out of time, I have never had a class that has been able to give a complete treatment for Taylor series. Taylor series can also open a can of worms, as there is so much to these, and they are so important in later mathematics that any treatment a first year class can give feels inadequate.

So with the time crunch in mind, along with the vast applications and topics, what is the key information to get to students? First, students need to understand that there are methods of non-linear approximation for functions that are very, very accurate. All of the theory behind why Taylor series work is beyond the scope of a first year class, but the computation of Taylor series is easy, and students should be comfortable with a couple of simple examples. Second, students should understand that Taylor series is how computation math gets done. Calculators and computer can't have a table for all the values for transcendental functions, and they, by definition, can't be defined by elementary functions. They can, however, be closely approximated by Taylor polynomials and that is exactly what computer math programs do. By focusing on the Taylor expansion of sine and e^x students can get a sense for the magnitude and scope of the topic.

MacLaurin series are an instance of Taylor series, and I have only ever heard them referred to in first year calculus books. I suspect that specific mention of them is not key to any course, now or in the future.